THE SECOND POST-NEWTONIAN EQUATIONS OF HYDRODYNAMICS
IN GENERAL RELATIVITY

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ABSTRACT

In this paper the equations of hydrodynamics governing a perfect fluid in the second post-Newtonian approximation to general relativity are derived; in this approximation all terms inclusive of $O(c^{-4})$ are retained consistently with Einstein's field equations. And the equation $T^{ij} = 0$ governing the energy-momentum tensor is also derived to $O(c^{-4})$.

The various conserved quantities to $O(c^{-4})$ are isolated with the aid of the Landau-Lifshitz complex $\Theta^{i}$. In particular, to obtain the conserved energy to $O(c^{-4})$ it is necessary to evaluate the metric coefficient $g_{ab}$ to $O(c^{-4})$, i.e., to an order appropriate to the third post-Newtonian approximation.

1. INTRODUCTION

In an earlier paper (Chandrasekhar 1965; this paper will be referred to hereinafter as Paper I) the equations of hydrodynamics governing a perfect fluid, in the first post-Newtonian approximation to general relativity, were derived; in this approximation all terms of order $c^{-2}$ are retained consistently with Einstein's field equations. In the present paper, the corresponding equations in the second post-Newtonian approximation are derived; in this approximation all terms inclusive of order $c^{-4}$ are retained. It will appear that in this second approximation the conserved quantities (except the total baryon number) can no longer be obtained by a simple inspection of the equation of motion

$$T^{ij}_{;j} = 0,$$

expressing the vanishing of the covariant divergence of the energy-momentum tensor $T^{ij}$ as it was possible in the first approximation. However, the equation

$$\Theta^{i} = 0,$$

expressing the vanishing of the ordinary divergence of the Landau-Lifshitz complex $\Theta^{i}$, enables us to determine the conserved quantities by a simple and direct algorithm (cf. Chandrasekhar 1969; this paper will be referred to hereinafter as Paper II).

In this paper, besides obtaining the explicit form of equation (1) in the second post-Newtonian approximation, the various conserved quantities are also explicitly determined.

II. THE INFORMATION PROVIDED BY THE EQUATIONS
OF THE FIRST POST-NEWTONIAN APPROXIMATION

The method by which we shall obtain the equations in the second post-Newtonian approximation is similar to the one by which the equations of the first post-Newtonian were obtained from the equations and relations valid in the Newtonian limit. Thus, with the improved knowledge of the metric coefficients provided by the first post-Newtonian approximation in Paper I, we can evaluate the components of the energy-momentum tensor to one higher order than was known before; this improved knowledge of the energy-momentum tensor will enable us to solve the field equations for the metric coefficients to one higher order; and this improved knowledge of the metric coefficients,
in turn, will enable us to obtain the equations governing the fluid in the second post-Newtonian approximation.

The coefficients of the metric tensor as determined in Paper I are

\[ g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4} (U^2 - 2\Phi), \quad g^{00} = 1 + \frac{2U}{c^2} + \frac{2}{c^4} (U^2 + 2\Phi), \]

\[ g_{\alpha\beta} = g^{\alpha\beta} = \frac{1}{c^2} P_\alpha, \quad g_{s\beta} = \left( 1 + \frac{2U}{c^2} \right) \delta_{s\beta}, \quad \text{and} \quad g^{s\beta} = \left( 1 - \frac{2U}{c^2} \right) \delta_{s\beta}, \tag{3} \]

where \( U \) denotes the Newtonian gravitational potential due to the prevailing distribution of \( \rho \), and \( \Phi \) and \( P_\alpha \) are further potentials determined as solutions of the equations

\[ \nabla^2 \Phi = -4\pi G \rho \phi, \quad \phi = v^2 + U + \frac{3}{2} \Pi + \frac{3}{2} \frac{p}{\rho}, \tag{4} \]

and

\[ \nabla^2 P_\alpha = 4\nabla^2 U_\alpha - \frac{3}{c^2} \frac{\partial^2 \Phi}{\partial t \partial x_\alpha} \nabla^2 x = -16\pi G \rho v_\alpha + \frac{\partial^2 U}{\partial t \partial x_\alpha}. \tag{5} \]

Also, it should be noted that the solution for the metric coefficients given in equations (3) is obtained in the gauge

\[ \frac{\partial P_\alpha}{\partial x_\mu} + 3 \frac{\partial U}{\partial t} = 4 \left( \frac{\partial U_\mu}{\partial x_\alpha} + \frac{\partial U}{\partial t} \right) = O \left( \frac{1}{c^2} \right). \tag{6} \]

The expressions for the components of the four velocity that follow from equations (3) are

\[ u^\alpha = 1 + \frac{1}{c^2} \left( \frac{3}{2} v^2 + U \right) + \frac{1}{c^4} \left( \frac{3}{2} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + 2\Phi - v_\mu P_\mu \right), \]

\[ u^\mu = \frac{1}{c} v_\mu \left[ 1 + \frac{1}{c^2} \left( \frac{3}{2} v^2 + U \right) + \frac{1}{c^4} \left( \frac{3}{2} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + 2\Phi - v_\mu P_\mu \right) \right], \tag{7} \]

\[ u_0 = 1 + \frac{1}{c^2} \left( \frac{3}{2} v^2 - U \right) + \frac{1}{c^4} \left( \frac{3}{2} v^4 + \frac{3}{2} v^2 U + \frac{1}{2} U^2 - 2\Phi \right), \]

and

\[ u_\alpha = - \frac{1}{c} v_\alpha + \frac{1}{c^3} \left[ P_\alpha - (\frac{3}{2} v^2 + 3U) v_\alpha \right]. \]

And from the definition

\[ T_{ij} = \rho (c^2 + \Pi + \frac{p}{\rho}) u_i u_j - \frac{1}{2} g_{ij}, \tag{8} \]

we deduce

\[ T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T = \frac{1}{2} \rho c^2 \delta_{\alpha\beta} + \rho \left[ u_\alpha u_\beta + \left( U + \frac{3}{2} \Pi - \frac{3}{2} \frac{p}{\rho} \right) \delta_{\alpha\beta} \right], \tag{9} \]

\[ T_{0\alpha} - \frac{1}{2} g_{0\alpha} T = - \rho c v_\alpha + \frac{1}{c} \rho \left[ \frac{3}{2} P_\alpha - \left( v^2 + 2U + \Pi + \frac{p}{\rho} \right) v_\alpha \right], \tag{10} \]

and

\[ T_{00} - \frac{1}{2} g_{00} T = \frac{1}{2} \rho c^2 + \rho \left( v^2 - U + \frac{3}{2} \Pi + \frac{3}{2} \frac{p}{\rho} \right) \]

\[ + \frac{1}{c^2} \rho \left[ v^4 + 2v^2 U + U^2 - 2\Phi + \Pi (v^2 - U) + \frac{p}{\rho} (v^2 - 3U) \right]. \tag{11} \]

With the foregoing information provided by the first post-Newtonian approximation, we can proceed to obtain the equations appropriate to the next higher approximation.
We now suppose that the metric coefficients in the second post-Newtonian approximation have the forms

\[ g_{\alpha \beta} = - \left(1 + \frac{2U}{c^2}\right) \delta_{\alpha \beta} + \frac{1}{c^4} Q_{\alpha \beta}, \]

\[ g_{00} = \frac{1}{c^4} P_\alpha + \frac{1}{c^6} Q_{00}, \]

and

\[ g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4} (U^2 - 2\Phi) + \frac{1}{c^6} Q_{00}, \]

where the \( Q_{ij} \)'s are certain coefficients to be determined.

**a) The Equation Determining \( Q_{\alpha \beta} \)**

With the assumed form for the metric coefficients, the \((\alpha, \beta)\)-component of the Ricci tensor can be evaluated to \(O(\epsilon^4)\). We find

\[ R_{\alpha \beta} = \frac{1}{c^2} \nabla^2 U \delta_{\alpha \beta} - \frac{1}{2c^4} \left[ \nabla^2 Q_{\alpha \beta} + \frac{\partial^2 Q_{\mu \nu}}{\partial x_\mu \partial x_\nu} - \frac{\partial}{\partial x_\mu} \left( \frac{\partial Q_{\alpha \mu}}{\partial x_\nu} + \frac{\partial Q_{\nu \mu}}{\partial x_\alpha} \right) \right] \]

\[ + \frac{1}{c^4} \left[ - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (U^2 + 2\Phi) + 2 \frac{\partial U}{\partial x_\alpha} \frac{\partial U}{\partial x_\beta} - \frac{1}{2} \frac{\partial}{\partial t} (\frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha}) \right] \]

\[ - \delta_{\alpha \beta} \left( \frac{\partial^2 U}{\partial t^2} + \nabla^2 U^2 \right). \]

And since (cf. eqs. [4] and [9])

\[ - \frac{8\pi G}{c^4} (T_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} T) = \frac{1}{c^4} \nabla^2 U \delta_{\alpha \beta} \]

\[ + \frac{1}{c^4} \left[ 2\nabla^2 \Phi \delta_{\alpha \beta} - 8\pi G \rho \left( v_\alpha v_\beta - v^2 \delta_{\alpha \beta} - 2 \frac{\rho}{\rho} \delta_{\alpha \beta} \right) \right], \]

the \((\alpha, \beta)\)-component of Einstein's field equation gives

\[ \nabla^2 Q_{\alpha \beta} - \frac{\partial}{\partial x_\alpha} \left( \frac{\partial Q_{\beta \mu}}{\partial x_\mu} \right) - \frac{\partial}{\partial x_\beta} \left( \frac{\partial Q_{\alpha \mu}}{\partial x_\mu} \right) = S_{\alpha \beta}, \]

where

\[ S_{\alpha \beta} = -2 \left( \nabla^2 \delta_{\alpha \beta} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) (U^2 + 2\Phi) + 16\pi G \rho \left( v_\alpha v_\beta - v^2 \delta_{\alpha \beta} - \frac{2}{\rho} \delta_{\alpha \beta} \right) \]

\[ - \frac{\partial}{\partial t} \left( \frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha} \right) - 2 \delta_{\alpha \beta} \frac{\partial U}{\partial t} + 4 \frac{\partial U}{\partial x_\alpha} \frac{\partial U}{\partial x_\beta}. \]

There is an integrability condition for the solvability of equation (15). Thus, by contracting equation (15), we obtain

\[ \nabla^2 Q_{\alpha \alpha} - \frac{\partial^2 Q_{\alpha \mu}}{\partial x_\alpha \partial x_\mu} = \frac{1}{2} S_{\alpha \alpha}; \]

while by differentiating it with respect to \( x_\alpha \), we obtain

\[ \frac{\partial}{\partial x_\alpha} \left( \nabla^2 Q_{\alpha \alpha} - \frac{\partial^2 Q_{\alpha \mu}}{\partial x_\alpha \partial x_\mu} \right) = \frac{\partial S_{\alpha \beta}}{\partial x_\beta}. \]
Accordingly, we must have
\[ \frac{\partial}{\partial x_\beta} (S_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta}S_{\gamma\gamma}) = 0. \]  
(19)

Now it can be verified that \( S_{\alpha\beta} \) defined in equation (16) does indeed satisfy the condition (19). Thus, by contracting equation (16) we find
\[ S_{\gamma\gamma} = -8
\nu^2(U^2 + 2\Psi) - 16\pi G(2\rho v^2 + 6P) + \frac{4}{(\partial U/\partial x_\mu)^2} \]
(20)
\[ = 32\pi G\rho(v^2 + 4U + \Pi) - 12\left(\frac{\partial U}{\partial x_\mu}\right)^2, \]
where we have made use of equation (4) defining \( \Phi \). A direct evaluation now leads to the result
\[ \frac{\partial}{\partial x_\beta} (S_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta}S_{\gamma\gamma}) = 16\pi G \left[ \frac{\partial}{\partial t} (\rho v_\alpha) + \frac{\partial}{\partial x_\beta} (\rho v_\alpha v_\beta) - \rho \frac{\partial U}{\partial x_\alpha} + \frac{\partial P}{\partial x_\alpha} \right]. \]
(21)

Since we are here concerned with the terms of the highest order that we are presently retaining, we are justified in evaluating these terms in the Newtonian limit; and in this limit the right-hand side of equation (21) does indeed vanish as required.

With the integrability condition (19) satisfied, the general solution of equation (15), as we shall now show, involves an arbitrary vector function. Thus, letting
\[ \frac{\partial Q_{\alpha\beta}}{\partial x_\beta} = \frac{1}{2} \frac{\partial Q_{\gamma\gamma}}{\partial x_\alpha} = w_\alpha, \]
(22)
we can rewrite equation (15) in the form
\[ \nabla^2 Q_{\alpha\beta} = S_{\alpha\beta} + \frac{\partial w_\alpha}{\partial x_\beta} + \frac{\partial w_\beta}{\partial x_\alpha}; \]
(23)
and the contraction of this equation yields
\[ \nabla^2 Q_{\gamma\gamma} = S_{\gamma\gamma} + 2 \frac{\partial w_\gamma}{\partial x_\gamma}. \]
(24)

With the aid of equations (23) and (24) we readily verify that, by virtue of the integrability condition being satisfied,
\[ \frac{\partial}{\partial x_\beta} \nabla^2 \left( Q_{\gamma\beta} - \frac{1}{2}\delta_{\gamma\beta}Q_{\gamma\gamma} \right) = \nabla^2 w_\beta; \]
(25)
and we recover the defining equation (22). Therefore, we may consider \( w_\alpha \) in equation (23) an arbitrary vector function in space-time.

b) The Equation Determining \( Q_{00} \)

With the assumed form for the metric coefficients, the \((0,0)\)-component of the Ricci tensor can be evaluated to \( O(c^{-3}) \). In evaluating it, we shall let (cf. eq. [6])
\[ 4 \left( \frac{\partial U}{\partial t} + \frac{\partial U_\mu}{\partial x_\mu} \right) + \frac{1}{c^2} \left( \frac{\partial Q_{00}}{\partial x_\alpha} - \frac{1}{2} \frac{\partial Q_{\gamma\gamma}}{\partial t} \right) = \frac{1}{c^2} w, \]
(26)
where \( w \) is, for the present, an unspecified function in space-time. We find
\[ R_{00} = -\frac{2}{c^2} \nabla^2 U_{00} - \frac{1}{2c^2} \left( \nabla^2 Q_{00} - \frac{\partial w}{\partial x_0} - \frac{\partial w_0}{\partial t} \right) \]
\[ + \frac{1}{c^2} \left( 4U \nabla^2 U_{00} + P_\mu \frac{\partial^2 U}{\partial x_\mu \partial x_0} - \frac{\partial U}{\partial x_\mu} \frac{\partial P_\mu}{\partial x_0} - 5 \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_0} \right), \]
(27)
SECOND POST-NEWTONIAN EQUATIONS

With the aid of equation (10), we now find that the \((0,a)\)-component of the field equation gives

\[
\nabla^2 Q_{0a} = -16\pi G\rho \left( v^2 + 4U + \Pi + \frac{\dot{P}}{\rho} \right) v_a + 8\pi G\rho P_a
\]

\[+ 2P_a \frac{\partial^2 U}{\partial x_a \partial x_a} - 2 \frac{\partial P_a}{\partial x_a} \frac{\partial U}{\partial x_a} - 10 \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} + \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial x_a}. \tag{28}\]

It remains to verify that equation (26) is consistent with equations (24) and (28). By straightforward reductions we find

\[
\nabla^2 \left[ 4 \left( \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x_n} \right) + \frac{1}{c^2} \left( \frac{\partial Q_{0n}}{\partial x_n} - \frac{1}{3} \frac{\partial Q_{n0}}{\partial t} \right) \right] = \frac{1}{c^2} \nabla^2 \omega
\]

\[+ 16\pi G \left[ \frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x_n} \left( \sigma v_n \right) + \frac{1}{c^2} \left( \frac{\partial U}{\partial t} - \frac{\partial P}{\partial t} \right) \right], \tag{29}\]

where

\[
\sigma = \rho + \frac{1}{c^2} \rho \left( v^2 + 2U + \Pi + \frac{\dot{P}}{\rho} \right). \tag{30}\]

In view of the equation of continuity in the first post-Newtonian approximation (Paper I, eq. [64]), the right-hand side of equation (29) reduces to \(\nabla^2 \omega / c^2\); and we recover the defining equation (26). Therefore, we may consider \(\omega\) in equation (28) an arbitrary function in space-time.

c) The Equation Determining \(Q_{00}\)

In evaluating the \((0,0)\)-component of the Ricci tensor, we find that we need the contravariant component \(g^{ab}\) of the metric tensor. We readily verify that

\[
g^{ab} = - \left( 1 - \frac{2U}{c^2} \right) \delta_{ab} - \frac{1}{c^2} \left( Q_{a0} + 4U^2 \delta_{a0} \right). \tag{31}\]

With the assumed form of the metric coefficients, \(R_{00}\) is determined to \(O(c^{-6})\); and we find

\[
R_{00} = \frac{1}{c^6} \nabla^2 U - \frac{1}{c^2} 8\pi G\rho \left( v^2 - U + \frac{1}{3} \Pi + \frac{3}{2} \frac{\dot{P}}{\rho} \right)
\]

\[+ \frac{1}{c^6} \left( \nabla^2 Q_{00} - 2w_\mu \frac{\partial U}{\partial x_\mu} - 2 \frac{\partial \omega}{\partial t} - 4 \frac{\partial W_\mu}{\partial x_\mu} \frac{\partial^2 U}{\partial x_\mu \partial x_\mu} \right)
\]

\[+ \frac{1}{c^2} \left( \Sigma_{\alpha\beta} + 4U^2 \delta_{\alpha\beta} \right) \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} + 2U \nabla^2 (U^2 - 2\Phi) + \frac{\partial U}{\partial x_\alpha} \frac{\partial P_\alpha}{\partial t}
\]

\[+ \frac{\partial U}{\partial x_\alpha} \frac{\partial}{\partial x_\alpha} (6\Phi - 5U^2) - 3 \left( \frac{\partial U}{\partial t} \right)^2 - 8 \frac{\partial U}{\partial x_\alpha} \left( \frac{\partial U}{\partial x_\alpha} - \frac{\partial U}{\partial x_\alpha} \right) \right], \tag{32}\]

where \(\nabla^2 W_\mu = w_\mu \) (see eq. [40] below) and \(\nabla^2 \Sigma_{\alpha\beta} = \Sigma_{\alpha\beta} \). The \((0,0)\)-component of the field equation now gives

\[
\nabla^2 Q_{00} = 2w_\mu \frac{\partial U}{\partial x_\mu} + 2 \frac{\partial \omega}{\partial t} + 4 \frac{\partial W_\mu}{\partial x_\mu} \frac{\partial^2 U}{\partial x_\mu \partial x_\mu}
\]

\[+ 16\pi G\rho \left[ v^2 \left( v^2 + 4U + \Pi + \frac{\dot{P}}{\rho} \right) - (U^2 + 2\Phi) \right]
\]

\[+ 2\Sigma_{\alpha\beta} \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} - 12U \left( \frac{\partial U}{\partial x_\alpha} \right)^2 + 12 \frac{\partial U}{\partial x_\alpha} \frac{\partial \Phi}{\partial x_\alpha} + 2 \frac{\partial U}{\partial x_\alpha} \frac{\partial P_\alpha}{\partial t}
\]

\[+ 6 \left( \frac{\partial U}{\partial t} \right)^2 - \frac{\partial P_\mu}{\partial x_\mu} \left( \frac{\partial P_\mu}{\partial x_\mu} - \frac{\partial P_\mu}{\partial x_\mu} \right). \tag{33}\]
d) The Choice of the Gauge \( w_a = w = 0 \)

We have seen that the solutions for the coefficients \( Q_{ij} \) involve four arbitrary functions: \( w \) and the vector function \( w_a \). This is consistent with the requirements of covariance that we have the freedom to effect arbitrary coordinate transformations that do not affect the choice already made in the first post-Newtonian approximation.

It is convenient to work in a particular gauge; and unless we explicitly state otherwise, the gauge we shall adopt will be based on the choice

\[ w_a = w = 0 \tag{34} \]

In this gauge

\[ \frac{\partial Q_{a\beta}}{\partial x_\beta} - \frac{1}{3} \frac{\partial Q_{s\sigma}}{\partial x_a} = 0 \tag{35} \]

and

\[ \frac{\partial P_{\mu}}{\partial x_\mu} + 3 \frac{\partial U}{\partial t} + \frac{1}{c^2} \left( \frac{\partial Q_{0\mu}}{\partial x_\mu} - \frac{1}{3} \frac{\partial Q_{s\sigma}}{\partial t} \right) = 0 ; \tag{36} \]

and we may further identify \( \Sigma_{\mu\nu} \) with \( Q_{\mu\nu} \) in equation (33). (Strictly speaking, we should write on the right-hand sides of eqs. [35] and [36] \( O(c^{-2}) \), and \( O(c^{-4}) \), respectively.)

In the chosen gauge, the equation governing \( Q_{s\sigma} \) is (cf. eqs. [20] and [24])

\[ \nabla^2 Q_{s\sigma} = 32\pi G \rho (v^2 + 4U + 2\Pi) - 12 \left( \frac{\partial U}{\partial x_\mu} \right)^2 . \tag{37} \]

An alternative form of this equation which we shall find useful is

\[ \nabla^2 (Q_{s\sigma} + 6U^2) = 16\pi G \rho (2v^2 + 2\Pi + 5U) . \tag{38} \]

If the occasion arises when we wish to determine the effect of the chosen gauge on the results which are obtained, we shall distinguish the quantities derived on the gauge (34) by asterisks, e.g., \( Q^*_{ij} \) instead of simply \( Q_{ij} \). Then, the solutions in a general gauge (the unasterisked quantities) are related to the solutions in the chosen gauge by the transformations

\[ Q_{a\beta} = Q^*_{a\beta} + \frac{\partial W_\alpha}{\partial x_\beta} + \frac{\partial W_\beta}{\partial x_\alpha} , \quad Q_{s\sigma} = Q^*_{s\sigma} + 2 \frac{\partial W_\sigma}{\partial x_\mu} , \]

\[ Q_{0a} = Q^*_{0a} + \frac{\partial W_\alpha}{\partial x_0} + \frac{\partial W_\sigma}{\partial t} , \quad \text{and} \quad Q_{00} = Q^*_{00} + 2 \frac{\partial W_\sigma}{\partial t} + 2W_0 , \tag{39} \]

where

\[ \nabla^2 W_\alpha = w_\alpha , \quad \nabla^2 W = w , \quad \text{and} \quad \nabla^2 W_0 = w_\mu \frac{\partial U}{\partial x_\mu} + 2 \frac{\partial W_\mu}{\partial x_\mu} \frac{\partial^2 U}{\partial x_\mu \partial x_\nu} . \tag{40} \]

**IV. Christoffel Symbols and Related Quantities in the Second Post-Newtonian Approximation**

With the metric coefficients given in equations (12) we readily find that

\[ g = - \left[ 1 + \frac{4U}{c^2} + \frac{1}{c^4} \left( 2U^2 - 4\Phi - Q_{s\sigma} \right) \right] , \tag{41} \]

\[ \sqrt{-g} = 1 + \frac{2U}{c^2} - \frac{1}{c^4} \left( U^2 + 2\Phi + \frac{1}{3} Q_{s\sigma} \right) , \tag{42} \]

\(^1\) We are grateful to Drs. F. B. Estabrook, B. K. Harrison, and H. D. Wahlquist for a full clarification of the relationship of the transformations (39) to the general covariance requirements of general relativity. It should, however, be emphasized that in this paper we mean by a "gauge transformation" the transformation (39) which enables us to pass from solutions for the metric coefficients with a particular choice of the functions \( w \) and \( w_a \) to a different choice of the same functions.
and
\[ \log \sqrt{-g} = \frac{2U}{c^2} - \frac{1}{c^4} (3U^2 + 2\Phi + \frac{1}{3}Q_{\mu\nu}) . \] (43)

From equation (43) it follows that
\[ y_a = \Gamma^i_{ia} = \frac{\partial \log \sqrt{-g}}{\partial x_a} = \frac{2}{c^2} \frac{\partial U}{\partial x_a} - \frac{1}{c^4} \frac{\partial f}{\partial x_a}, \]
\[ y_0 = \Gamma^i_{i0} = \frac{\partial \log \sqrt{-g}}{\partial x_0} = \frac{2}{c^2} \frac{\partial U}{\partial t} - \frac{1}{c^4} \frac{\partial f}{\partial t}, \] (44)

where
\[ f = 3U^2 + 2\Phi + \frac{1}{3}Q_{\mu\nu} . \] (45)

We shall further let (see eqs. [47] below)
\[ F = \frac{2}{3}U^3 - 4U\Phi + \frac{1}{3}Q_{\mu\nu} . \] (46)

Now evaluating the various Christoffel symbols with the known metric coefficients in the second post-Newtonian approximation (and in the chosen gauge) and including only those terms that we are entitled to retain, we find
\[ \Gamma^\nu_{\alpha\beta} = -\frac{1}{c^3} \frac{\partial U}{\partial t} + \frac{1}{c^4} \left( P_\mu \frac{\partial U}{\partial x_\mu} - 2 \frac{\partial \Phi}{\partial t} \right) \]
\[ + \frac{1}{c^6} \left[ \frac{\partial}{\partial t} \left( F + \frac{1}{2}P_\mu \frac{\partial P_\nu}{\partial x_\nu} \right) - P_\mu \frac{\partial}{\partial x_\mu} \left( U^2 - 2\Phi \right) + Q_{\mu\nu} \frac{\partial U}{\partial x_\mu} \right] , \]
\[ \Gamma^\nu_{0\alpha} = -\frac{1}{c^3} \frac{\partial U}{\partial x_\alpha} - \frac{2}{c^4} \frac{\partial \Phi}{\partial x_\alpha} + \frac{1}{c^6} \left[ \frac{\partial}{\partial x_\alpha} \left( F + \frac{1}{2}P_\mu \frac{\partial P_\nu}{\partial x_\nu} - \frac{3}{2}P_\mu \frac{\partial P_\nu}{\partial x_\nu} - P_\mu \frac{\partial U}{\partial t} \right) \right] , \]
\[ \Gamma^\nu_{\alpha\beta} = + \frac{1}{2c^3} \left( \frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha} + 2\delta_{\alpha\beta} \frac{\partial U}{\partial t} \right) \]
\[ + \frac{1}{c^5} \left[ U \left( \frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha} + 2\delta_{\alpha\beta} \frac{\partial U}{\partial t} \right) - \left( P_\alpha \frac{\partial U}{\partial x_\beta} + P_\beta \frac{\partial U}{\partial x_\alpha} - \delta_{\alpha\beta} \frac{\partial P_\mu}{\partial x_\mu} \frac{\partial U}{\partial x_\nu} \right) \right] \]
\[ + \frac{1}{2c^3} \left( \frac{\partial Q_{\alpha\beta}}{\partial x_\beta} + \frac{\partial Q_{\beta\alpha}}{\partial x_\alpha} - \frac{\partial Q_{\alpha\beta}}{\partial t} \right) , \] (47)
\[ \Gamma^\nu_{\alpha0} = -\frac{1}{c^2} \frac{\partial U}{\partial x_\alpha} + \frac{1}{c^4} \left[ 2 \frac{\partial}{\partial x_\alpha} \left( U^2 - \Phi \right) - \frac{\partial P_\alpha}{\partial t} \right] \]
\[ - \frac{1}{c^6} \left\{ 2U \left[ 2 \frac{\partial}{\partial x_\alpha} \left( U^2 - \Phi \right) - \frac{\partial P_\alpha}{\partial t} \right] + P_\alpha \frac{\partial U}{\partial t} + \frac{\partial Q_{\alpha0}}{\partial t} \right\} \]
\[ - \frac{1}{2c^3} \left( \frac{\partial P_\alpha}{\partial x_\alpha} - \frac{\partial P_\beta}{\partial x_\alpha} - 2\delta_{\alpha\beta} \frac{\partial U}{\partial t} \right) \]
\[ + \frac{1}{c^5} \left[ U \left( \frac{\partial P_\alpha}{\partial x_\beta} - \frac{\partial P_\beta}{\partial x_\alpha} - 2\delta_{\alpha\beta} \frac{\partial U}{\partial t} \right) - P_\alpha \frac{\partial U}{\partial x_\beta} - \frac{1}{2} \left( \frac{\partial Q_{\alpha0}}{\partial x_\alpha} - \frac{\partial Q_{\beta0}}{\partial x_\beta} + \frac{\partial Q_{\alpha0}}{\partial t} \right) \right] , \]
and
\[
\Gamma_{\alpha\beta\gamma} = \frac{1}{c^4} \left( \frac{\partial U}{\partial x_{\alpha}} \delta_{\beta\gamma} + \frac{\partial U}{\partial x_{\beta}} \delta_{\gamma\alpha} - \frac{\partial U}{\partial x_{\gamma}} \delta_{\alpha\beta} \right)
\]
\[
- \frac{1}{c^4} \left[ 2U \left( \frac{\partial U}{\partial x_{\alpha}} \delta_{\beta\gamma} + \frac{\partial U}{\partial x_{\beta}} \delta_{\gamma\alpha} - \frac{\partial U}{\partial x_{\gamma}} \delta_{\alpha\beta} \right) + \frac{1}{2} \left( \frac{\partial Q_{\alpha\beta}}{\partial x_{\gamma}} + \frac{\partial Q_{\alpha\gamma}}{\partial x_{\beta}} - \frac{\partial Q_{\beta\gamma}}{\partial x_{\alpha}} \right) \right].
\]

Also, in the chosen gauge
\[
\Gamma_{\alpha\nu} = \frac{1}{c^5} P_a \frac{\partial U}{\partial x_{\alpha}} \quad \text{and} \quad \Gamma_{\alpha\nu}^\alpha = -\frac{1}{c^5} \frac{\partial U}{\partial x_{\alpha}} + \frac{1}{c^4} \frac{\partial U^\nu}{\partial x_{\alpha}}.
\]

It should be noted that for the purposes of deriving the various equations in the second post-Newtonian approximation, we do not need the terms of \(O(c^{-3})\) in \(\Gamma_{\alpha\nu}^\alpha\) and the terms of \(O(c^{-4})\) in \(\Gamma_{\alpha\nu}\); i.e., for these particular Christoffel symbols the expressions derived for them with the knowledge provided by the first post-Newtonian approximation are sufficient. (However, we shall need the additional terms in § X below.)

V. THE EQUATIONS GOVERNING THE FLUID IN THE SECOND POST-NEWTONIAN APPROXIMATION

First we observe that with our present knowledge of the metric coefficients, the expressions for the contravariant components of the energy-momentum tensor are
\[
T^{00} = \rho c^2 + \rho (s^2 + 2U + \Pi)
\]
\[
+ \frac{1}{c^6} \rho \left[ v^4 + 6s^2U + 2U^2 + 4\Phi - 2v_\mu P_\mu + 2U\Pi + s^2 \left( \Pi + \frac{\rho}{\rho} \right) \right],
\]
\[
T^{0\alpha} = \rho c v_\alpha + \frac{1}{c} \rho v_\alpha \left( s^2 + 2U + \Pi + \frac{\rho}{\rho} \right)
\]
\[
+ \frac{1}{c^3} \rho \left\{ v_\alpha \left[ v^4 + 6s^2U + 2U^2 + 4\Phi - 2v_\mu P_\mu + (s^2 + 2U) \left( \Pi + \frac{\rho}{\rho} \right) \right] - \frac{\rho}{\rho} P_\alpha \right\},
\]
and
\[
T^{\alpha\beta} = \rho v_\alpha v_\beta + \rho \delta_{\alpha\beta} + \frac{1}{c^3} \left[ \rho v_\alpha v_\beta \left( s^2 + 2U + \Pi + \frac{\rho}{\rho} \right) - 2\rho U \delta_{\alpha\beta} \right]
\]
\[
+ \frac{1}{c^4} \left\{ \rho v_\alpha v_\beta \left[ v^4 + 6s^2U + 2U^2 + 4\Phi - 2v_\mu P_\mu + (s^2 + 2U) \left( \Pi + \frac{\rho}{\rho} \right) \right] \right\}
\]
\[
+ \rho \left( 4U^2 \delta_{\alpha\beta} + Q_{\alpha\beta} \right).
\]

The relevant equations governing the motions in the fluid in the second post-Newtonian approximation can be obtained by simply writing out the equation
\[
T_{\beta;\beta} = 0
\]
with the aid of equations (47) and (49)–(51). We need not, however, write down the 0-component of equation (52) since this equation is no more than the expression of the constancy of the volume integral of \(\rho v^0 \sqrt{-g}\). Evaluating this quantity with the aid of equations (7) and (42), we find
\[
\rho v^0 \sqrt{-g} = \rho \left[ 1 + \frac{1}{c^3} \left( \frac{3}{8} v^2 + 3U \right) + \frac{1}{c^4} \left( \frac{3}{8} v^4 + \frac{7}{3} s^2U + \frac{2}{3} U^2 - \frac{1}{3} Q_{\alpha\beta} - v_\mu P_\mu \right) \right].
\]
The reduction of the $a$-component of equation (52) is long but essentially straightforward. We obtain

$$
\left( \frac{d}{dt} + \text{div} \, \mathbf{v} \right) \left[ \rho \mathbf{v}_a + \frac{1}{c^2} \rho \left( \mathbf{v}_a \left( \mathbf{v}_a^2 + 6U + \Pi + \frac{\beta}{\rho} \right) - P_a \right) \right]
$$

$$
+ \frac{1}{c^4} \left\{ \rho \mathbf{v}_a \left[ \mathbf{v}_a^4 + 10\mathbf{v}_a^2 U + 13U^2 + 2\Phi - 2v_\mu P_\mu - \frac{1}{2}Q_{\mu\nu} + \left( \mathbf{v}_a^2 + 6U \right) \left( \Pi + \frac{\beta}{\rho} \right) \right] \right.
$$

$$
- \rho \left( Q_{\mu\nu} + v_\beta Q_{\mu\beta} \right) - \rho P_a \left( \mathbf{v}_a^2 + 4U + \Pi + \frac{\beta}{\rho} \right) \right\}
$$

$$
+ \frac{\partial}{\partial x_a} \left[ \rho \left( 1 + \frac{2U}{c^2} \right) \right] - \rho \frac{\partial U}{\partial x_a} - \frac{2}{c^2} \rho \left( \frac{\partial \Phi}{\partial x_a} + \frac{\partial \Phi}{\partial x_a} \right) + \frac{1}{c^2} \rho \frac{\partial P_\beta}{\partial x_a}
$$

$$
+ \frac{1}{c^4} \left\{ - \left( U^2 + 2\Phi + \frac{1}{2}Q_{\mu\nu} \right) \frac{\partial \phi}{\partial x_a} \right\}
$$

$$
+ \rho \left( 2v_\beta U - 3U^2 + 2\Phi + 2U \left( \Pi + \frac{\beta}{\rho} \right) - 2v_\mu P_\mu - \frac{1}{2}Q_{\mu\nu} \right) \frac{\partial U}{\partial x_a}
$$

$$
+ \rho \left( \mathbf{v}_a^2 + 4U + \Pi + \frac{\beta}{\rho} \right) \left( \frac{\partial P_\beta}{\partial x_a} - 2v_\alpha \frac{\partial U}{\partial x_a} - 2 \frac{\partial \Phi}{\partial x_a} \right)
$$

$$
+ \frac{1}{2} \rho \frac{\partial Q_{\alpha\beta}}{\partial x_a} + \rho \frac{\partial Q_{\alpha\beta}}{\partial x_a} + \frac{1}{2} \rho v_\alpha v_\beta \frac{\partial Q_{\gamma\delta}}{\partial x_a} \right\} = 0 .
$$

It is worth noticing here that equation (54) is gauge invariant, i.e., it is valid as it is written independently of the choice of the functions $w$ and $w_a$ in the solutions for $Q_{ij}$.

VI. $\Theta^{00}$ AND THE CONSERVED ENERGY IN THE FIRST POST-NEWTONIAN APPROXIMATION

It has been shown in Paper II in the context of the first post-Newtonian approximation that an algorithm for determining the conserved quantities is to evaluate the Landau-Lifshitz complex

$$
\Theta^{ij} = -g \left( T^{ij} + t^{ij} \right),
$$

where the symmetric "pseudo tensor" $t^{ij}$ is defined in Paper II, equations (32) and (33).

It will appear that in approximations higher than the first post-Newtonian the Landau-Lifshitz complex provides the sole means of determining the conserved quantities.

In this section we shall consider the implications of the conservation law

$$
\int \Theta^{00} dx = \int (-g) (T^{00} + \rho^{00}) dx .
$$

Evaluating $\rho^{00}$ with the aid of the Christoffel symbols listed in equations (47), we find

$$
\rho^{00} = \frac{1}{16\pi G} \left\{ -14 \left( \frac{\partial U}{\partial x_a} \right)^2 + \frac{4}{c^2} \frac{\partial U}{\partial x_a} \frac{\partial}{\partial x_a} (7U^2 + Q_{\mu\nu}) \right\}
$$

$$
+ \frac{1}{2c^2} \left\{ \frac{\partial P_\alpha}{\partial x_\alpha} \left( \frac{\partial P_\alpha}{\partial x_\alpha} + \frac{\partial P_\beta}{\partial x_\beta} \right) - 6 \left( \frac{\partial U}{\partial t} \right) \right\} .
$$
Now making use of equations (41), (49), and (57), we find

\[
\Theta^{00} = -g(T^{00} + \xi^{00}) = \rho c^2 + \rho(v^2 + 6U + \Pi) - \frac{7}{8\pi G} \left( \frac{\partial U}{\partial x_a} \right)^2 \\
+ \frac{1}{c^2} \rho \left[ v^4 + 6v^2U + 4U^2 - Q_{\mu \nu} - 2v_\mu P_\mu + 2U\Pi + v^2 \left( \Pi + \frac{\ddot{\rho}}{\rho} \right) \right] \\
+ \frac{4}{c^2} U \left[ \rho(v^2 + 2U + \Pi) - \frac{7}{8\pi G} \left( \frac{\partial U}{\partial x_a} \right)^2 \right] \\
+ \frac{1}{4\pi Gc^2} \left\{ \frac{\partial U}{\partial x_a} \frac{\partial}{\partial x_a} (7U^2 + Q_{\mu \nu}) + \frac{1}{8} \left[ \frac{\partial P_\alpha}{\partial x_\beta} \frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha} \frac{\partial P_\alpha}{\partial x_\beta} - 6 \left( \frac{\partial U}{\partial t} \right)^2 \right] \right\}.
\]

We have already noted in Paper II (eq. [49]) that

\[
- \frac{7}{8\pi G} \left( \frac{\partial U}{\partial x_a} \right)^2 = -\frac{7}{2} \rho U \quad (\text{mod } \text{div}) .
\]

Also,

\[
\frac{1}{4\pi G} \frac{\partial U}{\partial x_a} \frac{\partial Q_{\mu \nu}}{\partial x_\alpha} = -Q_{\mu \nu} \frac{\nabla^2 U}{4\pi G} = \rho Q_{\mu \nu} \quad (\text{mod } \text{div}) .
\]

Accordingly, we may write

\[
\Theta^{00} = \rho c^2 + \rho \left( v^2 + \frac{5}{2} U + \Pi \right) \\
+ \frac{1}{c^2} \rho \left[ v^4 + 10v^2U + 12U^2 + 6U\Pi + v^2 \left( \Pi + \frac{\ddot{\rho}}{\rho} \right) - 2v_\mu P_\mu \right] \\
+ \frac{1}{32\pi Gc^2} \left\{ \frac{\partial P_\alpha}{\partial x_\beta} \frac{\partial P_\alpha}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_\alpha} \frac{\partial P_\alpha}{\partial x_\beta} - 6 \left( \frac{\partial U}{\partial t} \right)^2 \right\}.
\]

Again, we have, modulo divergence,

\[
\frac{\partial P_\alpha}{\partial x_\beta} \frac{\partial P_\alpha}{\partial x_\beta} = -P_\alpha v^2 P_\alpha = 16\pi G \rho v_\alpha P_\alpha - P_\alpha \frac{\partial^2 U}{\partial t \partial x_\alpha} \\
= 16\pi G \rho v_\alpha P_\alpha + \frac{\partial P_\alpha}{\partial x_\alpha} \frac{\partial U}{\partial t} \\
= 16\pi G \rho v_\alpha P_\alpha - 3 \left( \frac{\partial U}{\partial t} \right)^2 \quad (\text{mod } \text{div})
\]

and

\[
\frac{\partial P_\alpha}{\partial x_\beta} \frac{\partial P_\beta}{\partial x_\alpha} = \frac{\partial P_\alpha}{\partial x_\alpha} \frac{\partial P_\beta}{\partial x_\beta} = 9 \left( \frac{\partial U}{\partial t} \right)^2 \quad (\text{mod } \text{div}).
\]

Hence we may write

\[
\Theta^{00} = \rho c^2 + \rho \left( v^2 + \frac{5}{2} U + \Pi \right) \\
+ \frac{1}{c^2} \rho \left[ v^4 + 10v^2U + 12U^2 + 6U\Pi + v^2 \left( \Pi + \frac{\ddot{\rho}}{\rho} \right) - 3v_\mu P_\mu \right] .
\]
From equations (53) and (64) we obtain
\[\mathcal{E} = \Theta^{00} - c^2 \rho \nabla^2 - g = \rho \left( \frac{1}{2} v^2 - \frac{1}{3} U + \Pi \right) + \frac{1}{c^2} \left[ \frac{5}{8} v^4 + \frac{3}{2} v^2 U + \frac{1}{2} U^2 + 6 U \Pi + \frac{1}{2} \rho \left( \frac{l}{\rho} \right)^2 + \frac{3}{2} Q_{\mu\nu} - \frac{1}{2} v_{\mu} P_{\mu} \right]. \tag{65}\]

In this expression for \(\mathcal{E}\) we may replace the term \(\frac{1}{2} \rho Q_{\mu\nu}\) by (cf. eq. [38])
\[\frac{1}{2} \rho Q_{\mu\nu} = - \frac{\nabla^2 U}{8 \pi G} (Q_{\mu\nu} + 6 U^2) - 3 \rho U^2 \]
\[= - \frac{U}{8 \pi G} \nabla^2 (Q_{\mu\nu} + 6 U^2) - 3 \rho U^2 \tag{66}\]
\[= - \rho (13 U^2 + 4 v^2 U + 4 U \Pi) \quad \text{(mod div)}. \]

Hence, we may define the energy in the first post-Newtonian approximation by
\[\mathcal{E} = \rho \left( \frac{1}{2} v^2 - \frac{1}{3} U + \Pi \right) \]
\[+ \frac{1}{c^2} \rho \left[ \frac{5}{8} v^4 + \frac{3}{2} v^2 U + \frac{1}{2} U^2 + 6 U \Pi + \frac{1}{2} \rho \left( \frac{l}{\rho} \right)^2 - \frac{1}{2} v_{\mu} P_{\mu} \right]; \tag{67}\]

and the integral of this quantity over the three-dimensional space gives the conserved total energy.

It is of interest to verify that the expression (67) for \(\mathcal{E}\) is, modulo divergence, the same as the one that was obtained in Paper I from a direct inspection of the equation of motion and assuming the condition for isentropic flow. The expression for \(\mathcal{E}\) given in Paper I, equation (166), is
\[\mathcal{E} = (\sigma - \frac{1}{2} \rho^*) v^2 + \rho^* \Pi - \frac{1}{2} \rho^* U^* \tag{68}\]
\[+ \frac{1}{c^2} \rho (- \frac{1}{2} v^4 + \frac{1}{2} v^2 U + \frac{1}{2} U^2 - \frac{1}{2} \rho^* \Pi - \frac{1}{2} v_{\mu} P_{\mu}), \]

where \(\sigma\) has the same meaning as in equation (30) and where
\[\rho^* = \rho + \frac{1}{c^2} \rho (\frac{1}{2} v^2 + 3 U) \tag{69}\]
and
\[U^* = U + \frac{1}{c^2} \Omega, \quad \text{where} \quad \nabla^2 \Omega = -4 \pi G \rho (\frac{1}{2} v^2 + 3 U). \]

Accordingly,
\[\rho^* U^* = \rho U + \frac{1}{c^2} \rho U (\frac{1}{2} v^2 + 3 U) + \frac{1}{c^2} \rho \Omega \tag{70}\]
\[= \rho U + \frac{2}{c^2} \rho U (\frac{1}{2} v^2 + 3 U) \quad \text{(mod div)}. \]

Replacing \(\sigma\), \(\rho^*\), and \(\rho^* U^*\) in equation (68) by their equivalents, we recover the expression (67) for \(\mathcal{E}\). Thus, in agreement with what was stated in Paper II, the conserved energy in the first post-Newtonian approximation is all that can be derived at this stage.
Evaluating the \((0,a)\)-component of the pseudo tensor in accordance with equations (32) and (33) of Paper II, we find

\[
\rho^a = \frac{1}{16\pi G c} \left\{ \frac{1}{2} \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} + 4 \frac{\partial U}{\partial x_a} \left( \frac{\partial P^a}{\partial x_a} - \frac{\partial P^a}{\partial x_a} \right) \right. \\
+ \frac{1}{c^2} \left[ \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} \left( 4\Phi - \frac{1}{2} Q_{ss} \right) \right. \\
- 2 \frac{\partial U}{\partial x_a} \frac{\partial Q_{ss}}{\partial t} + 4 \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} \\
+ \frac{\partial P_a}{\partial x_a} \frac{\partial \Phi}{\partial x_a} \left( 9U^2 + 2\Phi + Q_{ss} \right) \left[ \frac{\partial U}{\partial x_a} \right] \\
+ \frac{\partial P^a}{\partial x_a} \left( \frac{\partial Q_{ss}}{\partial x_a} - \frac{\partial Q_{ss}}{\partial x_a} \right) \left[ \frac{\partial U}{\partial x_a} \right] \right\}.
\]

With the aid of equations (41), (50), and (71), we now obtain

\[
\frac{1}{c} \Theta^a = -\frac{1}{c} \gamma (\mathbf{T}^a + \rho^a) \\
= \rho^a + \frac{1}{c^2} \rho^a \left( \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} \right) \\
+ \frac{1}{c^4} \rho \left\{ \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} \left( 10U^2 + 12U^2 + \left( \frac{\partial U}{\partial t} \right)^2 \right) \right. \\
- 2\nu\mu P^\mu - Q_{\mu\mu} \left\} - \frac{\partial \rho}{\partial x_a} P_a \right\} \\
+ \frac{1}{16\pi G c^2} \left[ \frac{1}{2} \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} + 4 \frac{\partial U}{\partial x_a} \left( \frac{\partial P^a}{\partial x_a} - \frac{\partial P^a}{\partial x_a} \right) \right. \\
+ \frac{\partial P_a}{\partial x_a} \frac{\partial \Phi}{\partial x_a} \left( 9U^2 + 2\Phi + Q_{ss} \right) \left[ \frac{\partial U}{\partial x_a} \right] \\
+ \frac{\partial P^a}{\partial x_a} \left( \frac{\partial Q_{ss}}{\partial x_a} - \frac{\partial Q_{ss}}{\partial x_a} \right) \left[ \frac{\partial U}{\partial x_a} \right] \right\}.
\]

After a lengthy reduction in which the equations governing \(P^a\) and \(Q_{\mu\mu}\) as well as the particular gauge conditions (35) and (36) are used, we find (cf. Paper II, eqs. [58] and [59])

\[
\frac{1}{c} \Theta^a = \pi^a + \frac{\partial \theta^a}{\partial x_a}.
\]
where

\[ \pi_a = \rho v_a + \frac{1}{c^2} \rho \left[ v_a \left( v^2 + 6U + \Pi + \frac{\rho}{\rho} \right) - P_a \right] \]

\[ + \frac{1}{c^2} \left\{ \rho v_a \left[ v^4 + 10v^2U + 13U^2 + 2\Phi + (v^2 + 6U) \left( \Pi + \frac{\rho}{\rho} \right) - 2\rho_p^\mu P_\mu - \frac{1}{2} Q_{\rho\rho} \right] \right. \]

\[ - \rho (Q_{\rho a} + v_\rho Q_{\rho a}) - \rho P_a \left( v^2 + 4U + \Pi + \frac{\rho}{\rho} \right) \right\} \]

\[ + \frac{1}{16\pi Gc^4} \left( 4P_\beta \frac{\partial U}{\partial x_\beta} \frac{\partial U}{\partial x_a} + \frac{\partial^2 U}{\partial x_a \partial x_a} \right) \]

and

\[ \theta_{a\beta} = \frac{1}{4\pi Gc^2} \left\{ 4U \left( \frac{\partial U_a}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_a} \right) + \frac{\delta_{a\beta}}{2} \frac{\partial U^2}{\partial t} - U \frac{\partial^2 X}{\partial t \partial x_a \partial x_\beta} \right. \]

\[ + \frac{\partial}{\partial x_\beta} \left[ U \left( \frac{1}{2} \frac{\partial X}{\partial t \partial x_a} - 4U \right) \right] \left\} + \frac{1}{16\pi Gc^4} \left[ (U^2 + 2\Phi) \left( \frac{\partial P_a}{\partial x_\beta} + \frac{\partial P_\beta}{\partial x_a} + 2\delta_{a\beta} \frac{\partial U}{\partial t} \right) \right. \]

\[ + 4Q_{\rho a} \frac{\partial U}{\partial x_\beta} \frac{\partial U}{\partial x_\beta} P_\mu \frac{\partial Q_{\rho a}}{\partial x_\mu} + \frac{\partial Q_{\rho a}}{\partial x_\mu} \delta_{a\beta} - \delta_{a\beta} \frac{\partial Q_{\rho a}}{\partial x_\rho} \right) \]

\[ + \frac{\partial U}{\partial t} \left( Q_{\rho a} - \delta_{a\beta} Q_{\rho \rho} \right) \]

\[ \left. - 4 \left( Q_{\rho a} \frac{\partial U}{\partial x_\beta} + Q_{\rho a} \frac{\partial U}{\partial x_\beta} - \delta_{a\beta} \left( P_\mu Q_{\rho \mu} - \frac{1}{2} P_\rho Q_{\rho \rho} \right) \right) \right\} \]

Since \( \theta_{a\beta}/c \) and \( \pi_a \) are equal modulo divergence and \( \theta_{a\beta} \) is moreover equal, modulo divergence, to a symmetric (pseudo) tensor, it is clear that they will yield the same total linear, as well as angular, momentum when integrated over the whole of the three-dimensional space.

We notice that the expression (74) for \( \pi_a \) contains two terms which are not confined to the volume occupied by the fluid. In § VIII we show how in a suitable gauge these terms can be transformed, modulo divergence, into quantities which are nonvanishing only in the volume occupied by the fluid.

Also, it may be noted here that the quantity in equation (54) on which \( \frac{d}{dt} + \text{div} \nu \) operates is precisely \( \pi_a \) except for the two terms which do not vanish outside the volume occupied by the fluid.

VIII. THE EFFECT OF THE CHOICE OF GAUGE ON THE CONSERVED QUANTITIES

Since we have evaluated the conserved quantities in §§ VI and VII in a particular gauge, it is of interest to see how they are affected by this choice.

We have already noticed in § IIId how the solutions for \( Q_{\rho a}, Q_{\rho \rho} \), and \( Q_{\rho a} \) in a general gauge are related to these same coefficients in the particular gauge \( \omega = \omega_a = 0 \). If in accordance with equations (39) and (40) we now distinguish the quantities derived in the chosen gauge by asterisks (hitherto not so distinguished), then by equations (41)-(44)

\[ - g = - g^* - \frac{2}{c^2} \frac{\partial W_\mu}{\partial x_\mu}, \quad \sqrt{-g} = \sqrt{-g^*} - \frac{1}{c^2} \frac{\partial W_\mu}{\partial x_\mu}, \]

\[ \gamma_0 = \gamma_0^* - \frac{1}{c^2} \frac{\partial W_\mu}{\partial t \partial x_\mu}, \quad \text{and} \quad \gamma_a = \gamma_a^* - \frac{1}{c^2} \frac{\partial W_\mu}{\partial x_a \partial x_\mu}. \]

Also, while \( \Gamma^\rho_{00}, \Gamma^\rho_{0a}, \) and \( \Gamma^\rho_{ab} \) are to the required orders \( O(c^{-6}) \) in the case of \( \Gamma^0_{00} \) and \( O(c^{-4}) \) in the cases of \( \Gamma^0_{0a} \) and \( \Gamma^0_{ab} \)—unaffected by the choice of gauge in the second post-Newtonian approximation, the remaining Christoffel symbols are affected; thus
Now re-evaluating \( \theta^{00} \), we find that in addition to the terms given in equation (57) (in which \( Q_{\mu \nu} \) is now replaced by \( Q^*_{\mu \nu} \)), we have

\[
\frac{1}{16\pi Gc^2} \left( -4 \frac{\partial U}{\partial x^a} w_a + 8 \frac{\partial U}{\partial x^a} \frac{\partial^2 W^a}{\partial x^a \partial x^a} \right)
\]

(78)

In \( \Theta^{00} \), in addition to the term (78) derived from \( \theta^{00} \) we have \(-2\rho \partial W^a/c^2 \partial x^a\) derived from the factor \(-g\); thus, altogether, the additional term in \( \Theta^{00} \) is

\[
- \frac{1}{c^2} \frac{\partial W^a}{\partial x^a},
\]

(79)

The additional term in \( c^2 \rho u^a \sqrt{-g} - g \), derived from the factor \( \sqrt{-g} \), is

\[
- \frac{1}{c^2} \frac{\partial W^a}{\partial x^a}.
\]

(80)

Hence \( \Theta^{00} - c^2 \rho u^a \sqrt{-g} - g \) is gauge invariant, as it should be, since this quantity expresses the conserved energy in the first post-Newtonian approximation and cannot, therefore, depend on the choice of gauge in the second post-Newtonian approximation.

Considering next \( \rho^a \), we find on re-evaluation that in addition to the terms given in equation (71) (in which \( Q_{\sigma \tau} \) and \( Q_{0a} \) are now replaced by \( Q^*_{\sigma \tau} \) and \( Q^*_{0a} \) to distinguish that these quantities are now defined in the special gauge \( \psi = 0 \)), we have

\[
\frac{1}{16\pi Gc^2} \left[ - \frac{\partial U}{\partial t} \frac{\partial^2 W^a}{\partial x^a \partial x^a} - 4 \frac{\partial U}{\partial x^a} \frac{\partial^2 W^a}{\partial x^a \partial t} + 2 \frac{\partial P_a}{\partial x^a} \frac{\partial^2 W^a}{\partial x^a \partial x^a} - \frac{\partial P_{\alpha}}{\partial x^\alpha} \frac{\partial^2 W^a}{\partial x^a \partial x^\alpha} \right]
\]

\[
+ 4 \frac{\partial U}{\partial x^a} \frac{\partial}{\partial t} \left( \frac{\partial W^a}{\partial x^a} - \frac{\partial W^a}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^a} \right) + \frac{\partial P_{\alpha}}{\partial x^\alpha} \frac{\partial}{\partial t} \left( \frac{\partial W^a}{\partial x^a} - \frac{\partial W^a}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^a} \right)
\]

\[
- \omega \frac{\partial P_a}{\partial x^a} + w_a \frac{\partial U}{\partial t}.
\]

(81)

It can be readily verified that with these additional terms \( Q^*_{\mu \nu} \) and \( Q^*_{0a} \), which now occur in equation (71), can be replaced by \( Q_{\mu \nu} \) and \( Q_{0a} \) in accordance with the transformation (39) and that the only terms not so absorbed are

\[
\frac{1}{16\pi Gc^2} \left( - \omega \frac{\partial P_a}{\partial x^a} + w_a \frac{\partial U}{\partial t} \right).
\]

(82)
SECOND POST-NEWTONIAN EQUATIONS

\[
\frac{1}{16\pi G c^4} \left[ 4 \frac{\partial U}{\partial x_a} w + w_\beta \frac{\partial P_\beta}{\partial x_a} - \frac{\partial}{\partial x_\beta} (w_\beta P_a) \right].
\] (83)

In the expression (83) we have retained the divergence of \(-w_\beta P_a\) since this is not symmetric in \(a\) and \(\beta\).

With the choice

\[
w = -P_\beta \frac{\partial U}{\partial x_\beta} \quad \text{and} \quad w_\beta = -\frac{1}{6} \frac{\partial Q_{\beta\epsilon}}{\partial x_\beta},
\] (84)

the terms in \(\pi_a\), not confined to the volume occupied by the fluid, become

\[
\frac{1}{16\pi G c^4} \left[ 4 P_\beta \frac{\partial U}{\partial x_\beta} \frac{\partial U}{\partial x_a} + \frac{1}{2} Q_{\epsilon\delta} \frac{\partial^2 U}{\partial t \partial x_a} + 4 w \frac{\partial U}{\partial x_a} + w_\beta \frac{\partial P_\beta}{\partial x_a} - \frac{\partial}{\partial x_\beta} (w_\beta P_a) \right]
\]

\[
= -\frac{1}{96\pi G c^4} \left[ \frac{\partial Q_{\epsilon\sigma}}{\partial x_\sigma} \frac{\partial P_\beta}{\partial x_a} - 3 Q_{\epsilon\delta} \frac{\partial^2 U}{\partial t \partial x_a} - \frac{\partial}{\partial x_\beta} \left( \frac{\partial Q_{\epsilon\sigma}}{\partial x_\sigma} P_a \right) \right]
\]

\[
= -\frac{1}{96\pi G c^4} \left[ \frac{\partial}{\partial x_\beta} \left( Q_{\epsilon\sigma} \frac{\partial P_\beta}{\partial x_\sigma} + \frac{\partial P_a}{\partial x_\beta} \right) - \frac{\partial}{\partial x_\beta} (Q_{\epsilon\sigma} P_\sigma) \right],
\] (84')

where in passing from the second to the third line we have made use of equation (6).

Thus with the choice of the gauge (84), we can reduce \(\pi_a\) modulo divergence to the volume occupied by the fluid. And the conserved angular momentum can also be defined in terms of the same reduced linear momentum since the final expression in (84') is, modulo divergence, symmetric in \(a\) and \(\beta\).

IX. THE EQUATION \(\Theta^{\alpha\ell} = 0\)

To write the equation governing \(\Theta^{\alpha\ell}\), namely,

\[
\frac{1}{c} \frac{\partial}{\partial t} \left[ (-g)(T^{\alpha\alpha} + t^{\alpha\epsilon}) \right] + \frac{\partial}{\partial x_\beta} \left[ (-g)(T^{\alpha\beta} + t^{\alpha\beta}) = 0 \right],
\] (85)

in the different approximations, we need \(t^{\alpha\beta}\) to different orders: to \(O(c^{-2})\) in the first post-Newtonian approximation and to \(O(c^{-4})\) in the second post-Newtonian approximation. Certain peculiarities that are noteworthy appear in the evaluation of \(t^{\alpha\beta}\) to these required orders; we shall clarify their nature in the framework of the first post-Newtonian approximation.

a) The Equation \(\Theta^{\alpha\ell} = 0\) in the First Post-Newtonian Approximation

As we have noted, in the context of the first post-Newtonian approximation, \(t^{\alpha\beta}\) is needed to \(O(c^{-2})\). Since \(t^{ij}\) is quadratic in the Christoffel symbols and appears with a factor \(c^4\), it is clear that to evaluate \(t^{\alpha\beta}\) consistently to \(O(c^{-2})\), we should want to know the necessary Christoffel symbols to \(O(c^2)\). While some of the Christoffel symbols (e.g., \(\Gamma^{\alpha\beta}_{\alpha\beta}\) and \(\Gamma^{\alpha\beta}_{\alpha\beta}\)) are known to the required orders, some others (e.g., \(\Gamma^{\alpha\beta}_{\alpha\beta}\)) are not. However, since the Christoffel symbols listed in equations (47) (with our present knowledge of the second post-Newtonian approximation) provide the needed information, we can evaluate \(t^{\alpha\beta}\) consistently to \(O(c^{-2})\). And when the evaluation is
carried out, a remarkable thing happens: all the terms of $O(c^{-2})$ in $\tau^{\alpha\beta}$ which derive from an explicit knowledge of the second post-Newtonian approximation cancel separately. In other words, we should formally have obtained the correct final result had we simply ignored (blindly!) all the terms in $\tau^{\alpha\beta}$ of $O(c^{-2})$ which depend on terms in the Christoffel symbols known only in the second post-Newtonian approximation. By a careful inspection of why this cancellation occurs in the first post-Newtonian approximation, one can convince oneself that the same cancellation must also occur in the higher approximations; e.g., the terms of $O(c^{-4})$ in $\tau^{\alpha\beta}$ which depend on the knowledge of the Christoffel symbols in the third post-Newtonian approximation will cancel.

Returning to $\tau^{\alpha\beta}$, we have to $O(c^{-2})$

$$\tau^{\alpha\beta} = \frac{1}{16\pi G} \left(1 - \frac{4U}{c^2}\right) \left[4 \frac{\partial U}{\partial x_\alpha} \frac{\partial U}{\partial x_\beta} - 2\delta_{\alpha\beta} \left(\frac{\partial U}{\partial x_\mu}\right)^2\right]$$

$$+ \frac{1}{16\pi G c^2} \left[8 \left(\frac{\partial U}{\partial x_\alpha} \frac{\partial \Phi}{\partial x_\beta} + \frac{\partial U}{\partial x_\beta} \frac{\partial \Phi}{\partial x_\alpha}\right) - 8\delta_{\alpha\beta} \frac{\partial U}{\partial x_\alpha} \frac{\partial \Phi}{\partial x_\beta}\right]$$

$$+ 4 \left(\frac{\partial P_\alpha}{\partial x_\alpha} \frac{\partial P_\beta}{\partial t} + \frac{\partial P_\beta}{\partial x_\beta} \frac{\partial P_\alpha}{\partial t} - \delta_{\alpha\beta} \frac{\partial U}{\partial x_\alpha} \frac{\partial P_\mu}{\partial t}\right) - 8\delta_{\alpha\beta} \left(\frac{\partial U}{\partial t}\right)^2 \tag{86}$$

$$+ \frac{\partial P_\alpha}{\partial x_\alpha} \left(\frac{\partial P_\beta}{\partial x_\beta} - \frac{\partial P_\beta}{\partial x_\alpha}\right) + \frac{\partial P_\beta}{\partial x_\beta} \left(\frac{\partial P_\alpha}{\partial x_\alpha} - \frac{\partial P_\alpha}{\partial x_\beta}\right)$$

$$+ \frac{1}{c} \frac{\partial \Theta^{\alpha\beta}}{\partial t} + \frac{\partial \Theta^{\alpha\beta}}{\partial x_\beta} = 0 \tag{87}$$

and the corresponding expression for $\Theta^{\alpha\beta}$ (see eq. [90] below) can be written down with the aid of the known expressions for $T^{\alpha\beta}$ and $g$. The further reduction of the equation

$$\frac{1}{c} \frac{\partial \Theta^{\alpha\beta}}{\partial t} + \frac{\partial \Theta^{\alpha\beta}}{\partial x_\beta} = 0 \tag{87}$$

with $\Theta^{\alpha\beta}$ as given in equation (57) of Paper II, yields the equation

$$\left(\frac{d}{dt} + \text{div } \mathbf{v}\right)\left\{ \rho v_\alpha + \frac{1}{c^2} \rho \left[ v_\alpha \left(\frac{v^2}{2} + U + \Pi + \frac{p}{\rho}\right) - P_\alpha\right]\right\} = -\rho \frac{\partial U}{\partial x_\alpha}$$

$$+ \frac{\partial}{\partial x_\alpha} \left[\left(1 + \frac{2U}{c^2}\right) \frac{p}{c^2} \rho \frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right] + \frac{1}{c^2} \rho v_\beta \frac{\partial P_\beta}{\partial x_\alpha} = 0 \tag{88}$$

We observe that this equation is identical with the equation $T^{\alpha\beta}_{\alpha\beta} = 0$ in the first post-Newtonian approximation as reduced in Paper I (eq. [68]).

b) The Equation $\Theta^{\alpha\beta}_{\alpha\beta} = 0$ in the Second Post-Newtonian Approximation

In the second post-Newtonian approximation, $\tau^{\alpha\beta}$ is needed to $O(c^{-4})$. To evaluate $\tau^{\alpha\beta}$ to this order, one should strictly want to know the Christoffel symbols to $O(c^{-6})$. The symbols listed in equations (47) do not provide this information for all of them. But as we have explained in § a above, the terms in $\tau^{\alpha\beta}$ of $O(c^{-4})$ which depend on terms in the Christoffel symbols known only in the third post-Newtonian approximation will cancel. Therefore, ignoring these contributions to $\tau^{\alpha\beta}$, we find that the terms of $O(c^{-4})$ in $16\pi G\tau^{\alpha\beta}$ are
\[8U^2 + 32\Phi \frac{\partial U}{\partial x_a} \frac{\partial U}{\partial x_b} - 8 \frac{\partial \Phi}{\partial x_a} \frac{\partial \Phi}{\partial x_b} - 40U \left( \frac{\partial U}{\partial x_a} \frac{\partial \Phi}{\partial x_b} + \frac{\partial U}{\partial x_b} \frac{\partial \Phi}{\partial x_a} \right)\]

\[-\frac{\partial U}{\partial x_a} \frac{\partial}{\partial x_b} (P_x^2) - \frac{\partial U}{\partial x_b} \frac{\partial}{\partial x_a} (P_x^2) - 2 \frac{\partial U}{\partial x_a} \frac{\partial}{\partial x_b} (P_x P_y)\]

\[-\left( \frac{\partial P_a}{\partial x_a} + \frac{\partial P_b}{\partial x_b} \right) \frac{\partial}{\partial t} \left( U^2 + 2\Phi + \frac{1}{4}Q_{xx} \right) + 4 \frac{\partial U}{\partial t} \left( P_a \frac{\partial U}{\partial x_a} + P_b \frac{\partial U}{\partial x_b} \right)\]

\[+ 4U \left( \frac{\partial P_a}{\partial x_a} \frac{\partial P_a}{\partial x_a} + \frac{\partial P_b}{\partial x_b} \frac{\partial P_b}{\partial x_b} \right) - 4U \left( \frac{\partial P_a}{\partial x_a} \frac{\partial P_b}{\partial x_b} + \frac{\partial P_b}{\partial x_a} \frac{\partial P_a}{\partial x_b} \right)\]

\[+ 2 \frac{\partial U}{\partial t} \left( P_a \frac{\partial P_a}{\partial x_a} + P_b \frac{\partial P_b}{\partial x_b} \right) + 2P_a \left( \frac{\partial U}{\partial x_a} \frac{\partial P_a}{\partial x_a} + \frac{\partial U}{\partial x_b} \frac{\partial P_a}{\partial x_b} \right)\]

\[-\frac{\partial P_a}{\partial t} \frac{\partial}{\partial x_b} \left( 8U^2 + \frac{1}{2}Q_{xx} \right) - \frac{\partial P_b}{\partial t} \frac{\partial}{\partial x_a} \left( 8U^2 + \frac{1}{2}Q_{xx} \right) - \frac{\partial U}{\partial t} \frac{\partial Q_{ax}}{\partial x_a}\]

\[-\frac{\partial Q_{xx}}{\partial x_a} \frac{\partial}{\partial x_b} \left( U^2 + 2\Phi \right) - \frac{\partial Q_{xx}}{\partial x_b} \frac{\partial}{\partial x_a} \left( U^2 + 2\Phi \right) - \frac{\partial Q_{ax}}{\partial x_a} \frac{\partial}{\partial x_b} \left( 3U^2 + 6\Phi + Q_{xx} \right)\]

\[+ 4 \frac{\partial U}{\partial x_a} \left( Q_{aa} \frac{\partial U}{\partial x_b} + Q_{ba} \frac{\partial U}{\partial x_a} \right) + \left( \frac{\partial Q_{aa}}{\partial x_a} + \frac{\partial Q_{bb}}{\partial x_b} \right) \frac{\partial}{\partial t} \left( U^2 + 2\Phi + \frac{1}{2}Q_{xx} \right)\]

\[-Q_{ab} \left( \frac{\partial U}{\partial x_a} \frac{\partial U}{\partial x_b} \right) + \left( \frac{\partial Q_{aa}}{\partial x_a} - \frac{\partial Q_{bb}}{\partial x_b} \right) \frac{\partial}{\partial t} \left( U^2 + 2\Phi \right) - \frac{\partial Q_{ab}}{\partial x_a} \frac{\partial}{\partial x_b} \left( 3U^2 + 6\Phi + Q_{xx} \right)\]

\[+ \frac{1}{4} \frac{\partial Q_{xx}}{\partial x_a} \frac{\partial Q_{xx}}{\partial x_b} + \frac{\partial Q_{ab}}{\partial x_a} \frac{\partial Q_{ab}}{\partial x_b} + \frac{1}{2} \frac{\partial Q_{ab}}{\partial x_a} \frac{\partial Q_{ba}}{\partial x_b}\]

\[+ 4 \frac{\partial U}{\partial x_a} \left( \frac{\partial Q_{aa}}{\partial x_a} - \frac{\partial Q_{bb}}{\partial x_b} \right) + 4 \frac{\partial U}{\partial t} \left( \frac{\partial Q_{aa}}{\partial x_a} - \frac{\partial Q_{bb}}{\partial x_b} \right)\]

\[\left( \frac{\partial Q_{aa}}{\partial x_a} - \frac{\partial Q_{bb}}{\partial x_b} \right)\]

\[+ \delta_{ab} \left[ -16\Phi \left( \frac{\partial U}{\partial x_a} \right)^2 + 8 \left( \frac{\partial \Phi}{\partial x_a} \right)^2 + 48U \frac{\partial U}{\partial x_a} \frac{\partial \Phi}{\partial x_b} + 20U \left( \frac{\partial U}{\partial t} \right)^2 - \frac{4U}{\partial t} \frac{\partial \Phi}{\partial t} \right]\]

\[-4P_a \frac{\partial U}{\partial t} \frac{\partial U}{\partial x_a} + \frac{\partial U}{\partial x_a} \frac{\partial P_a}{\partial x_a} - 2P_a \frac{\partial P_a}{\partial x_a} \frac{\partial U}{\partial x_a} + 2U \frac{\partial P_a}{\partial x_a} \frac{\partial P_a}{\partial x_a}\]

\[+ \frac{\partial P_a}{\partial t} \frac{\partial}{\partial x_a} \left( 8U^2 + Q_{xx} \right) + 2 \frac{\partial Q_{xx}}{\partial x_a} \frac{\partial}{\partial x_a} \left( U^2 + 2\Phi \right) - 2Q_{uu} \frac{\partial U}{\partial x_a}\]

\[+ 2 \frac{\partial U}{\partial x_a} \left( \frac{\partial Q_{uu}}{\partial x_a} - \frac{\partial Q_{uu}}{\partial x_a} \right) + \frac{\partial U}{\partial t} \frac{\partial Q_{xx}}{\partial x_a} + \frac{1}{4} \left( \frac{\partial Q_{xx}}{\partial x_a} \right)^2\]

\[-\frac{\partial P_a}{\partial x_a} \left( \frac{\partial Q_{xx}}{\partial x_a} - \frac{\partial Q_{xx}}{\partial x_a} \right) - \frac{1}{4} \frac{\partial Q_{uu}}{\partial x_a} \frac{\partial Q_{uu}}{\partial x_a} + \frac{1}{2} \frac{\partial Q_{uu}}{\partial x_a} \frac{\partial Q_{uu}}{\partial x_a} \right].\]
With the aid of equations (41), (51), and (89) we now find
\[
\Theta^\beta = \rho v^\alpha v_\beta + \rho \delta^\alpha_\beta + \frac{1}{c^2} \left[ \rho v^\alpha v_\beta \left( v^2 + 6U + \Pi + \frac{p}{\rho} \right) + 2p \mu \delta^\alpha_\beta \right]
\]
\[
+ \frac{1}{c^2} \left\{ \rho v^\alpha v_\beta \left[ v^4 + 10v^2U + 12U^2 + (v^2 + 6U) \left( \Pi + \frac{p}{\rho} \right) - 2v_\mu P_\mu - Q_{\nu\nu} \right] \right\}
\]
\[
+ p \left[ Q_{\alpha\beta} - (2U^2 + 4\Phi + Q_{\nu\nu}) \delta_{\alpha\beta} \right] \}
\]
\[
+ \frac{1}{16\pi G} \left[ \frac{4}{4} \frac{\partial U}{\partial x_{\alpha}} \frac{\partial U}{\partial x_{\beta}} - 2\delta_{\alpha\beta} \left( \frac{\partial U}{\partial x_{\mu}} \right)^2 \right]
\]
\[
+ \frac{1}{c^2} \left\{ 8 \left( \frac{\partial U}{\partial x_{\alpha}} \frac{\partial \Phi}{\partial x_{\beta}} + \frac{\partial U}{\partial x_{\alpha}} \frac{\partial \Phi}{\partial x_{\alpha}} \right) + 4 \left( \frac{\partial U}{\partial x_{\alpha}} \frac{\partial P_\beta}{\partial t} + \frac{\partial U}{\partial x_{\beta}} \frac{\partial P_\alpha}{\partial t} \right)
\]
\[
- \left( \frac{\partial P_\alpha}{\partial x_{\beta}} + \frac{\partial P_\beta}{\partial x_{\alpha}} \right) \frac{\partial}{\partial t} \left( U^2 + 2\Phi + \frac{1}{2}Q_{\nu\nu} \right) + 4 \frac{\partial U}{\partial t} \left( P_{\alpha} \frac{\partial U}{\partial x_{\beta}} + P_{\beta} \frac{\partial U}{\partial x_{\alpha}} \right)
\]
\[
+ 2 \frac{\partial U}{\partial x_{\alpha}} \left( P_{\alpha} \frac{\partial P_\beta}{\partial x_{\alpha}} + P_{\beta} \frac{\partial P_\alpha}{\partial x_{\alpha}} \right) + 2P_{\beta} \left( \frac{\partial U}{\partial x_{\alpha}} \frac{\partial P_\beta}{\partial x_{\alpha}} + \frac{\partial U}{\partial x_{\beta}} \frac{\partial P_\alpha}{\partial x_{\alpha}} \right)
\]
\[
+ \frac{\partial Q_{\nu\nu}}{\partial x_{\alpha}} \left[ \frac{1}{2} \frac{\partial P_\beta}{\partial t} + \frac{\partial}{\partial x_{\beta}} (U^2 + 2\Phi) \right] - \frac{\partial Q_{\nu\nu}}{\partial x_{\beta}} \left[ \frac{1}{2} \frac{\partial P_\alpha}{\partial t} + \frac{\partial}{\partial x_{\alpha}} (U^2 + 2\Phi) \right]
\]
\[
+ \frac{\partial}{\partial x_{\alpha}} \left( \frac{Q_{\nu\nu}}{x_{\alpha}} + \frac{Q_{\alpha\nu}}{x_{\alpha}} \right) + \left( \frac{\partial Q_{\nu\nu}}{\partial x_{\alpha}} + \frac{\partial Q_{\alpha\nu}}{\partial x_{\alpha}} \right) \frac{\partial}{\partial x_{\alpha}} \left( U^2 + 2\Phi + \frac{1}{2}Q_{\nu\nu} \right)
\]
\[
+ \left( \frac{\partial P_\alpha}{\partial x_{\alpha}} - \frac{\partial P_\beta}{\partial x_{\beta}} \right) \frac{\partial Q_{\beta\alpha}}{\partial x_{\alpha}} - \frac{\partial P_\alpha}{\partial x_{\alpha}} \frac{\partial Q_{\beta\alpha}}{\partial x_{\alpha}} + \left( \frac{\partial P_\beta}{\partial x_{\alpha}} - \frac{\partial P_\beta}{\partial x_{\alpha}} \right) \frac{\partial Q_{\alpha\beta}}{\partial x_{\beta}} - \frac{\partial Q_{\alpha\beta}}{\partial x_{\beta}}
\]
\[
+ \frac{\partial P_\beta}{\partial x_{\alpha}} \frac{\partial Q_{\beta\alpha}}{\partial t} + \frac{\partial P_\mu}{\partial x_{\alpha}} \frac{\partial Q_{\alpha\mu}}{\partial t} - \frac{\partial P_\beta}{\partial x_{\beta}} \frac{\partial Q_{\alpha\beta}}{\partial t} - \frac{1}{4} \frac{\partial Q_{\alpha\beta}}{\partial x_{\beta}} \frac{\partial Q_{\beta\alpha}}{\partial x_{\alpha}} + \frac{1}{2} \frac{\partial Q_{\gamma\mu}}{\partial x_{\alpha}} \frac{\partial Q_{\mu\gamma}}{\partial x_{\beta}}
\]
\[
+ 4 \frac{\partial U}{\partial x_{\alpha}} \left( \frac{\partial Q_{\alpha\nu}}{\partial t} - \frac{1}{2} \frac{\partial Q_{\nu\nu}}{\partial t} \right) + 4 \frac{\partial U}{\partial x_{\beta}} \left( \frac{\partial Q_{\beta\nu}}{\partial t} - \frac{1}{2} \frac{\partial Q_{\nu\nu}}{\partial t} \right)
\]
\[
+ \frac{\partial Q_{\alpha\nu}}{\partial x_{\alpha}} \frac{\partial Q_{\nu\beta}}{\partial x_{\beta}} - \left( \frac{\partial Q_{\alpha\nu}}{\partial x_{\alpha}} \frac{\partial Q_{\nu\beta}}{\partial x_{\beta}} + \frac{\partial Q_{\beta\nu}}{\partial x_{\beta}} \frac{\partial Q_{\nu\beta}}{\partial x_{\alpha}} \right)
\]
\[
\right)
\]
+ δab \left[ (28U^2 - 8Φ + 2Q_{μν}) \left( \frac{∂U}{∂x_μ} \right)^2 + 8 \left( \frac{∂Φ}{∂x_μ} \right)^2 + 16U \frac{∂U}{∂x_μ} \frac{∂Φ}{∂x_μ} \right] (90)

- 2 \frac{∂U}{∂t} \frac{∂}{∂t} (U^2 + 2Φ) - 4P_μ \frac{∂U}{∂x_μ} \frac{∂U}{∂x_μ} + \frac{∂U}{∂x_μ} \frac{∂P_μ}{∂x_μ} - 2P_μ \frac{∂P_μ}{∂x_μ} \frac{∂U}{∂x_μ} + \frac{∂Q_{μν}}{∂x_μ} \frac{∂U}{∂x_μ}

+ 2 \frac{∂Q_{μν}}{∂x_μ} \frac{∂}{∂x_μ} (U^2 + 2Φ) + \frac{∂Q_{μν}}{∂x_μ} \frac{∂P_μ}{∂t} + \frac{∂Q_{μν}}{∂x_μ} \frac{∂U}{∂t} - 2Q_{μν} \frac{∂U}{∂x_μ} \frac{∂U}{∂x_μ}

+ 2 \frac{∂U}{∂x_μ} \left( \frac{∂Q_{μν}}{∂x_μ} - 2 \frac{∂Q_{μν}}{∂t} \right) - \frac{∂P_μ}{∂x_μ} \left( \frac{∂Q_{μν}}{∂x_μ} - \frac{∂Q_{μν}}{∂x_μ} + \frac{∂Q_{μν}}{∂x_μ} \right)

+ \frac{1}{2} \left( \frac{∂Q_{μν}}{∂x_μ} \right)^2 - \frac{1}{4} \frac{∂Q_{μν}}{∂x_μ} \frac{∂Q_{μν}}{∂x_μ} + \frac{1}{2} \frac{∂Q_{μν}}{∂x_μ} \frac{∂Q_{μν}}{∂x_μ} \right] .

With Θ^{αβ} and Q^{αβ} given by equations (72) and (90), respectively, we find that equation (87), after considerable simplifications, reduces to the same equation (54). Thus in this second approximation also the equations satisfied by Θ^{ij} and T^{ij} are the same (see Notes added in proof).

X. THE CONSERVED ENERGY IN THE SECOND POST-NEWTONIAN APPROXIMATION

To obtain the conserved energy in the second post-Newtonian approximation, we must evaluate $\rho_0 v^0$ and $\sigma^μ v^μ = -g$ to $O(c^{-6})$ (i.e., in the third post-Newtonian approximation) in order that their difference may be known to $O(c^{-6})$. As we shall see presently, the evaluation of $\rho_0$ and $\sigma^μ v^μ = -g$ to $O(c^{-6})$ requires a knowledge of $g_{αβ}$ to $O(c^{-6})$.

Letting, then,

$$ g_{αβ} = - \left( 1 + \frac{2U}{c^4} \right) δ_{αβ} + \frac{1}{c^4} Q_{αβ} + \frac{1}{c^4} Q_{αβ}^{(3)} , $$

we find

$$ -g = 1 + \frac{4U}{c^4} + \frac{1}{c^4} (2U^2 - 4Φ - Q_{μν}) $$

$$ + \frac{1}{c^4} (-4U^3 - 24UΦ - 2UQ_{μν} - Q_{μν}^{(3)} + P_{μ}^{2} + Q_{00}) , $$

$$ \sqrt{-g} = 1 + \frac{2U}{c^4} - \frac{1}{c^4} (U^2 + 2Φ + \frac{1}{2}Q_{μν}) $$

$$ + \frac{1}{c^4} (-8UΦ + \frac{1}{2}P_{μ}^{2} + \frac{1}{2}Q_{00} - \frac{1}{2}Q_{μν}^{(3)} ) , $$

and

$$ u^0 = 1 + \frac{1}{c^4} (\frac{1}{2}v^2 + U) + \frac{1}{c^4} (\frac{3}{2}v^4 + \frac{1}{2}v^2 U + \frac{1}{2} U^2 + 2Φ - v_μ P_μ) $$

$$ + \frac{1}{2} \left[ \frac{5}{16} v^6 + \frac{27}{8} v^4 U + \frac{21}{4} v^2 U^2 - \frac{1}{3} U^3 + 3(v^2 + 2U)Φ $$

$$ - 3(\frac{1}{2}v^2 + U) v_μ P_μ - v_μ Q_{μ0} - \frac{1}{2} v_μ v_ν Q_{μν} - \frac{1}{2} Q_{00} \right] . $$

With the aid of these equations we find

$$ \rho_0 v^0 \sqrt{-g} = \rho + \frac{1}{c^2} \rho (\frac{1}{2}v^2 + 3U) + \frac{1}{c^2} \rho (\frac{3}{2}v^4 + \frac{7}{2}v^2 U + \frac{3}{2} U^2 - v_μ P_μ - \frac{1}{2} Q_{μν}) $$

$$ + \frac{1}{c^6} \rho \left[ \frac{5}{16} v^6 + \frac{33}{8} v^4 U + \frac{39}{4} v^2 U^2 - \frac{1}{3} U^3 + 2v^2 Φ $$

$$ - (\frac{3}{2}v^2 + 5U) v_μ P_μ + \frac{1}{2} P_μ^{2} - \frac{1}{2}(\frac{1}{2}v^2 + U) Q_{μν} $$

$$ - v_μ Q_{μ0} - \frac{1}{2} v_μ v_ν Q_{μν} - \frac{1}{2} Q_{μν}^{(3)} \right] . $$

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Next, we note that to $O(c^{-1})$

$$T^{00} = \rho c^2 + \rho (v^2 + 2U + \Pi)$$

$$+ \frac{1}{c^2} \rho \left[ v^2 + 6v^2 + 2U^2 + 4\Phi + 2U\Pi + v^2 \left( \Pi + \frac{\rho}{\rho} \right) - 2\rho P \right]$$

$$+ \frac{1}{c^2} \rho \left[ v^2 + 10v^2 + 16v^2U^2 + 8(v^2 + 2U)\Phi + 2(U^2 + 2\Phi)\Pi \right]$$

$$+ (v^2 + 6v^2 - 2\rho P) \left( \Pi + \frac{\rho}{\rho} \right) - 4(v^2 + 2U)\rho P$$

$$- 2\rho Q_{\rho 0} - \rho v^2 Q_{\rho \nu} - Q_{00} \right].$$

And the terms of $O(c^{-3})$ in $\phi^{00}$ are found to be

$$\frac{1}{16\pi Gc^4} \left\{-28(7U^2 + 2\Phi) \left( \frac{\partial U}{\partial x_{\mu}} \right)^2 - 4P \frac{\partial U}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} - 2P \frac{\partial P}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} \right\}$$

$$- 2P \frac{\partial P}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} - 16U \frac{\partial U}{\partial x_{\mu}} \frac{\partial Q_{x\nu}}{\partial x_{\nu}} - 6Q_{\mu\nu} \left[ \frac{\partial U}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} + 2\delta_{\mu\nu} \left( \frac{\partial U}{\partial x_{\sigma}} \right)^2 \right]$$

$$- \frac{\partial P}{\partial x_{\mu}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} + \left( \frac{\partial P}{\partial x_{\mu}} + \frac{\partial P}{\partial x_{\nu}} \right) \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} - \frac{1}{4} \frac{\partial Q_{\sigma\tau}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}}$$

$$+ \frac{1}{4} \frac{\partial Q_{\mu\nu}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} - \frac{1}{4} \frac{\partial Q_{\sigma\tau}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} + 4 \frac{\partial U}{\partial x_{\mu}} \frac{\partial Q_{\sigma\tau}}{\partial x_{\sigma}} \right\}$$

(97)

Combined with the terms of lower orders given in equation (57), we find that the terms of $O(c^{-4})$ in $\Theta^{00}$ are

$$\frac{1}{c^4} \rho \left[ v^2 + 14v^2U + 42v^2U^2 + 8U^3 + 4v^2\Phi + 12U^3 \Pi \right]$$

$$+ (v^2 + 10v^2 - 2\rho P) \left( \Pi + \frac{\rho}{\rho} \right) - 4(v^2 + 4U)\rho P + P^2$$

$$- (v^2 + 4U + \Pi)Q_{x\sigma} - v^2 Q_{x\nu} - 2\rho Q_{\rho 0} \right]$$

$$+ \frac{1}{16\pi Gc^4} \left\{-4P \frac{\partial U}{\partial x_{\mu}} \frac{\partial P}{\partial x_{\sigma}} - \frac{\partial U}{\partial x_{\mu}} \frac{\partial P}{\partial x_{\sigma}} - 2P \frac{\partial P}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} - 12U \left( \frac{\partial U}{\partial t} \right)^2 \right\}$$

$$+ 2U \frac{\partial P}{\partial x_{\tau}} \left( \frac{\partial P}{\partial x_{\mu}} + \frac{\partial P}{\partial x_{\nu}} \right) + 2Q_{x\sigma} \left( \frac{\partial U}{\partial x_{\mu}} \right)^2 - 6Q_{\mu\nu} \frac{\partial U}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}}$$

$$+ \frac{\partial Q_{\rho 0}}{\partial x_{\rho}} \left( \frac{\partial P}{\partial x_{\mu}} + \frac{\partial P}{\partial x_{\nu}} \right) - \frac{\partial P}{\partial x_{\nu}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} + \frac{1}{4} \frac{\partial Q_{\mu\nu}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}}$$

$$- \frac{1}{4} \frac{\partial Q_{\sigma\tau}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} - \frac{1}{4} \frac{\partial Q_{\sigma\tau}}{\partial x_{\sigma}} \frac{\partial Q_{\mu\nu}}{\partial x_{\nu}} + \frac{5}{2} \frac{\partial U}{\partial x_{\mu}} \frac{\partial Q_{\rho 0}}{\partial x_{\rho}} - \frac{1}{2} \frac{\partial Q_{x\sigma}}{\partial x_{\sigma}} \right\}.$$
These terms can be reduced, modulo divergence, to
\[
\frac{1}{c^4} \rho \left[ v^6 + 14v^4U + 42v^2U^2 + 8U^3 + 4v^2 \Phi + 12U^2 \Pi + \frac{3}{2} P_\mu^2 \right.
\]
\[
+ \left. (v^4 + 10v^2U - 2v_\mu P_\mu) \left( \Pi + \frac{\dot{v}}{\rho} \right) - 4(v^2 + 4U) v_\mu P_\mu \right.
\]
\[
- \frac{1}{2} (v^2 + 4U + \Pi) Q_{\sigma \sigma} - v_\mu Q_{\mu 0} - v_\mu v_\nu Q_{\mu \nu} \right]
\]
\[
+ \frac{1}{16\pi Gc^4} \left[ - 4P_\mu \frac{\partial U}{\partial x_\mu} - 2P_\mu \frac{\partial P_\nu}{\partial x_\mu} \frac{\partial U}{\partial x_\nu} + 2U \frac{\partial P_\mu}{\partial x_\nu} \frac{\partial P_\nu}{\partial x_\mu} \left( \frac{\partial P_\mu}{\partial x_\nu} + \frac{\partial P_\nu}{\partial x_\mu} \right) \right.
\]
\[
- 12U \left( \frac{\partial U}{\partial t} \right)^2 - \frac{5}{2} Q_{\epsilon \sigma} \left( \frac{\partial U}{\partial x_\mu} \right)^2 - 6Q_{\rho \sigma} \frac{\partial U}{\partial x_\rho} \frac{\partial U}{\partial x_\sigma} + \frac{1}{4} \frac{\partial Q_{\mu \nu}}{\partial x_\rho} \frac{\partial Q_{\rho \sigma}}{\partial x_\mu} \right.
\]
\[
+ \frac{1}{2} \frac{\partial U}{\partial t} \left( \frac{\partial Q_{\mu \nu}}{\partial x_\rho} + \frac{1}{2} \frac{\partial Q_{\nu \sigma}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{\sigma \rho}}{\partial x_\mu} \right) \right]
\]
\[
\left. - \frac{1}{4} (2U + \Pi) Q_{\sigma \sigma} - \frac{3}{2} v_\mu v_\nu Q_{\mu \nu} + \frac{3}{2} Q_{\sigma \sigma} \right].
\]

Now making use of the earlier result (67) and including the contribution of \( O(c^{-4}) \) that follows from equations (95) and (99), we find
\[
\Theta^{00} - c^2 \rho u^0 \sqrt{-g} = \rho \left( \frac{1}{16\pi Gc^4} - \frac{1}{2} U + \Pi \right)
\]
\[
+ \frac{1}{c^2} \rho \left[ \frac{5}{8} v^4 + \frac{5}{2} v^2 U - \frac{5}{2} U^2 + 2U^2 \Pi + v^3 \left( \Pi + \frac{\dot{v}}{\rho} \right) - \frac{3}{2} v_\mu P_\mu \right]
\]
\[
+ \frac{1}{c^4} \left\{ \rho \left[ \frac{1}{16} v^4 + \frac{9}{8} v^2 U + \frac{12}{8} v^2 U^2 + \frac{17}{2} U^3 + 2v^2 \Phi + 12U^2 \Pi \right.
\]
\[
+ (v^2 + 10vU - 2v_\mu P_\mu) \left( \Pi + \frac{\dot{v}}{\rho} \right) - \left( \frac{5}{2} v^2 + 11U \right) v_\mu P_\mu + \frac{1}{4} P_\mu^2
\]
\[
- \frac{1}{2} (2U + \Pi) Q_{\sigma \sigma} - \frac{3}{2} v_\mu v_\nu Q_{\mu \nu} + \frac{3}{2} Q_{\sigma \sigma} \right] \right] \right]
\]
\[
+ \frac{1}{16\pi Gc^4} \left[ - 4P_\mu \frac{\partial U}{\partial x_\mu} - 2P_\mu \frac{\partial P_\nu}{\partial x_\mu} \frac{\partial U}{\partial x_\nu} + 2U \frac{\partial P_\mu}{\partial x_\nu} \frac{\partial P_\nu}{\partial x_\mu} \left( \frac{\partial P_\mu}{\partial x_\nu} + \frac{\partial P_\nu}{\partial x_\mu} \right) \right.
\]
\[
- 12U \left( \frac{\partial U}{\partial t} \right)^2 - \frac{5}{2} Q_{\epsilon \sigma} \left( \frac{\partial U}{\partial x_\mu} \right)^2 - 6Q_{\rho \sigma} \frac{\partial U}{\partial x_\rho} \frac{\partial U}{\partial x_\sigma} + \frac{1}{4} \frac{\partial Q_{\mu \nu}}{\partial x_\rho} \frac{\partial Q_{\rho \sigma}}{\partial x_\mu} \right.
\]
\[
+ \frac{1}{2} \frac{\partial U}{\partial t} \left( \frac{\partial Q_{\mu \nu}}{\partial x_\rho} + \frac{1}{2} \frac{\partial Q_{\nu \sigma}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{\sigma \rho}}{\partial x_\mu} \right) \right]
\]
\[
\left. - 4U \frac{\partial}{\partial t} \left( \frac{\partial Q_{\mu \nu}}{\partial x_\rho} + \frac{1}{2} \frac{\partial Q_{\nu \sigma}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{\sigma \rho}}{\partial x_\mu} \right) \right] \right].
\]

The integral of this last expression over the whole of the three-dimensional space gives the required conserved energy in the second post-Newtonian approximation. But we observe that it includes the third post-Newtonian term \( \frac{1}{2} \rho Q_{\sigma \sigma}^{(3)} \); and we must eliminate its appearance. For this last purpose we need the equation which governs \( Q_{\sigma \sigma}^{(3)} \); i.e., we must obtain the third post-Newtonian approximation to \( g_{\alpha \beta} \).

\( a) \text{ The Equation Determining } Q_{\alpha \beta}^{(3)} \)

With the metric coefficient \( g_{\alpha \beta} \) given in equation (91), we find that the terms of \( O(c^{-6}) \) in \( R_{\alpha \beta} \) are
\[ R_{ab}^{(3)} = -\frac{1}{2} \nabla^2 Q_{ab}^{(3)} + \frac{1}{2} \frac{\partial}{\partial x_a} \left( \frac{\partial Q_{ab}^{(3)}}{\partial x_a} - \frac{1}{2} \frac{\partial Q_{ax}^{(3)}}{\partial x_a} \right) + \frac{1}{2} \frac{\partial}{\partial x_a} \left( \frac{\partial Q_{ab}^{(3)}}{\partial x_a} - \frac{1}{2} \frac{\partial Q_{ax}^{(3)}}{\partial x_a} \right) \]

\[ + S_{ab} U + \frac{1}{2} \frac{\partial^2 Q_{aa}}{\partial x_a \partial x_a} + Q_{\mu \nu} \frac{\partial^2 U}{\partial x_\mu \partial x_\nu} \delta_{ab} + 2 \frac{\partial U}{\partial x_a} \frac{\partial Q_{ab}}{\partial x_a} + Q_{\mu \nu} \frac{\partial^2 U}{\partial x_\mu \partial x_\nu} \]

\[ - \frac{\partial}{\partial x_a} \left( Q_{\beta \mu} \frac{\partial U}{\partial x_\mu} \right) - \frac{\partial}{\partial x_\beta} \left( Q_{\alpha \mu} \frac{\partial U}{\partial x_\mu} \right) + \frac{1}{2} \left( \frac{\partial U}{\partial x_a} \frac{\partial Q_{ab}}{\partial x_a} + \frac{\partial U}{\partial x_\beta} \frac{\partial Q_{ab}}{\partial x_\beta} \right) \]

\[ + \frac{1}{2} \frac{\partial^2 Q_{ab}}{\partial t^2} - \frac{\partial}{\partial t} \left( \frac{\partial Q_{ab}}{\partial x_a} + \frac{\partial Q_{ab}}{\partial x_\beta} \right) + \frac{1}{2} \left( \frac{\partial P_{\alpha}}{\partial x_a} \frac{\partial P_{\beta}}{\partial x_\beta} + \frac{\partial P_{\alpha}}{\partial x_\beta} \frac{\partial P_{\beta}}{\partial x_a} \right) \]

\[ + P_{\mu} \frac{\partial^2 P_{\mu}}{\partial x_a \partial x_\beta} - \frac{1}{2} P_{\mu} \frac{\partial}{\partial x_a} \left( \frac{\partial P_{\alpha}}{\partial x_\beta} + \frac{\partial P_{\beta}}{\partial x_\alpha} \right) - \frac{\partial P_{\mu}}{\partial t} \frac{\partial U}{\partial x_a} \delta_{ab} - 2P_{\mu} \frac{\partial^2 U}{\partial t \partial x_\beta} \]

\[ - U \frac{\partial}{\partial t} \left( \frac{\partial P_{\alpha}}{\partial x_\beta} + \frac{\partial P_{\beta}}{\partial x_\alpha} \right) + \frac{\partial}{\partial t} \left( P_{\alpha} \frac{\partial U}{\partial x_\beta} + P_{\beta} \frac{\partial U}{\partial x_\alpha} \right) + \frac{1}{2} \frac{\partial U}{\partial x_\beta} \left( \frac{\partial P_{\alpha}}{\partial x_\alpha} + \frac{\partial P_{\beta}}{\partial x_\beta} \right) \]

\[ + \frac{1}{6} \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} + 2 \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} + \delta_{ab} \left( \frac{\partial U}{\partial t} \right)^2 - 2\delta_{ab} U \frac{\partial^2 U}{\partial t^2} \]

\[ - 4U \frac{\partial^3 U}{\partial x_\alpha \partial x_\beta} - 4 \frac{\partial^3 U}{\partial x_\alpha \partial x_\beta} - \frac{\partial U}{\partial x_\beta} \frac{\partial U}{\partial x_\alpha} \left( U^2 + 2\Phi \right) \delta_{ab} \] \[ \text{(101)} \]

while the terms of \( O(c^{-2}) \) in \( T_{ab} - \frac{1}{2} \delta_{ab} T \) are

\[ (T_{ab} - \frac{1}{2} \delta_{ab} T)^{(2)} = \rho v_{ab} \left( v^2 + 6U + \Pi + \frac{\Phi}{\rho} \right) - \rho (v_a P_{\beta} + v_{\beta} P_a) \]

\[ + \delta_{ab} \rho U \left( \Pi - \frac{\Phi}{\rho} \right) - \frac{1}{2} \rho Q_{ab} \] \[ \text{(102)} \]

Making use of these results, we find that the equation governing \( Q_{ab}^{(3)} \) can be brought to the form

\[ \nabla^2 Q_{ab}^{(3)} - \frac{\partial}{\partial x_a} \left( \frac{\partial Q_{ab}^{(3)}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{ax}^{(3)}}{\partial x_\mu} \right) - \frac{\partial}{\partial x_\beta} \left( \frac{\partial Q_{ab}^{(3)}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{ax}^{(3)}}{\partial x_\mu} \right) = S_{ab}^{(3)} \] \[ \text{(103)} \]

where

\[ S_{ab}^{(3)} = 2S_{ab} U + \frac{8}{3} \nabla^2 U \delta_{ab} + 4 \frac{\partial^2 U^3}{\partial x_\alpha \partial x_\beta} - 4U \left( \frac{\partial U}{\partial x_\mu} \right)^2 \delta_{ab} \]

\[ + 2 \frac{\partial}{\partial t} \left( \frac{\partial U}{\partial t} \right)^2 \delta_{ab} - 4U \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} \delta_{ab} - 8U \frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\beta} - 8 \Phi \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} \]

\[ - 4 \frac{\partial U}{\partial x_\alpha} \frac{\partial \Phi}{\partial x_\beta} + 2P_{\mu} \frac{\partial P_{\mu}}{\partial x_\alpha} \frac{\partial P_{\mu}}{\partial x_\beta} + \frac{\partial P_{\mu}}{\partial x_\alpha} \frac{\partial P_{\mu}}{\partial x_\beta} + \frac{\partial P_{\mu}}{\partial x_\beta} \frac{\partial P_{\mu}}{\partial x_\alpha} \]

\[ + \frac{\partial U}{\partial x_\alpha} \left( \frac{\partial P_{\mu}}{\partial x_\beta} + \frac{\partial P_{\mu}}{\partial x_\alpha} \right) - 2U \frac{\partial}{\partial t} \left( \frac{\partial P_{\alpha}}{\partial x_\beta} + \frac{\partial P_{\beta}}{\partial x_\alpha} \right) - 4 \frac{\partial \Phi}{\partial x_a} \frac{\partial \Phi}{\partial x_\beta} \delta_{ab} \]

\[ - 2 \frac{\partial P_{\mu}}{\partial x_\alpha} \frac{\partial U}{\partial x_\beta} \delta_{ab} + 2 \frac{\partial}{\partial t} \left( P_{\alpha} \frac{\partial U}{\partial x_\beta} + P_{\beta} \frac{\partial U}{\partial x_\alpha} \right) - P_{\mu} \frac{\partial}{\partial x_\alpha} \left( \frac{\partial P_{\mu}}{\partial x_\beta} + \frac{\partial P_{\mu}}{\partial x_\alpha} \right) \]

\[ + \frac{\partial^2 Q_{ab}}{\partial x_\alpha \partial x_\beta} + 2Q_{\mu \nu} \frac{\partial^2 U}{\partial x_\mu \partial x_\nu} \delta_{ab} + 4 \frac{\partial U}{\partial x_\alpha} \frac{\partial Q_{ab}}{\partial x_\beta} + 2Q_{\mu \nu} \frac{\partial^2 U}{\partial x_\mu \partial x_\nu} \]

\[ + \frac{\partial U}{\partial x_\alpha} \frac{\partial Q_{ab}}{\partial x_\beta} + \frac{\partial U}{\partial x_\beta} \frac{\partial Q_{ab}}{\partial x_\alpha} - 2 \frac{\partial}{\partial x_a} \left( Q_{\beta \mu} \frac{\partial U}{\partial x_\mu} \right) - 2 \frac{\partial}{\partial x_\beta} \left( Q_{\alpha \mu} \frac{\partial U}{\partial x_\mu} \right) \]

\[ + \frac{\partial}{\partial t} \left( \frac{\partial Q_{ab}}{\partial t} - \frac{\partial Q_{ab}}{\partial x_a} - \frac{\partial Q_{ab}}{\partial x_\beta} \right) \]

\[ + 16\pi G \rho \left[ v_a v_\beta \left( v^2 + 6U + \Pi + \frac{\Phi}{\rho} \right) - v_a P_{\beta} - v_{\beta} P_a + U \left( \Pi - \frac{\Phi}{\rho} \right) \delta_{ab} - \frac{1}{2} Q_{ab} \right]. \]
Equation (103) governing \( Q^{(2)}_{ab} \) must be considered together with equation (15) governing \( Q_{ab} \), and when they are considered together, their solvability simultaneously requires that the equation

\[
\frac{\partial S_{ab}}{\partial x_\beta} - \frac{1}{2} \left( \frac{\partial S_{\alpha\gamma}}{\partial x_\alpha} + c^2 \left( \frac{\partial S_{ab}^{(3)}}{\partial x_\beta} - \frac{1}{2} \frac{\partial S_{\alpha\gamma}^{(3)}}{\partial x_\alpha} \right) \right) = 0
\]

be satisfied as an integrability condition. It can be verified (though the verification is long and tedious) that this integrability condition is indeed satisfied by virtue of the equation of motion (88) in the first post-Newtonian approximation.

In view of the integrability condition (105) being satisfied, we can extend the gauge condition (35), used in the second post-Newtonian approximation, to

\[
\frac{\partial Q_{ab}^{(3)}}{\partial x_\beta} - \frac{1}{2} \frac{\partial Q_{\alpha\gamma}^{(3)}}{\partial x_\alpha} + \frac{1}{c^2} \left( \frac{\partial Q_{ab}^{(3)}}{\partial x_\beta} - \frac{1}{2} \frac{\partial Q_{\alpha\gamma}^{(3)}}{\partial x_\alpha} \right) = 0
\]

in the third post-Newtonian approximation. With this choice of the gauge condition, the equation determining \( Q_{ab}^{(3)} \) takes the form

\[
\nabla^2 Q_{ab}^{(3)} = S_{ab}^{(3)}.
\]

Now contracting equations (104) and (107), we obtain

\[
\nabla^2 (Q_{\alpha\gamma}^{(3)} - 12U^3) = -48U \left( \frac{\partial U}{\partial x_\mu} \right)^2 - 6 \left( \frac{\partial U}{\partial t} \right)^2 + \frac{\partial P_\mu}{\partial x_\nu} \frac{\partial P_\nu}{\partial x_\mu} + \frac{\partial P_\mu}{\partial x_\nu} \frac{\partial P_\nu}{\partial x_\mu} + 4Q_{\nu\mu} \frac{\partial^2 U}{\partial x_\nu \partial x_\mu} + 4 \frac{\partial U}{\partial x_\mu} \frac{\partial Q_{\alpha\gamma}}{\partial x_\nu} - 2 \frac{\partial}{\partial t} \left( \frac{\partial Q_{\nu\mu}}{\partial x_\mu} - \frac{1}{2} \frac{\partial Q_{\alpha\gamma}}{\partial t} \right)
+ 16\pi G \rho \left[ 2\Theta + 16\Theta^2 U + 17U^2 + 8U^2 + 2\Theta \left( \Pi + \frac{\rho}{\frac{\rho}{\rho}} \right) \right]
- 4\nu_\mu P_\mu - Q_{\alpha\gamma}.
\]

b) The Conserved Energy in the Second Post-Newtonian Approximation

Returning to equation (100), we may replace (modulo divergence) the term \( \frac{1}{2} \rho Q_{\alpha\gamma}^{(3)} \) which occurs in it by

\[
\frac{1}{2} \rho Q_{\alpha\gamma}^{(3)} = \frac{1}{2} \rho (Q_{\alpha\gamma}^{(3)} - 12U^3) + 6\rho U^3
\]

\[
= - \frac{\nabla^2 U}{8\pi G} (Q_{\alpha\gamma}^{(3)} - 12U^3) + 6\rho U^3
\]

\[
= - \frac{U}{8\pi G} \nabla^2 (Q_{\alpha\gamma}^{(3)} - 12U^3) + 6\rho U^3 \quad \text{(mod div)},
\]

where we may further substitute for \( \nabla^2 (Q_{\alpha\gamma}^{(3)} - 12U^3) \) the expression on the right-hand side of equation (108). With this replacement and after some further reductions, modulo divergence, we find that the conserved energy in the second post-Newtonian approximation is the integral of the quantity,
\[ \mathcal{E} = \rho \left( \frac{5}{8} v^2 - \frac{1}{2} U + \Pi \right) \]
\[ + \frac{1}{c^2} \rho \left[ \frac{5}{8} v^4 + \frac{5}{8} v^2 U^{\frac{1}{2}} + 2 U \Pi + v^2 \left( \Pi + \frac{\mathcal{E}}{\rho} \right) - \frac{3}{2} v \rho P_{\mu} \right] \]
\[ + \frac{1}{c^4} \rho \left[ \frac{1}{16} v^6 + 2 v^4 U + \frac{7}{8} v^2 U^2 - \frac{25}{8} U^3 + 2 v^2 \Phi - \frac{5}{2} U^2 \Pi \right] \]
\[ + (v^4 + 6v^2 U - 2v \rho P_{\mu}) \left( \Pi + \frac{\mathcal{E}}{\rho} \right) - \left( \frac{v^2}{2} + 3U \right) P_{\mu} v_{\mu} + \frac{1}{2} P_{\mu} \right] \]
\[ + \frac{1}{8} (7U - 2\Pi) Q_{\nu \nu} - \frac{3}{2} v_{\nu} v_{\nu} Q_{\mu \nu} \]
\[ + \frac{1}{16 \pi G c^2} \left[ -4 P_{\mu} \frac{\partial U}{\partial x_{\mu}} \frac{\partial U}{\partial t} - 2 P_{\mu} \frac{\partial P_{\nu}}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} + 2 Q_{\mu \nu} \frac{\partial U}{\partial x_{\mu}} \frac{\partial U}{\partial x_{\nu}} \right] \]

over the whole of the three-dimensional space. We observe that this expression for \( \mathcal{E} \), like the one for \( \pi \) (eq. [74]), includes terms that do not vanish outside the volume occupied by the fluid; but these terms in \( \mathcal{E} \), unlike the analogous terms in \( \pi \), depend explicitly on the metric coefficients in this approximation.

\section*{XI. CONCLUDING REMARKS}

The present paper completes the solution of Einstein's field equations to the extent that is necessary to solve any problem in the hydrodynamics of self-gravitating perfect fluids with proper allowance for the effects of general relativity to \( O(c^{-4}) \). Also, the various integrals which the equations admit have been isolated to the same order. The information on the metric coefficients which is needed to obtain the relevant equations of motion in the different approximations is summarized in Table 1. As is indicated in this table, it is expected that the effects of gravitational radiation on the behavior of the system will first manifest itself when the next half-step is successfully taken. But it appears that entirely new considerations will be needed before we can properly take this next half-step. We hope to be able to return to these considerations in the near future.
One of us (S. C.) is grateful to Drs. Kip S. Thorne and James W. Bardeen for many fruitful discussions on topics related to this paper.

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Notes added in proof: 1. By an accidental omission of a term in the evaluation of $e^{\theta}$ (eq. [89]), the equation $\Theta_{i,j} = 0$ at first did not agree with equation (54). That this discrepancy might be due to an error in the evaluation of $e^{\theta}$ was suggested to us by Drs. F. B. Estabrook, B. K. Harrison, and H. D. Wahlquist. We are immensely grateful to them for their interest and for their critical examination of a preprint version of this paper.

2. Professor J. Stachel has kindly informed us that the gauge transformation (84) was independently discovered by Dr. Elliott Krefetz while reading an earlier preprint version of this paper. We are also grateful to Professor Stachel for some clarifying remarks.

3. Since writing this paper it has been possible to extend the analysis along lines initiated by Drs. W. L. Burke and Kip Thorne (but much more in the spirit of an early paper by A. Trautman [Bulletin de l'Académie Polonaise des Sciences, 6, 627–633, 1958]) to obtain the equations of the $2\frac{1}{2}$-post-Newtonian approximation in which the terms representing radiation reaction appear explicitly. The results of this investigation will be published separately by one of us (S. C.).

REFERENCES

———. 1969, ibid., 158, 45 (Paper II).