THE INSTABILITY OF THE CONGRUENT DARWIN ELLIPSOIDS

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Received January 9, 1969

ABSTRACT

In this paper a class of synchronous coupled oscillations of the congruent Darwin ellipsoids is considered; and it is shown that two of the five modes of oscillation belonging to this class excite instabilities along the entire Darwin sequence.

I. INTRODUCTION

In an earlier paper (Chandrasekhar 1964; this paper will be referred to hereafter as Paper I) devoted to the equilibrium and the stability of the Darwin ellipsoids, it was shown that under the specifications of Darwin's problem there are precisely two cases for which his problem allows mathematically consistent solutions. The two cases are (1) when one of the components is a Maclaurin spheroid and the other is a homogeneous "satellite" of infinitesimal mass rotating synchronously with the spheroid in a dynamically allowed circular orbit and (2) when the two components are congruent and of equal mass and describe, likewise, a dynamically allowed circular orbit. The problem of the stability in the first case presents no difficulty since perturbations of the satellite, in view of its infinitesimal mass, will not affect the central Maclaurin spheroids. But in the second case, a general treatment of the stability of the system does present special difficulties. Thus, while it is a simple problem to determine the point of onset of dynamical instability via a natural mode of oscillation of either component by itself, with the other remaining fixed, the problem, whether under their mutual tidal influence coupled modes of oscillation may exist and which may be more relevant to the onset of instability in the system, is not as simple. The latter problem was left unresolved in Paper I, though it was emphasized that a treatment of such coupled oscillations is essential if we are to pass judgment on the validity, or otherwise, of Darwin's surmises: that "the configuration of limiting stability" is the one for which the total angular momentum of the system (comprising the orbital angular momentum as well as that of the two components) attains its minimum value, or that "the limit of partial stability" is attained for that configuration for which the angular momentum, "representing all that part of the moment of momentum which is liable to variation when tides cannot be raised" in the other component, similarly attains a minimum value.

In this paper, we shall investigate a class of synchronous coupled oscillations of two congruent Darwin ellipsoids and show that two of the five modes of oscillation belonging to this class excite instabilities along the entire Darwin sequence.

The ideas underlying the present treatment are best clarified by first considering the isolation of a point along the Darwin sequence where identical quasi-static deformations of both components can be effected simultaneously without violating any of the conditions for equilibrium. A point where such quasi-static deformations of both components is possible is clearly different from the "Roche limit" as determined in Paper I (§ Illc), where the deformation of only one of the components was considered while the other was kept fixed. It would appear that the point which we wish to isolate now is strictly more analogous to the Roche limit as conventionally determined along a Roche sequence; indeed, as we shall see, it occurs precisely at the distance of closest approach of the two components.
II. THE STRICT ANALOGUE OF THE ROCHE LIMIT ALONG THE DARWIN SEQUENCE
OF CONGRUENT ELLIPSOIDS

Under conditions of hydrostatic equilibrium (in a rotating frame of reference) the
equation governing each of the two congruent components is (Paper I, eqs. [95]–[101])

\[-\frac{1}{\pi G \rho} \frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left\{- \sum_{j=1}^{3} A_j x_j^2 + a_1 (x_1^2 + x_2^2) + \frac{3}{2} \left[(a_2 + a_3) x_1^2 - a_2 x_2^2 - a_3 x_3^2\right]\right\} = 0,
\]

where

\[A_j = a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_j^2 + u) \Delta(u)}, \quad a_j = 2a_1 a_2 a_3 \int_{R^2 - a_1^2}^\infty \frac{du}{(a_j^2 + u) \Delta(u)},\]

and

\[\Delta^2(u) = (a_2^2 + u)(a_3^2 + u)(a_3^2 + u)\,.
\]

Since the pressure must vanish identically on the surface of the ellipsoid, we must
have

\[p = p_c \left(1 - \sum_{j=1}^{3} \frac{x_j^2}{a_j^2}\right),\]

where \(p_c\) denotes the pressure at the center of the ellipsoid. Inserting this expression
for the pressure in equation (1), we find that its satisfaction requires

\[\frac{2p_c}{\pi G \rho^2} = a_2(2A_1 - 2a_1 - a_2 - a_3) = a_3(2A_2 - 2a_1 + a_2) = a_3(2A_3 + a_2).
\]

It can be readily verified that these equations are entirely equivalent to the equations
used in Paper I (namely, eqs. [99]–[102]) to determine the equilibrium figures of the
Darwin ellipsoids.

We now ask the question: Is there a point along the sequence where equation (1)
allows neighboring solutions? Or, to put the question more precisely: Is there a point
along the sequence where the two congruent ellipsoids can be deformed simultaneously
into neighboring congruent ellipsoids, at the same distance \(R\) but with infinitesimally
differing axes, without in any way violating equation (1) governing their equilibrium?

It is clear that, formulated in this manner, the point we wish to identify and isolate
is entirely equivalent to the Roche limit along a Roche sequence (cf. Chandrasekhar
1963; see particularly § V of this paper); for in the Roche problem, the object tidally
distorting the body under consideration is, by assumption, a rigid sphere; accordingly,
its structure will be unaffected by any perturbation to which the Roche ellipsoid may
be subject. Therefore, the question of where, along a Roche sequence, a Roche ellipsoid
can be quasi-statically deformed into a neighboring ellipsoid without affecting its
equilibrium is equivalent to the question we have now asked in the more general con-
text of the Darwin sequence. However, it is important to emphasize that since equation
(1) governs both ellipsoids, we are considering identical and simultaneous deformations
of both ellipsoids.

Returning to equation (1), we now observe that at a point where quasi-static defor-

\[\text{That } R \text{ should be kept constant for such quasi-static deformations follows from a consideration of}
\]

\[\text{the equations governing the centers of the ellipsoids. An explicit demonstration of this requirement is}
\]

\[\text{given in the Appendix.}
\]
motions of the kind described are possible, not only must equation (1) be satisfied, its first variation must also be satisfied. And the variational equation is

\[- \frac{1}{\pi G \rho^2} \frac{\partial}{\partial x_i} \delta \rho + \frac{\partial}{\partial x_i} \left\{ - \sum_{j=1}^3 \delta A_j x_j^2 + \delta a_1 (x_1^2 + x_2^2) + \frac{1}{2} \left[ (\delta a_2 + \delta a_3) x_1^2 - \delta a_2 x_2^2 - \delta a_3 x_3^2 \right] \right\} = 0 , \]

where \( \delta A_j \) and \( \delta a_j \) are the first-order changes in the respective quantities arising from infinitesimal changes \( \delta a_j \) in the semi-axes of the ellipsoid compatible with the requirement that the volume of the ellipsoid is preserved. This last requirement implies that

\[ \frac{\delta a_1}{a_1} + \frac{\delta a_2}{a_2} + \frac{\delta a_3}{a_3} = 0 . \]

Since the distribution of pressure in an ellipsoid must always be given by an expression of the form given in equation (4), it follows that

\[ \delta \rho = \delta \rho_c \left( 1 - \sum_{j=1}^3 \frac{x_j^2}{a_j^2} \right) + 2 \rho_c \sum_{j=1}^3 \frac{x_j^2}{a_j^2} \delta a_j . \]

Inserting this expression for \( \delta \rho \) in equation (6), we find that the following equations must be satisfied:

\[ \frac{1}{\pi G \rho^2} \left( \delta \rho_c - 2 \rho_c \frac{\delta a_1}{a_1} \right) + a_1^2 (-\delta A_1 + \delta a_1 + \frac{2}{3} \delta a_2 + \frac{1}{3} \delta a_3) = 0 , \]

\[ \frac{1}{\pi G \rho^2} \left( \delta \rho_c - 2 \rho_c \frac{\delta a_2}{a_2} \right) + a_2^2 (-\delta A_2 + \delta a_1 + \frac{1}{3} \delta a_2 - \frac{2}{3} \delta a_3) = 0 , \]

\[ \frac{1}{\pi G \rho^2} \left( \delta \rho_c - 2 \rho_c \frac{\delta a_3}{a_3} \right) + a_3^2 (-\delta A_3 + \frac{2}{3} \delta a_2 - \frac{1}{3} \delta a_3) = 0 . \]

From the integrals defining \( A_j \) and \( a_j \) we find (remembering that \( a_1 a_2 a_3 \) remains constant for the variations considered) that

\[ \delta A_i = - \left( 3 A_i a_i^2 \frac{\delta a_i}{a_i} + A_i a_j^2 \frac{\delta a_j}{a_j} + A_i a_k^2 \frac{\delta a_k}{a_k} \right) \quad (i \neq j \neq k) , \]

and

\[ \delta a_1 = Q_{11} a_1^2 \frac{\delta a_1}{a_1} - a_1 a_2^2 \frac{\delta a_2}{a_2} - a_1 a_3^2 \frac{\delta a_3}{a_3} , \]

\[ \delta a_2 = Q_{21} a_1^2 \frac{\delta a_1}{a_1} - 3 a_1 a_2 a_2^2 \frac{\delta a_2}{a_2} - a_1 a_3 a_3^2 \frac{\delta a_3}{a_3} , \]

and

\[ \delta a_3 = Q_{31} a_1^2 \frac{\delta a_1}{a_1} - a_2 a_2 a_2^2 \frac{\delta a_2}{a_2} - 3 a_1 a_3 a_3^2 \frac{\delta a_3}{a_3} , \]

where

\[ Q_{11} = q_1 - 3 a_{11} , \quad Q_{21} = q_2 - a_{21} , \quad Q_{31} = q_3 - a_{31} , \]

\[ q_i = \frac{4 a_1 a_2 a_3}{(R^2 + a_i^2 - a_i^2)^2} , \]

and

\[ \Delta(\lambda) = R (R^2 + a_1^2 - a_1^2)^{1/2} (R^2 + a_2^2 - a_2^2)^{1/2} . \]
Also, in equations (12) and (13), $A_{ij}$ and $a_{ij}$ are the two-index symbols defined in the usual fashion:

$$A_{ij} = a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_i^2 + u)(a_j^2 + u)\Delta(u)}$$

and

$$a_{ij} = 2a_1 a_2 a_3 \int_{R^2 - a^2}^\infty \frac{du}{(a_i^2 + u)(a_j^2 + u)\Delta(u)} .$$

In view of equations (12) and (13), equations (9)-(11), together with equation (7), provide a system of four linear homogeneous equations for $a_i/\alpha_1$, $a_i/\alpha_2$, $a_i/\alpha_3$, and $\delta p_e/\pi G \rho^2$; and the condition for a nontrivial solution, namely, that the determinant of the system vanishes, will determine the desired point along the Darwin sequence. By a direct evaluation of the determinant along the sequence, it is found that the point occurs where

$$R = 2.8430 , \quad \Omega^2 = 0.12888 , \quad a_2/a_1 = 0.61842 , \quad a_3/a_1 = 0.56312 ,$$

$$d_1 = 1.4214 , \quad d_2 = 0.87900 , \quad \text{and} \quad d_3 = 0.80040 .$$

(18)

As one might have expected, the separation between the components is the least at this point. In the Appendix, we describe an alternative method of isolating this same point based on the virial equations of § VI below.

III. ON THE NATURE OF THE SYNCHRONOUS OSCILLATIONS TO BE STUDIED

The synchronous coupled modes of oscillation that we wish to consider are the natural generalizations of the quasi-static deformations considered in § II. The oscillations are to be associated, then, with identical deformations of the two ellipsoids, effected simultaneously, so that they remain congruent at all times. Precisely, we shall suppose that during the oscillations (1) the lengths of the semi-axes of the ellipsoid vary periodically, (2) the directions of the major axes of the two ellipsoids also vary periodically so that they are not always aligned, and finally (3) the directions of the least axes remain unchanged, perpendicular to the orbital plane. In addition, we shall find that under these circumstances the centers of the two ellipsoids also execute synchronous oscillations in such a manner that they always remain at the opposite ends of a diameter passing through their common center of mass.

A deformation of the ellipsoid, consistent with the foregoing assumptions, can be effected by a Lagrangian displacement $\xi$ with the components

$$\xi_1 = L_{1;0} + L_{1;1} x_1 + L_{1;2} x_2 ,$$

(19)

$$and$$

$$\xi_2 = L_{2;0} + L_{2;1} x_1 + L_{2;2} x_2 ,$$

$$\xi_3 = L_{3;3} x_3 .$$

By the application of the displacement (19), the boundary of the ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

(20)

becomes

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 2 \left( \frac{\xi_1 x_1}{a_1^2} + \frac{\xi_2 x_2}{a_2^2} + \frac{\xi_3 x_3}{a_3^2} \right) = 1 ;$$

(21)
or equivalently, to the first order in the displacement,

$$\frac{(x_1 - L_{1,0})^2}{a_1^2(1 + 2L_{1,1})} + \frac{(x_2 - L_{2,0})^2}{a_2^2(1 + 2L_{2,2})} + \frac{x_3^2}{a_3^2(1 + 2L_{3,3})} - 2x_1x_2\left(\frac{L_{1,2}}{a_1^2} + \frac{L_{2,1}}{a_2^2}\right) = 1. \tag{22}$$

Equation (22) represents an ellipsoid with its center at \((L_{1,0}, L_{2,0}, 0)\) and semi-axes of lengths

\[ a_1(1 + L_{1,1}), \quad a_2(1 + L_{2,2}), \quad \text{and} \quad a_3(1 + L_{3,3}); \tag{23} \]

further, the major axis of the ellipsoid is inclined to the \(x_1\)-axis by an angle

\[ \delta \theta = \frac{a_1^2 + a_2^2}{a_1 a_2} \frac{a_3^2}{a_3^2}. \tag{24} \]

It will be convenient to express the foregoing results, on the deformation of the ellipsoid (20) by the displacement (19), in terms of the virials

\[ V_i = \int_V \rho \xi_i dx, \quad V_{ij} = \int_V \rho \xi_i \xi_j dx, \quad \text{and} \quad V_{ij} = V_{ij} + V_{ij}. \tag{25} \]

The coefficients \(L_{ij}\) in the expressions for the components of \(\xi\) are related to \(V_{ij}\) by

\[ L_{ij} = \frac{5V_{ij}}{Ma_i^2}, \tag{26} \]

where \(M\) denotes the mass of the ellipsoid. Thus, by the application of the displacement (19), the lengths of the semi-axes are altered by the amounts

\[ \frac{\delta a_j}{a_j} = \frac{5V_{ij}}{2Ma_i^2}, \tag{27} \]

while the ellipsoid is rotated about the \(x_3\)-axis by the angle

\[ \delta \theta = \frac{5}{M} \frac{V_{12}}{a_1^2 - a_2^2}. \tag{28} \]

At the same time, the center of the ellipsoid is displaced to the point

\[ \frac{1}{M} (V_1, V_2, 0). \tag{29} \]

IV. THE TIDAL POTENTIAL BETWEEN TWO SLIGHTLY MISALIGNED ELLIPSOIDS

In Paper I, the tidal effect of an ellipsoid on another directly facing it (so that the major axes of the two are aligned) was expressed in terms of the first and the second derivatives of its potential at the center of the other. When the ellipsoids are slightly misaligned, as in Figure 1, it is clear that in order to express the tidal effects of one of the ellipsoids on the other, we need the first and the second derivatives of its external potential at a point \((R \cos \delta \theta, -R \sin \delta \theta, 0)\) slightly off the continuation of its major axis.

Quite generally, the potential of an ellipsoid at an external point is given by

\[ \Psi = -\frac{2}{3}GM \int \left( \sum_{j=1}^{3} \frac{x_j^2}{a_j^3 + u} - 1 \right) \frac{du}{\Delta(u)}, \tag{30} \]
where $\Delta(u)$ has the same meaning as in equation (3) and $\lambda$ is the positive root of the equation
\[
\sum_{j=1}^{3} \frac{x_j^2}{a_j^2 + \lambda} = 1.
\] (31)

We shall need the first and the second derivatives of $\mathcal{B}$ given by equations (30) and (31) at the point $(R \cos \delta \theta, -R \sin \delta \theta, 0)$ to the first order in $R \delta \theta$. To this order, the solution of equation (31) for $\lambda$ is given by
\[
\lambda = R^2 - a^2 + O(R^2 \delta \theta);
\] (32)
i.e., the value of $\lambda$, to the order required, is the same as for the point $(R,0,0)$ on the continuation of the axis. It can be readily deduced from this fact that the same is also true of $\partial \mathcal{B} / \partial x_1$ and $\partial^2 \mathcal{B} / \partial x_1 \partial x_2$; i.e., the expressions for these quantities, to the first order

\[
\left( R \cos \delta \theta, -R \sin \delta \theta \right).
\]

Fig. 1.—The relative positions of the Darwin ellipsoids in synchronous oscillation

in $R \delta \theta$, are the same as those given in Paper I, equations (89)–(92), for the point $(R,0,0)$. However, $\partial \mathcal{B} / \partial x_2$ and $\partial^2 \mathcal{B} / \partial x_1 \partial x_2$ differ from zero—as they are at $(R,0,0)$—by quantities of the first order in $R \delta \theta$; and these we must retain. We have
\[
\frac{\partial \mathcal{B}}{\partial x_2} = -\frac{3}{2} GM x_2 \int_{\lambda}^{\infty} \frac{du}{(a^2 + u)\Delta(u)}
\] (33)
and
\[
\frac{\partial^2 \mathcal{B}}{\partial x_1 \partial x_2} = \frac{3}{2} GM x_2 \frac{\partial \lambda}{\lambda} \frac{\partial \lambda}{\partial x_1}.
\] (34)

By making use of equation (88) of Paper I, equation (34) becomes
\[
\frac{\partial^2 \mathcal{B}}{\partial x_1 \partial x_2} = \frac{3 GM x_2 x_3}{a^2 + \lambda} \left[ \sum_{j=1}^{3} \frac{x_j^2}{(a_j^2 + \lambda)^2} \right]^{-1}.
\] (35)

Hence, to the order required,
\[
\frac{1}{\pi G \rho} \left( \frac{\partial \mathcal{B}}{\partial x_2} \right)_{R \cos \delta \theta, -R \sin \delta \theta, 0} = a_2 R \delta \theta + O(R^2 \delta \theta) ,
\] (36)

and
\[
\frac{1}{\pi G \rho} \left( \frac{\partial^2 \mathcal{B}}{\partial x_1 \partial x_2} \right)_{R \cos \delta \theta, -R \sin \delta \theta, 0} = \frac{4 R a_1 a_2 a_3 \delta \theta}{(R^2 + a_2^2 + a_1^2)^{3/2}} (R^2 + a_3^2 - a_1^2)^{1/2} + O(R^2 \delta \theta).
\] (37)

V. EQUATIONS GOVERNING THE FLUID MOTIONS IN THE ELLIPSOIDS APPROPRIATE FOR TREATING SYNCHRONOUS OSCILLATIONS

In writing the equations governing the fluid motions in either of the two congruent ellipsoids, we shall use a frame of reference that is somewhat more general than the one

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adopted in Paper I, § II. As in Paper I, we shall refer the motions to a frame rotating
with an angular velocity $\Omega$ about the common center of mass and at right angles to the
orbital plane; but in contrast to Paper I, we shall now let $\Omega$ be a function of time which
we shall specify presently. We shall also allow for a possible misalignment of the major
axes of the two ellipsoids. On these assumptions, equation (5) of Paper I is replaced by

$$
\rho \frac{du_i}{dt} = - \frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left[ \mathcal{Q} + \frac{3}{2} \sum_{j=1}^{3} \left( \frac{\partial \mathcal{Q}'}{\partial x_j^2} \right)_* x_j^2 + \frac{1}{2} \Omega^2 (x_1^2 + x_2^2) 
+ \left( \frac{\partial \mathcal{Q}'}{\partial x_1 \partial X_2} \right)_* x_1 x_2 - \left[ \left( \frac{\partial \mathcal{Q}'}{\partial x_1} \right)_* + \frac{1}{2} \Omega^2 \right] x_1 - \left( \frac{\partial \mathcal{Q}'}{\partial x_2} \right)_* x_2 \right] 
+ 2 \rho \Omega \epsilon_{iib} u_i - \rho \frac{d\Omega}{dt} \epsilon_{iin} (\frac{1}{2} R - x)_i,
$$

where by the asterisks, after the parentheses enclosing the various partial derivatives
of the external potential $\mathcal{Q}'$ of the other ellipsoid, we indicate that they are to be evaluated
at the point $(R \cos \delta \theta, -R \sin \delta \theta, 0)$ in its coordinate frame $(X_1, X_2, X_3)$ (see Paper
I, eq. [2]).

Since we now contemplate the possibility of the ellipsoids varying their relative orien-
tations and separation, as well as their shapes (though retaining their ellipsoidal figures)
we must allow for the possibility that the various partial derivatives of $\mathcal{Q}'$ are functions
of time. It is on this account that we have let $\Omega$ be a function of time so that we can
specify (as in Paper I, eq. [6])

$$
\Omega^2(t) = - \frac{2}{R} \left( \frac{\partial \mathcal{Q}'}{\partial x_1} \right)_* = 2a_1(\pi G \rho) .
$$

With $\Omega^2(t)$ specified in this manner, we can rewrite equation (38) in the form

$$
\rho \frac{du_i}{dt} = - \frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \left[ \mathcal{Q} + \frac{1}{2} \beta_{lm} x_l x_m \left( \frac{\partial \mathcal{Q}'}{\partial x_j} \right)_* x_j \right] 
+ 2 \rho \Omega \epsilon_{iib} u_i - \rho \frac{d\Omega}{dt} \epsilon_{iin} (\frac{1}{2} R - x)_i,
$$

where, under the circumstances contemplated, the nonvanishing components of $\beta_{lm}$ are

$$
\beta_{11} = \Omega^2 + \left( \frac{\partial \mathcal{Q}'}{\partial x_1^2} \right)_*, \quad \beta_{22} = \Omega^2 + \left( \frac{\partial \mathcal{Q}'}{\partial x_2^2} \right)_*, \\
\beta_{33} = \left( \frac{\partial \mathcal{Q}'}{\partial x_3^2} \right)_* = - \left( \frac{\partial \mathcal{Q}'}{\partial x_1^2} \right)_* + \frac{1}{2} \Omega^2, \quad \text{and} \quad \beta_{12} = \left( \frac{\partial \mathcal{Q}'}{\partial x_1 \partial x_2} \right)_* .
$$

a) The Virial Equations of the First and the Second Orders

The forms of the virial equations, appropriate to the problem on hand, can be derived
from equation (40) in the usual fashion. We have

$$
\frac{d}{dt} \int_V \rho u_i dx = \beta_{im} I_m - M \left( \frac{\partial \mathcal{Q}'}{\partial X_i} \right)_* \delta_{i2} + 2 \Omega \epsilon_{iib} \int_V \rho u_i dx \\
- \frac{1}{2} M \frac{d\Omega}{dt} \epsilon_{iib} R_i + \frac{d\Omega}{dt} \epsilon_{iin} I_i
$$

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and
\[
\frac{d}{dt} \int \rho u_i x_j dx = 2 \dot{X}_{ij} + \nabla_{ij} + \beta_{im} I_{mj} + 2 \Omega e_{i8} \int \rho u_i x_j dx
\]
\[
- \delta_{ij} \left( \frac{\partial \Psi}{\partial X_j} \right)_e I_j + \frac{d \Omega}{dt} \epsilon_{ij} R_i I_j + \delta_i \Pi ,
\]
where the various symbols have their standard meanings.

Equations (42) and (43) allow the same solutions for equilibrium as those derived in Paper I. And the circumstances for equilibrium are the same: there are no internal motions, \( \Omega \) is a constant, and the major axes of the two ellipsoids are aligned so that \( \beta_{im} \) is diagonal and \( (\partial \Psi'/\partial X_j)_e = 0 \). The equations governing equilibrium are
\[
\mathcal{B}_{11} + \beta_{11} I_{11} = \mathcal{B}_{22} + \beta_{22} I_{22} = \mathcal{B}_{33} + \beta_{33} I_{33} = -\Pi (44)
\]

and
\[
\beta_{11} = 2a_1 + a_2 + a_3 , \quad \beta_{22} = 2a_1 - a_3 , \quad \beta_{33} = -a_1 , \quad \text{and} \quad \Omega^2 = 2a_1 . (45)
\]

In writing equations (44) and (45), we have suppressed a factor \( \pi G \rho \) in the expressions for the potentials; this suppression is equivalent to measuring frequencies in the unit \((\pi G \rho)^{1/2}\). This convention will be adopted in the rest of this paper.

VI. THE VIRIAL EQUATIONS GOVERNING THE SYNCHRONOUS OSCILLATIONS OF THE DARWIN ELLIPSOIDS

Let the equilibrium ellipsoids, determined in accordance with equations (44) and (45), execute synchronous oscillations of the kind described in § III; and let the ensuing motions, in each of the two ellipsoids, be derived from the same Lagrangian displacement
\[
\xi(x) e^{it} ,
\]
where \( \lambda \) is a parameter whose characteristic values are to be determined. For such synchronous oscillations, not only do the components of the potential-energy tensor \( \mathcal{B}_{ij} \) vary, the coefficients \( \beta_{ij} \), determining the tidal effects of the other ellipsoid, also vary. The virial equations governing these synchronous oscillations follow from the appropriately linearized forms of equations (42) and (43). We have
\[
\lambda^2 V_i = \beta_{im} V_m - \left( \frac{\partial \Psi}{\partial X_j} \right)_e M \delta_{ij} + 2 \lambda \Omega e_{i8} V_1 + 2 \lambda \delta \Omega M R \delta_{i2} , (47)
\]

and
\[
\lambda^2 V_{ij} = 2 \lambda \Omega e_{i8} V_{1;j} - \lambda \delta \Omega e_{i8} I_{ij}
\]
\[
= \delta \mathcal{B}_{ij} + \beta_{im} V_{mj} + \delta \beta_{im} I_{mj} + \delta_i \delta \Pi , (48)
\]
where it should be noted that we have not included the terms \( \delta \beta_{im} I_{im} \) and \( \epsilon_{i8} I_{d} \delta \Omega / dt \) in equation (47) and the terms \( (\partial \Psi'/\partial X_j)_e I_{ij} \delta_{i2} \) and \( \epsilon_{i8} R_i I_{d} \delta \Omega / dt \) in equation (48) since these are of the second order in the displacement.

a) The Reduction of the Second-Order Virial Equations

For the synchronous coupled oscillations of the kind we are considering, the only nonvanishing virials are \( V_1, V_2, V_{1;1}, V_{2;2}, V_{3;3}, V_{1;2}, \) and \( V_{2;1} \) since we have assumed in the derivation of equation (40) that the direction of the \( x_3 \)-axis remains invariant and also that the centers of the two ellipsoids execute no oscillatory motions in that direction. And remembering also that in the equilibrium figures \( I_{ij} \) and \( \beta_{ij} \) are diagonal and
that the only nonvanishing off-diagonal element among the \( \delta \beta_{ij} \)'s is \( \delta \beta_{13} \), we find that equation (48) for the different components gives

\[
\frac{1}{2} \lambda^2 V_{33} = \delta \beta_{33} + \beta_{33} V_{33} + \delta \beta_{33} I_{33} + \delta \Pi ,
\]

\[
\frac{1}{2} \lambda^2 V_{11} - 2 \eta V_{12} = \delta \beta_{11} + \beta_{11} V_{11} + \delta \beta_{11} I_{11} + \delta \Pi ,
\]

\[
\frac{1}{2} \lambda^2 V_{22} + 2 \eta V_{12} = \delta \beta_{22} + \beta_{22} V_{22} + \delta \beta_{22} I_{22} + \delta \Pi ,
\]

\[
\lambda^2 V_{1:2} - \lambda \Omega V_{22} - \lambda \delta \Omega I_{22} = -(2B_{12} - \beta_{11}) V_{12} + \delta \beta_{12} I_{22} ,
\]

and

\[
\lambda^2 V_{2:1} + \lambda \Omega V_{11} + \lambda \delta \Omega I_{11} = -(2B_{12} - \beta_{22}) V_{12} + \delta \beta_{12} I_{11} ,
\]

where we have substituted for \( \delta \beta_{12} \) its known expression (Paper I, eqs. [21]–[23]).

After eliminating \( \delta \Pi \) appropriately from equations (49)–(51) and combining equations (52) and (53) suitably, we obtain

\[
(\lambda^2 + 4B_{12} - \beta_{11} - \beta_{22}) V_{12} + \lambda \Omega (V_{11} - V_{22}) + \lambda \delta \Omega (I_{11} - I_{22}) = \delta \beta_{12} (I_{11} + I_{22}) ,
\]

\[
\lambda^2 (V_{1:2} - V_{2:1})
\]

\[
= \lambda \Omega (V_{11} + V_{22}) + \lambda \delta \Omega (I_{11} + I_{22}) + (\beta_{11} - \beta_{22}) V_{12} - \delta \beta_{12} (I_{11} - I_{22}) ,
\]

\[
\frac{1}{2} \lambda^2 (V_{11} + V_{22}) + 2 \lambda \Omega (V_{1:2} - V_{2:1}) - \lambda^2 V_{33} = \delta \beta_{11} + \delta \beta_{22} - 2 \delta \beta_{33} ,
\]

\[
+ \beta_{11} V_{11} + \beta_{22} V_{22} - 2 \delta \beta_{33} V_{33} + \delta \beta_{11} I_{11} + \delta \beta_{22} I_{22} - 2 \delta \beta_{33} I_{33} ,
\]

and

\[
\frac{1}{2} \lambda^2 (V_{11} - V_{22}) - 2 \lambda \Omega V_{12} = \delta \beta_{11} + \delta \beta_{22} + \beta_{11} V_{11} - \beta_{22} V_{22} + \delta \beta_{11} I_{11} - \delta \beta_{22} I_{22} .
\]

These four equations must be supplemented by the further condition (cf. eqs. [7] and [27]),

\[
\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0 .
\]

Next, eliminating \((V_{1:2} - V_{2:1})\) between equations (55) and (56) and substituting for \( \delta \beta_{11} + \delta \beta_{22} - 2 \delta \beta_{33} \) and \( \delta \beta_{11} - \delta \beta_{22} \) their known expressions (Paper I, eqs. [45] and [46]), we obtain the following equations:

\[
(\frac{1}{2} \lambda^2 + 3B_{11} - B_{12} - \beta_{11}) V_{11} + (- \frac{1}{2} \lambda^2 - 3B_{22} + B_{12} + \beta_{22}) V_{22}
\]

\[
+ (B_{13} - B_{23}) V_{33} - 2 \lambda \Omega V_{12} - \delta \beta_{11} I_{11} + \delta \beta_{22} I_{22} = 0 ,
\]

\[
(\frac{1}{2} \lambda^2 + 3B_{11} + B_{12} - 2B_{13} + 2 \Omega^2 - \beta_{11}) V_{11} + (\frac{1}{2} \lambda^2 + 3B_{22} + B_{12} - 2B_{23}) V_{22}
\]

\[
+ 2 \Omega^2 - \beta_{22}) V_{22} + (- \lambda^2 - 6B_{33} + B_{13} + B_{23} + 2 \beta_{33}) V_{33} + 2 (\beta_{11} - \beta_{22}) \frac{\Omega}{\lambda} V_{12}
\]

\[
+ 2 \lambda \delta \Omega (I_{11} + I_{22}) - \frac{2 \Omega}{\lambda} (I_{11} - I_{22}) \delta \beta_{12} - \delta \beta_{11} I_{11} - \delta \beta_{22} I_{22} + 2 \delta \beta_{33} I_{33} = 0 ,
\]

and

\[
\lambda \Omega V_{11} - \lambda \Omega V_{22} + (\lambda^2 + 4B_{12} - \beta_{11} - \beta_{22}) V_{12} + \lambda \delta \Omega (I_{11} - I_{22})
\]

\[
- \delta \beta_{12} (I_{11} + I_{22}) = 0 .
\]

It remains to express the \( \delta \beta_{ij} \)'s that occur in these equations in terms of the \( V_i \)'s and \( V_{ij} \)'s.
Considering first the variations in the diagonal components of $\beta_{ij}$ and in $\Omega$, we have (cf. eqs. [45])

$$\delta \beta_{11} = 2\delta a_1 + \delta a_2 + \delta a_3, \quad \delta \beta_{22} = 2\delta a_1 - \delta a_2, \quad \delta \beta_{33} = -\delta a_3,$$

and $\Omega \delta \Omega = \delta a_1$, (62)

where the $\delta a_j$'s denote the changes in the $a_j$'s consequent to changes in the semi-axes ($a_j$) of the ellipsoids and in the separation ($R$) between their centers. Accordingly,

$$\delta a_j = \sum_{k=1}^3 \frac{\partial a_j}{\partial a_k} \delta a_k + \frac{\partial a_j}{\partial R} \delta R,$$

(63)

In equations (13), we have already given expressions for the changes in the $a_j$'s consequent to the changes in the $a_j$'s; to these changes we must now add the changes resulting from the change in $R$. From the definition of $a_j$, it readily follows that

$$\frac{\partial a_j}{\partial R} = -q_j R,$$

(64)

where $q_j$ has the same meaning as in equation (15). Also, from the fact that during the oscillations the centers of the two ellipsoids approach or recede in phase, we conclude that (cf. eqs. [19] and [29])

$$\delta R = -2L_{1;0} = -\frac{2}{M} V_1.$$

(65)

From equations (13), (27), and (63)–(65) we now obtain

$$\delta a_1 = \frac{1}{2M} \left[ 5(Q_{11} V_{11} - a_{12} V_{22} - a_{13} V_{33}) + 4q_1 R V_1 \right],$$

$$\delta a_2 = \frac{1}{2M} \left[ 5(Q_{22} V_{11} - 3a_{23} V_{22} - a_{23} V_{33}) + 4q_2 R V_1 \right],$$

(66)

and

$$\delta a_3 = \frac{1}{2M} \left[ 5(Q_{33} V_{11} - a_{32} V_{22} - 3a_{33} V_{33}) + 4q_3 R V_1 \right].$$

These relations, when inserted into equations (62), provide the required expressions for $\delta \beta_{11}$, $\delta \beta_{22}$, $\delta \beta_{33}$, and $\Omega \delta \Omega$ in terms of $V_{11}$, $V_{22}$, $V_{33}$, and $V_1$. And equations (28) and (37) provide the further relation

$$\delta \beta_{12} = -\frac{5}{M} \frac{4R a_1 a_2 a_3}{(R^2 + a_1^2 - a_2^2)^{3/2}} \frac{V_{12}}{(R^2 + a_2^2 - a_3^2)^{1/2}} (a_1^2 - a_2^2).$$

(67)

Finally, substituting in equations (59)–(61) for the $\delta \beta_{ij}$'s and $\delta \Omega$ their expressions in terms of the virials, we obtain

$$\left[ \frac{1}{2} \lambda^2 + 3B_{11} - B_{12} - \beta_{11} - (a_1^2 - a_2^2)Q_{11} \right] V_{11}$$

$$+ \left[ -\frac{1}{2} \lambda^2 - 3B_{22} + B_{12} + \beta_{22} + (a_1^2 - a_2^2)a_{12} + \frac{3}{2}(a_1^2 + a_2^2)a_{22} + \frac{1}{2} a_1^2 a_{23} \right] V_{22}$$

$$+ \left[ B_{13} - B_{23} + (a_1^2 - a_2^2)a_{13} + \frac{1}{2}(a_1^2 + a_2^2)a_{23} + \frac{3}{2} a_1^2 a_{33} \right] V_{33} - 2\lambda^2 \Omega V_{13}/\lambda$$

$$- \frac{2}{5} R [2(a_1^2 - a_2^2)q_1 + (a_1^2 + a_2^2)q_2 + a_1^2 q_3] V_1 = 0,$$

(68)
\[
\begin{align*}
&\left[\frac{1}{3}\lambda^2 + 3B_{11} + B_{12} - 2B_{13} + 2\Omega^2 - \beta_{11} - \frac{1}{2}(a_1^2 - a_2^2)Q_{12} - \frac{1}{2}(a_1^2 + 2a_3^2)Q_{13}\right]V_{11} \\
&+ \left[\frac{1}{3}\lambda^2 + 3B_{22} + B_{12} - 2B_{23} + 2\Omega^2 - \beta_{22} + \frac{3}{2}(a_1^2 - a_3^2)a_{22} + \frac{1}{2}(a_1^2 + 2a_3^2)a_{32}\right]V_{22} \\
&+ \left[\lambda^2 - 6B_{33} + B_{13} + B_{23} + 2\beta_{33} + \frac{1}{2}(a_1^2 - a_2^2)a_{33} + \frac{3}{2}(a_1^2 + 2a_3^2)a_{33}\right]V_{33} \\
&+ 2\Omega\left[\beta_{11} - \beta_{22} + \frac{4Ra_1a_2a_3}{(R^2 + a_1^2 - a_2^2)(R^2 + a_3^2 - a_2^2)^{1/2}}\right]V_{12} \\
&- \frac{2}{3}\lambda R[(a_1^2 - a_2^2)q_2 + (a_1^2 + 2a_3^2)q_3]V_1 = 0 ,
\end{align*}
\]

and

\[
\begin{align*}
&\left[\lambda^2 + \frac{1}{2}(a_1^2 - a_2^2)q_{11}\right]V_{11} - \frac{1}{2}\left[\frac{1}{3}\lambda^2 + \frac{1}{2}(a_1^2 - a_2^2)a_{12}\right]V_{22} = \frac{(a_1^2 - a_2^2)a_{13}}{2\Omega} V_{33} \\
&+ \left[\lambda^2 + 4B_{12} - \beta_{11} - \beta_{22} + \frac{4Ra_1a_2a_3(a_3^2 + a_2^2)/(a_1^2 - a_2^2)}{(R^2 + a_2^2 - a_1^2)(R^2 + a_3^2 - a_2^2)^{1/2}}\right]V_{12} \\
&+ \frac{2}{5\Omega}(a_1^2 - a_3^2)Rq_1V_1 = 0 .
\end{align*}
\]

b) The Reduction of the First-Order Virial Equation

Returning to equation (47) and substituting for \( (\partial R^2/\partial X_2) \) \( \frac{\partial}{\partial \Omega} \) in accordance with equations (28), (36), (62), and (66), we obtain

\[
\lambda^2 V_i - 2\lambda\Omega e_{1i2}V_1 - \beta_{im}V_m + \frac{5a_2R}{a_1^2 - a_2^2} V_{12}e_{12} \\
- \frac{\lambda}{2\Omega}\left[\frac{1}{2}(Q_{11}V_{11} - a_{12}V_{22} - a_{13}V_{33}) + 2q_1RV_1\right]e_{12} = 0 .
\]

For the type of oscillations we are considering, \( V_3 = 0 \); and the equations governing \( V_1 \) and \( V_2 \) are

\[
(\lambda^2 - \beta_{11})V_1 - 2\lambda\Omega V_2 = 0 ,
\]

and

\[
(\lambda^2 - \beta_{22})V_2 + 2\lambda\left[\Omega^2 - \frac{1}{2}q_1R^2\right]V_1 + \frac{5a_2R}{a_1^2 - a_2^2} V_{12} \\
- \frac{5}{4}\lambda R\left[Q_{11}V_{11} - a_{12}V_{22} - a_{13}V_{33}\right] = 0 .
\]

VII. THE DYNAMICAL INSTABILITY OF THE DARWIN ELLIPSOIDS

The required characteristic equation for \( \lambda^2 \) follows from setting the determinant of equations (68)-(70), (72), and (73) equal to zero; and the square of the characteristic frequencies (\( \sigma^2 \)) determined with its aid are listed in Table 1. From these calculations the unexpected result emerges that the Darwin ellipsoid is unstable along its entire sequence by two of the five modes of coupled oscillation.

It will be observed that ordinary instability (in which the amplitudes increase exponentially with time) is replaced by overstability (in which the amplitudes of oscillations increase exponentially) at a certain determinate point. Besides, one of the two unstable modes becomes neutral at a point close to the Roche limit. Thus setting \( \lambda = 0 \) in the secular determinant, we find that it reduces to
\begin{equation}
\begin{pmatrix}
3B_{11} - B_{12} - \beta_{11} \\
- (a_1^2 - a_2^2)Q_{11} \\
- \frac{1}{2}(a_1^2 + a_2^2)Q_{12} \\
- \frac{1}{2}(a_1^2 + 2a_2^2)Q_{13}
\end{pmatrix}
\begin{pmatrix}
-3B_{22} + B_{12} + \beta_{22} \\
+ (a_1^2 - a_2^2)a_{12} \\
+ \frac{3}{2}(a_1^2 + a_2^2)a_{22} \\
+ \frac{1}{2}(a_1^2 + 2a_2^2)a_{23}
\end{pmatrix}
\begin{pmatrix}
B_{13} - B_{23} \\
+ (a_1^2 - a_2^2)a_{13} \\
+ \frac{3}{2}(a_1^2 + a_2^2)a_{23} \\
+ \frac{3}{2}(a_1^2 + 2a_2^2)a_{23}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
2(\beta_{11} - \beta_{22}) \\
2(\beta_{11} - \beta_{22})
\end{pmatrix}
+ \frac{8Ra_1a_2a_3}{(R^2 + a_2^2 - a_1^2)^{3/2}(R^2 + a_3^2 - a_1^2)^{1/2}}
\begin{pmatrix}
3B_{11} + B_{12} - 2B_{13} \\
3B_{22} + B_{12} - 2B_{23} \\
+ 2\Omega^2 - \beta_{11} \\
+ 2\Omega^2 - \beta_{22}
\end{pmatrix}
\begin{pmatrix}
3B_{11} + B_{12} - 2B_{13} \\
3B_{22} + B_{12} - 2B_{23} \\
+ 2\Omega^2 - \beta_{11} \\
+ 2\Omega^2 - \beta_{22}
\end{pmatrix}
\begin{pmatrix}
-6B_{33} + B_{13} + B_{23} \\
-6B_{33} + B_{13} + B_{23} \\
+ 2\beta_{33} \\
+ 2\beta_{33}
\end{pmatrix}
+ \frac{4Ra_1a_2a_3(a_1^2 + a_2^2)/(a_1^2 - a_2^2)}{(R^2 + a_2^2 - a_1^2)^{3/2}(R^2 + a_3^2 - a_1^2)^{1/2}}
\begin{pmatrix}
\Omega^2 + \frac{1}{2}(a_1^2 - a_2^2)Q_{11} \\
- \Omega^2 - \frac{1}{2}(a_1^2 - a_2^2)a_{12} \\
- \Omega^2 - \frac{1}{2}(a_1^2 - a_2^2)a_{12} \\
- \frac{1}{2}(a_1^2 - a_2^2)a_{13}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{a_1^2} \\
\frac{1}{a_2^2} \\
\frac{1}{a_2^2} \\
\frac{1}{a_3^2}
\end{pmatrix}
\begin{pmatrix}
- \frac{1}{2}(a_1^2 - a_2^2)a_{13} \\
+ \frac{1}{2}(a_1^2 - a_2^2)a_{13} \\
+ \frac{1}{2}(a_1^2 - a_2^2)a_{13} \\
+ \frac{1}{2}(a_1^2 - a_2^2)a_{13}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
= 0 .
\end{equation}
### TABLE 1

**Characteristic Frequencies of Synchronous Oscillations of the Darwin Ellipsoids**

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$R$</th>
<th>$\Omega^2 / G \rho$</th>
<th>$d_1$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$\sigma_1^2$</th>
<th>$\sigma_2^2$</th>
<th>$\sigma_3^2$</th>
<th>$\sigma_4^2$</th>
<th>$\sigma_5^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5°</td>
<td>4.4572</td>
<td>0.03023</td>
<td>1.0416</td>
<td>0.9933</td>
<td>0.9665</td>
<td>2.1398</td>
<td>1.1388</td>
<td>0.85091</td>
<td>-0.02512</td>
<td>0.01665</td>
</tr>
<tr>
<td>10°</td>
<td>3.7457</td>
<td>0.05125</td>
<td>1.0767</td>
<td>0.9858</td>
<td>0.9421</td>
<td>2.1191</td>
<td>1.1899</td>
<td>0.71887</td>
<td>-0.04659</td>
<td>0.01968</td>
</tr>
<tr>
<td>15°</td>
<td>3.4667</td>
<td>0.06302</td>
<td>1.1039</td>
<td>0.9790</td>
<td>0.9253</td>
<td>2.0950</td>
<td>1.2220</td>
<td>0.63782</td>
<td>-0.06316</td>
<td>0.00546</td>
</tr>
<tr>
<td>10°08458°*</td>
<td>3.4569</td>
<td>0.06559</td>
<td>1.1051</td>
<td>0.9787</td>
<td>0.9246</td>
<td>2.0938</td>
<td>1.2233</td>
<td>0.63453</td>
<td>-0.06390</td>
<td>-0.06390</td>
</tr>
<tr>
<td>10°11°</td>
<td>3.4551</td>
<td>0.06570</td>
<td>1.1053</td>
<td>0.9786</td>
<td>0.9245</td>
<td>2.0936</td>
<td>1.2236</td>
<td>0.63394</td>
<td>-0.06636</td>
<td>-0.06171</td>
</tr>
<tr>
<td>10°12°</td>
<td>3.4438</td>
<td>0.06637</td>
<td>1.1068</td>
<td>0.9782</td>
<td>0.9236</td>
<td>2.0923</td>
<td>1.2251</td>
<td>0.63008</td>
<td>-0.07127</td>
<td>-0.05856</td>
</tr>
<tr>
<td>12°</td>
<td>3.2682</td>
<td>0.07817</td>
<td>1.1343</td>
<td>0.9707</td>
<td>0.9082</td>
<td>2.0639</td>
<td>1.2537</td>
<td>0.56315</td>
<td>-0.11372</td>
<td>-0.04910</td>
</tr>
<tr>
<td>12°5°</td>
<td>3.2276</td>
<td>0.08132</td>
<td>1.1424</td>
<td>0.9683</td>
<td>0.9040</td>
<td>2.0551</td>
<td>1.2614</td>
<td>0.54548</td>
<td>-0.12443</td>
<td>-0.04798</td>
</tr>
<tr>
<td>15°</td>
<td>3.0657</td>
<td>0.09003</td>
<td>1.1867</td>
<td>0.9550</td>
<td>0.8924</td>
<td>2.0067</td>
<td>1.2968</td>
<td>0.46278</td>
<td>-0.18013</td>
<td>-0.04241</td>
</tr>
<tr>
<td>16°</td>
<td>3.0166</td>
<td>0.10134</td>
<td>1.2062</td>
<td>0.9488</td>
<td>0.8738</td>
<td>1.9858</td>
<td>1.3092</td>
<td>0.43211</td>
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</tr>
<tr>
<td>18°</td>
<td>2.9398</td>
<td>0.11082</td>
<td>1.2485</td>
<td>0.9352</td>
<td>0.8564</td>
<td>1.9430</td>
<td>1.3297</td>
<td>0.37424</td>
<td>-0.25155</td>
<td>-0.03384</td>
</tr>
<tr>
<td>20°</td>
<td>2.8874</td>
<td>0.11860</td>
<td>1.2959</td>
<td>0.9197</td>
<td>0.8391</td>
<td>1.9016</td>
<td>1.3423</td>
<td>0.32008</td>
<td>-0.30060</td>
<td>-0.02644</td>
</tr>
<tr>
<td>22°</td>
<td>2.8559</td>
<td>0.12449</td>
<td>1.3489</td>
<td>0.9023</td>
<td>0.8216</td>
<td>1.8655</td>
<td>1.3435</td>
<td>0.26848</td>
<td>-0.34937</td>
<td>-0.01751</td>
</tr>
<tr>
<td>24°</td>
<td>2.8433</td>
<td>0.12836</td>
<td>1.4085</td>
<td>0.8831</td>
<td>0.8040</td>
<td>1.8389</td>
<td>1.3300</td>
<td>0.21826</td>
<td>-0.39665</td>
<td>-0.00669</td>
</tr>
<tr>
<td>24°4035°†</td>
<td>2.8429</td>
<td>0.12888</td>
<td>1.4214</td>
<td>0.8790</td>
<td>0.8004</td>
<td>1.8004</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24°4065°‡</td>
<td>2.8429</td>
<td>0.12889</td>
<td>1.4215</td>
<td>0.8790</td>
<td>0.8004</td>
<td>1.8004</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25°0434§</td>
<td>2.8429</td>
<td>0.12900</td>
<td>1.4245</td>
<td>0.8780</td>
<td>0.7995</td>
<td>1.8340</td>
<td>1.3240</td>
<td>0.20574</td>
<td>-0.40809</td>
<td>-0.00360</td>
</tr>
</tbody>
</table>

* The point of onset of overstability.

† The Roche limit.

‡ The ellipsoids are in contact.

§ A neutral mode occurs at this point.
A numerical solution of this equation yields

$$R = 2.8438, \quad \Omega^2 = 0.12954, \quad a_2/a_1 = 0.60479, \quad a_3/a_1 = 0.55093,$$

$$\tilde{a}_1 = 1.4425, \quad \tilde{a}_2 = 0.87238, \quad \text{and} \quad \tilde{a}_3 = 0.79468.$$  

(75)

Comparing the results given in equations (18) and (75), we observe that, as in the case of the Roche sequences, the Roche limit occurs at a point slightly earlier along the sequence than does the neutral point. However, in this instance the neutral point occurs subsequent to contact, i.e., in the nonphysical branch of the solutions. Also, in contrast to the case of the Roche ellipsoids, the mode in question is unstable prior to neutrality.

### Table 2

<table>
<thead>
<tr>
<th>$\phi_0$</th>
<th>$V_{11}/ V_1$</th>
<th>$V_{12}/ V_1$</th>
<th>$V_{13}/ V_1$</th>
<th>$V_{14}/ V_1$</th>
<th>$V_{15}/ V_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^\circ 1$</td>
<td>0.0704</td>
<td>-0.0329</td>
<td>-0.0199</td>
<td>-0.0838</td>
<td>-0.7541</td>
</tr>
<tr>
<td>$10^\circ 2$</td>
<td>0.0709</td>
<td>-0.0331</td>
<td>-0.0200</td>
<td>-0.0875</td>
<td>-0.9002</td>
</tr>
<tr>
<td>$12^\circ$</td>
<td>0.0715</td>
<td>-0.0333</td>
<td>-0.0201</td>
<td>-0.0825</td>
<td>-0.6382</td>
</tr>
<tr>
<td>$12^\circ 5$</td>
<td>0.0728</td>
<td>-0.0340</td>
<td>-0.0204</td>
<td>-0.0929</td>
<td>-0.1031</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>0.0976</td>
<td>-0.0430</td>
<td>-0.0249</td>
<td>-0.0976</td>
<td>-0.2073</td>
</tr>
<tr>
<td>$16^\circ$</td>
<td>0.1082</td>
<td>-0.0482</td>
<td>-0.0272</td>
<td>-0.1669</td>
<td>-0.1793</td>
</tr>
<tr>
<td>$18^\circ$</td>
<td>0.1059</td>
<td>-0.0459</td>
<td>-0.0263</td>
<td>-0.1036</td>
<td>-0.1699</td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>0.1201</td>
<td>-0.0527</td>
<td>-0.0293</td>
<td>-0.1921</td>
<td>-0.1948</td>
</tr>
<tr>
<td>$22^\circ$</td>
<td>0.1536</td>
<td>-0.0602</td>
<td>-0.0335</td>
<td>-0.1368</td>
<td>-0.3284</td>
</tr>
<tr>
<td>$24^\circ$</td>
<td>0.2008</td>
<td>-0.0806</td>
<td>-0.0422</td>
<td>-0.3786</td>
<td>-0.2832</td>
</tr>
<tr>
<td>$24^\circ 5$</td>
<td>0.1757</td>
<td>-0.0658</td>
<td>-0.0365</td>
<td>-0.1515</td>
<td>-0.0622</td>
</tr>
<tr>
<td>$25^\circ$</td>
<td>0.2473</td>
<td>-0.0951</td>
<td>-0.0491</td>
<td>-0.4970</td>
<td>-0.3300</td>
</tr>
<tr>
<td>$26^\circ$</td>
<td>0.2252</td>
<td>-0.0760</td>
<td>-0.0423</td>
<td>-0.1828</td>
<td>+ 0.2814</td>
</tr>
<tr>
<td>$28^\circ$</td>
<td>0.3837</td>
<td>-0.1339</td>
<td>-0.0682</td>
<td>-0.8833</td>
<td>-46.92</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>0.2820</td>
<td>-0.0845</td>
<td>-0.0479</td>
<td>-0.2163</td>
<td>+ 0.4837</td>
</tr>
<tr>
<td>$32^\circ$</td>
<td>0.6347</td>
<td>-0.1981</td>
<td>-0.1012</td>
<td>-1.719</td>
<td>-74.06</td>
</tr>
<tr>
<td>$34^\circ$</td>
<td>0.3465</td>
<td>-0.0911</td>
<td>-0.0530</td>
<td>-0.2518</td>
<td>+ 0.6129</td>
</tr>
<tr>
<td>$36^\circ$</td>
<td>1.2239</td>
<td>-0.3265</td>
<td>-0.1751</td>
<td>-4.181</td>
<td>-148.3</td>
</tr>
<tr>
<td>$38^\circ$</td>
<td>0.4196</td>
<td>-0.0955</td>
<td>-0.0576</td>
<td>-0.2894</td>
<td>+ 0.7038</td>
</tr>
<tr>
<td>$40^\circ$</td>
<td>0.0841</td>
<td>-0.9740</td>
<td>-0.5234</td>
<td>-22.89</td>
<td>-673.1</td>
</tr>
<tr>
<td>$44^\circ$</td>
<td>0.4392</td>
<td>-0.0962</td>
<td>-0.0586</td>
<td>-0.2992</td>
<td>+ 0.7230</td>
</tr>
<tr>
<td>$46^\circ$</td>
<td>0.0860</td>
<td>-1.8567</td>
<td>-1.0077</td>
<td>-61.89</td>
<td>-1736</td>
</tr>
</tbody>
</table>

*For each value of $\phi_0$, the entries along the first row belong to $\sigma^4$ while those along the second row belong to $\sigma^5$ (see Table 1).

Finally, in Table 2 the proper solutions belonging to the two unstable modes are listed.

In conclusion, I wish to record my indebtedness to Professor Maurice J. Clement of the University of Toronto for subjecting the analysis of this paper to careful scrutiny and for correcting an earlier serious oversight. I am also grateful to Dr. Morris Aizenman for providing solutions for the Darwin figures more extensive than those given in Paper I; and to Miss Donna Elbert for her assistance with the numerical work.

The research reported in this paper has in part been supported by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago.

* Contact occurs where

$$R = 2.8429, \quad \Omega^2 = 0.12889, \quad a_2/a_1 = 0.61836, \quad a_3/a_1 = 0.56306,$$

$$\tilde{a}_1 = 1.4215, \quad \tilde{a}_2 = 0.87897, \quad \text{and} \quad \tilde{a}_3 = 0.80037.$$
APPENDIX

AN ALTERNATIVE METHOD FOR LOCATING THE ROCHE LIMIT

We shall briefly indicate how the "Roche limit," determined in § II directly from the equation governing equilibrium, can also be determined with the aid of the virial equations (54)–(57), (72), and (73). Thus setting $\lambda = 0$ in equations (54)–(57), as is appropriate for quasi-static deformations, we find that these equations are identically satisfied if

$$\delta \omega_{11} + \delta \omega_{22} + 2\delta \omega_{33} + \beta_{11}V_{11} + \beta_{22}V_{22} - 2\beta_{33}V_{33}$$

$$+ \delta \beta_{11}I_{11} + \delta \beta_{22}I_{22} - 2\delta \beta_{33}I_{33} = 0 \quad (A1)$$

and

$$\delta \omega_{11} - \delta \omega_{22} + \beta_{11}V_{11} - \beta_{22}V_{22} + \delta \beta_{11}I_{11} - \delta \beta_{22}I_{22} = 0 \quad (A2)$$

where the last of these conditions implies that $\delta \beta_{12} = 0$ (cf. eq [67]). Under these same circumstances it follows from equations (72) and (73) that

$$V_{12} = 0 \quad (A3)$$

in agreement with the assumption made in § II that the centers of the ellipsoids do not suffer any displacements.

With the aid of the known expressions for $\delta \omega_{11} + \delta \omega_{22} - 2\delta \omega_{33}$ and $\delta \omega_{11} - \delta \omega_{22}$ and equations (62) and (66) (with $V_{1}$ set equal to zero in the last group of equations), equations (A1) and (A2) become a pair of linear homogeneous equations for $V_{1}$, $V_{2}$, and $V_{3}$; and these two equations, together with the solenoidal condition (58), lead to the following determinantal equation for locating the desired point:

$$\begin{vmatrix}
\delta \omega_{11} + \delta \omega_{22} - 2\delta \omega_{33} + \beta_{11}V_{11} + \beta_{22}V_{22} - 2\beta_{33}V_{33} \\
+ \delta \beta_{11}I_{11} + \delta \beta_{22}I_{22} - 2\delta \beta_{33}I_{33} \\
\delta \omega_{11} - \delta \omega_{22} + \beta_{11}V_{11} - \beta_{22}V_{22} + \delta \beta_{11}I_{11} - \delta \beta_{22}I_{22}
\end{vmatrix} = 0$$

$$V_{1} = V_{2} = 0 \quad (A4)$$
It is found that equation (A5) determines the same point as equation (18).

REFERENCES

———. 1964, ibid., 140, 599.
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