INVERSION OF THE LIMB-DARKENING EQUATION
USING THE PRONY ALGORITHM

O. R. WHITE
University of Hawaii

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ABSTRACT

King's interesting application of the Prony algorithm to the inversion of the limb-darkening equation is modified and generalized so that it can be applied rigorously to limb-darkening curves, spectral line profiles, and multiplet line intensities. Since King's formulation requires physically inaccessible observations, a change of variable is introduced in the limb-darkening integral to avoid this difficulty. Calculations on noisy data confirm the self-limiting property of this inversion method. An approximate method for constructing the smooth S(r) distribution from the slab solution is presented.

I. INTRODUCTION

King (1964) applies the Prony algorithm (Whittaker and Robinson 1932) to a quadrature approximation to the emergent intensity integral in order to estimate the depth variation of the source function from intensity measurements of a stellar or planetary atmosphere. In this method the atmosphere is visualized as a series of slabs, each of constant source function but of variable thickness. Since the Prony algorithm is relatively simple and free from constraint on the form of the source function, King's application of it is of interest in the inversion problem often encountered in the analysis of astronomical intensity data. Of even greater interest, however, is the remarkable property (King 1965b) that the number of physically possible slabs in the solution is found empirically to decrease with the accuracy of the observations, and that the weights of the physically inadmissible solutions are small. This self-limiting property of King's inversion scheme in the presence of observational noise sets it apart from other, more common, inversion methods.

The limb-darkening integral can be readily integrated for the slab model to give the following quadrature approximation:

\[ I_0(\mu) = \int_0^\infty S(\tau_0) \exp(-\tau_0/\mu) d\tau_0/\mu \approx S_0 + \sum_k \Delta S_k \exp(-\tau_0/\mu), \] 

where \( S_0 \) is the boundary value of the source function, \( \Delta S_k \) is the source function increment between slabs \( k - 1 \) and \( k \), \( \tau_0 \) is the optical depth of the boundary between slabs \( k - 1 \) and \( k \), and \( \mu \) is the usual limb-darkening variable \( \cos \theta \).

Given a set of \( 2m \) observations \( I_0(\mu_j) \) at \( 2m \) positions \( \mu_j \), the Prony algorithm allows solution of the \( 2m \) non-linear simultaneous equations (1) for the \( m \) weights \( \Delta S_k \) and the \( m \) slab-boundary variables \( \exp(-\tau_0) \). However, the boundary value of the source function \( S_0 \) must be known a priori. In the next section, a generalized form of equation (1) will be developed for using the Prony solution in spectral line analysis.

II. GENERALIZATION OF THE THEORY

Of the several topics to be discussed here, the first is to establish the correspondence between equation (1) and the more familiar set of equations found in Gaussian quadrature problems:

\[ a_l = \sum_{k=1}^{m} a_k x_k^l, \quad l = 0, 1, 2, \ldots, 2m - 1 \]
In this non-linear approach the parameters \( a_k \) and \( x_\mu \) are derived from measurements of \( a_\ell \) at the specified values of \( \ell \) by means of the Prony algorithm. The essential restriction is that the observations be equally spaced in the variable \( \ell \). The quadrature form for the emergent intensity integral (1) takes the same form as equation (2) when we write

\[
I_i(\mu_j) - S_0 = \sum_{k=1}^{m} [\Delta S_k \exp(-\tau_{ik}/\mu_0)] [\exp(-\tau_{ik}/A)] A(1/\mu_j - 1/\mu_0),
\]

where the subscript \( i \) refers to the frequency \( \nu_i \). The quantity \( \mu_0 \) is the maximum value of cosine \( \theta \) for which we have observations, and \( A \) is a scaling parameter determined by the minimum \( \mu \) and the desired number of slabs. The reasons for introducing these parameters will be discussed later. If we choose \( \mu_j \) such that

\[
A(1/\mu_j - 1/\mu_0) = l = 0,1,2, \ldots, 2m - 1
\]

for a given \( A \) and \( \mu_0 \), the correspondence between equations (2) and (3) is complete.

Up to this point only the \( \mu \) variation of \( I_\ell \) has been considered, but in theory the shape of either a line profile or a limb-darkening curve gives some information on the depth variation of the source function. Therefore, a generalized inversion scheme should also be applicable to line profiles where the frequency in the line plays a role conceptually equivalent to \( \theta \) in the limb-darkening case. In order to include frequency variables, equation (3) is rewritten in terms of a reference optical depth \( \tau_\ell \) at frequency \( \nu_\ell \), i.e.,

\[
I_i(\mu_j) - S_0 = \sum_{k=1}^{m} [\Delta S_k \exp(-\tau_{ik}/\mu_0)] [\exp(-\tau_{ik}/A)] A(\tau_{ik}/\mu_0 - 1/\mu_0).
\]

The sum (4) appears now in the form required by the Prony algorithm if we define

Observable: \( a_i = I_i(\mu_j)|_l - S_0 \),

Weights: \( a_k = \Delta S_k \exp(-\tau_{ik}/\mu_0) \),

Division variable: \( x_k = \exp(-\tau_{ik}/A) \),

Quantizing condition: \( l = A(\tau_{ik}/\mu_0 - 1/\mu_0) = 0,1,2, \ldots, 2m - 1 \).

Given \( S_0 \), \( A \), \( \mu_0 \), and the number of slabs \( m \), the quantizing condition (8) specifies the positions \( \mu_j \) or frequencies \( \nu_j \) at which \( I_i(\mu_j) \) must be measured. The Prony solution then yields \( a_k \) and \( x_k \) from which \( \Delta S_k \) and \( \tau_{ik} \) can be obtained for constructing the slab model of \( S(\tau_\ell) \).

Since the algorithm requires that \( l \) be independent of \( k \) or, equivalently, independent of the reference optical depth \( \tau_\ell \), the optical depth ratio \( (\tau_{ik}/\tau_{rk}) \) appearing in the quantizing condition (8) must likewise be constant with depth. Such depth independence is possible only in cases where either the observed and reference frequencies are the same, or the frequency and depth dependences of the absorption coefficient are separable, i.e., when \( \tau_{ik} \) can be written in the form \( \phi_i \tau_{rk} \). The frequency function \( \phi_i \) is defined as the absorption coefficient ratio \( k_i/k_0 \), where \( k_0 \) is commonly taken as the absorption coefficient at the center of the spectral line, at the band head of a molecular series, or at the center of the most opaque member of a series of multiplet lines. When the frequency dependence of the total absorption coefficient is thus known, the general quantizing condition (8) reduces to the practical form

\[
l = \frac{A}{\mu_0} \left( \frac{\mu_0 \phi_i}{\mu_j \phi_r} - 1 \right) = 0,1,2, \ldots, 2m - 1.
\]
King (1964, 1965a, b) discusses both frequency and cosine θ quantization, but he uses a quantizing condition of the form

\[
\frac{\phi_i}{\phi_j} = l = 0, 1, 2, \ldots, 2m - 1.
\]  

(10)

This equation is equivalent to the new formula (9) with \( A \) set to unity and \( 1/\mu_0 \) set to zero. As Twomey (1965) points out, the condition \( 1/\mu = 0 \) is physically impossible, since by definition \( \mu \) is equal to or less than unity. The requirement that the absorption coefficient ratio \( \phi_i/\phi_r \) equal zero presents a similar problem, because it is not possible to specify unambiguously the frequency at which the absorption coefficient is exactly zero. The formulation presented here avoids the physically inadmissible condition \( \phi_i/\mu_0 \phi_r = 0 \) by a change of variable in equation (1) that leads to the quadrature formula (3). Dr. G. D. Finn has suggested an alternate procedure in which the discrete variable \( l \) starts at unity instead of at zero, but one can show that this procedure is formally equivalent to the change of variable used here.

Twomey (1965) also emphasizes that King's quantizing condition is not practical because the necessary \( \mu \) values cluster very near the limb where observations tend to be very uncertain. To avoid this difficulty for reasonable numbers of slabs, the scaling parameter \( A \) has been introduced in equation (3). Dr. J. I. F. King suggested an alternate form of the equations that shows best the role of \( A \). Let \( \rho_0 = 1/\mu_0 \) and \( \Delta \rho = 1/A \) in the quadrature formula (4), and we obtain

\[
I_i(\mu_j) - S_0 = \sum_{k=1}^{m} \Delta S_k \exp\left[-\tau_k(\rho_0 + \Delta \rho l)\right] \quad (l = 0, 1, 2, \ldots, 2m - 1).
\]  

(11)

From this we see that \( A \) is the reciprocal increment in the variable \( 1/\mu \), and that \( 1/\mu_0 \) is the starting value of this variable. As such, the value of \( A \) can be construed as determining the depth of focus, and that of \( \mu_0 \) as giving the depth of field of the Prony solution. Given the number of slabs desired and the minimum and maximum values of cosine \( \theta \), the most efficient value of \( A \) is given by the quantizing condition (9) with the maximum value of \( l \), i.e.,

\[
\Delta \rho = 1/A = (\phi_i/\phi_r \mu_{\text{min}} - 1/\mu_0)/(2m - 1).
\]  

(12)

III. INFERENCE OF THE SOURCE FUNCTION FROM THREE TYPES OF OBSERVATION

Consider separately the form of the quantizing condition (9) appropriate to the analyses of (a) limb-darkening curves; (b) single line profiles; and (c) intensities in multiplet lines.

a) Limb-darkening Curves

In the limb-darkening case where the frequency is fixed and equal to the reference frequency \( \phi_r = \phi_i \), the quantizing condition (9) reduces to

\[
\mu_j = A \mu_0/(A + \mu_0) \quad (l = 0, 1, 2, \ldots, 2m - 1).
\]  

(13)

For non-zero \( A \) values it is clear that \( \mu_j \) lies within its proper range of zero to one for all values of \( l \). Regardless of the precision of the intensity measurements, equation (13) shows that the maximum number of slabs allowed in the solution is fixed by the smallest value of \( \mu \) observationally attainable. In order to allow a reasonable number of slabs in solutions from data limited in \( \mu \), the scaling constant \( A \) has been introduced to change the distribution of \( \mu_j \).

When \( A = 1 \) and the limb-darkening measurements lie in the readily observable range \( 1 \geq \mu \geq 0.2 \), no more than two slabs will be permitted in the solution, even
though the photometric accuracy may justify four or five slabs. However, as the parameter $A$ is increased, the allowed $\mu_j$ values become finer spaced in the more accessible region $\mu \geq 0.2$; and the number of possible slabs thus increases. For example, when $A = 2$, a four-slab model is at least possible for observations lying in the range $1 \geq \mu \geq 0.222$. Obviously, $A$ cannot be increased indefinitely so as to better resolve the source function variation with depth, because the observational errors limit the amount of recoverable information.

The new quantizing formula (13) has been applied to the same example used by King (1964) in which he inverted synthetic limb-darkening curves computed from a source function of the form $S(\tau) = 1 - \exp(-\tau)$. Figure 1 shows King’s original three-slab model based on the quantizing condition (13) with $A = 1$ and $\mu_0 \to \infty$ as compared to the more realistic representation with $A = 1$ and $\mu_0 = 1$. It is of some interest to note that the third step in King’s solution is equal to the asymptotic value of the source function at great depths. Physically, one would not expect to infer this asymptotic value because the very deep parts of a real atmosphere do not contribute significantly to the emergent intensity. Judging from this, the selection of $\mu_j$ values with $\mu_j < 1$ appears to be more correct, simply because it yields source function values lying intermediate to the boundary and asymptotic limits. Of course, these difficulties with King’s solution stem from the physically inadmissible condition $1/\mu_0 = 0$.

Calculations have also been made to show the effects of varying $A$, $\mu_0$, and $m$ on the final solution for $S(\tau)$. Figure 2 shows the slab solutions for $m = 5$ and two values of $A$ and $\mu_0$. As $A$ increases, the slab boundaries move deeper into the atmosphere. Likewise, as $\mu_0$ decreases, the boundaries fall nearer the atmospheric surface. As a rule the value taken halfway up each step face lies within 10 per cent of the true source function at the slab boundary. This half-step estimation rule for $S(\tau)$ and solutions for several values $A$, $\mu_0$, and $m$ allow one to obtain approximate $S(\tau)$ values at many depths and, thus, to obtain some measure of the smooth $S(\tau)$ distribution.

The effect of noise on the solutions has been investigated by adding a normally distributed perturbation to the $I(\mu)$ values computed from the source function, $1 - \exp(-\tau)$. For given noise amplitude $\sigma$ and sampling parameters $A$ and $\mu_0$, the Prony solutions were obtained for $1, 2, 3, \ldots, 8$ slabs. Solutions for up to eight slabs are numerically stable for double precision (fourteen reliable significant figures) calculations on an IBM 360 model 50 computer. The calculations on noisy data do show the self-limiting proper-
ty reported by King (1965b). The number of physically realistic slab solutions is given
in Table 1 for various values of the noise parameter \( f \), i.e., \( \sigma = f I(\mu = 1) \).

Note that in computing Table 1, values of \( m \) up to 8 have been used; therefore, in
many cases we have asked for far more slabs than the accuracy of the data justifies.
In such cases, the inadmissible solutions give negative or complex boundary depths as
well as pairs of repeated values for the boundary variable \( s_k = \exp(-\tau_k/A) \). Those
solutions with negative or complex boundary positions usually have small weights or
small \( \Delta S_k \) values, in which case they do not contribute significantly to the final step
function. However, in the case of the repeated boundaries the associated weights are
equal in absolute value but of opposite algebraic sign, so that they cancel each other in
the final solution.

![Slab models computed for ten observed \( \mu \) positions and four sets of parameters: (a) \( A = 1, \mu_0 = 0.5 \); (b) \( A = 1, \mu_0 = 1 \); (c) \( A = 2, \mu_0 = 0.5 \); (d) \( A = 2, \mu_0 = 1 \).](image)

**Fig. 2.**—Slab models computed for ten observed \( \mu \) positions and four sets of parameters: (a) \( A = 1, \mu_0 = 0.5 \); (b) \( A = 1, \mu_0 = 1 \); (c) \( A = 2, \mu_0 = 0.5 \); (d) \( A = 2, \mu_0 = 1 \).

<table>
<thead>
<tr>
<th>( f )</th>
<th>No of Slabs</th>
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<tr>
<td>0 1</td>
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<tr>
<td>0 01</td>
<td>2</td>
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<tr>
<td>0 00001</td>
<td>4</td>
</tr>
</tbody>
</table>

**TABLE 1**

**DEPENDENCE OF NUMBER OF REALISTIC SLAB SOLUTIONS ON THE RMS DEVIATION**
b) Single Line Profiles

Since it is often much easier to obtain observations of a single line profile in a fixed direction $\mu$ than to determine the limb-darkening curve at fixed frequency, we shall cast the quantizing condition (9) in the form necessary to infer $S(r)$ from the line profile alone. However, in attempting to analyze a single spectral line in this manner, it must be assumed that the source function is frequency independent, that the frequency dependence ($\phi$) of the absorption coefficient is known, and further that $\phi$ does not change with depth. Under these conditions the Prony algorithm may be applied through the quantizing condition

$$\frac{\phi_i}{\phi_r} = \frac{\mu(A + \mu \mu_l)}{\mu_0 A} \quad (l = 0, 1, 2, \ldots, 2m - 1). \quad (14)$$

Since the frequency function $\phi$ is assumed to be known, the condition (14) then unambiguously specifies the frequencies $\nu_i$ at which intensities must be measured once the reference frequency $\nu_r$ is chosen. Physically, the quantizing condition (14) implies observations equally spaced in the relative absorption coefficient difference $(\phi_i - \phi_r)/\phi_r$. The scaling parameter $A$ now determines the distribution of the observed frequencies over the spectral line.

c) Multiplet Lines

The preceding two sections define the slab model inversion for observations in which the data appear as continuous functions of either frequency or $\mu$. Another important class of problems arises in determination of the depth dependence of the source function from observations of the central intensities of multiplet lines having a common source function and arising from a common lower level. In such cases, the optical depths in the line centers are proportional to the $g_f/\lambda_o$ values of the lines. Since the absorption coefficient ratio at observable frequencies is not a continuous variable, the quantizing condition must now be used in its general form

$$\frac{\phi_i}{\phi_r} = \frac{A + \mu \mu_l}{\mu_0 A} \quad (l = 0, 1, 2, \ldots, 2m - 1), \quad (15)$$

where $\phi_i/\phi_r$ refers to the opacity ratio at the centers of two lines in the series, i.e.,

$$\phi_i/\phi_r = (\lambda g_f)_i/(\lambda g_f)_r.$$ 

Then, for a given set of multiplet lines, $\mu_l$ must be adjusted until condition (15) is satisfied for given $\mu_0$ and $A$ values. Observationally, this means that the central intensities for the different lines must be measured at different positions on the disk. This type of multiplet analysis is also applicable to frequencies other than the line centers, provided the frequencies are chosen so as to preserve the absorption coefficient ratio given by the $f$-values.

IV. CONCLUSION

King's (1964) application of the Prony algorithm to the inversion of the limb-darkening equation has been modified so as to make it rigorously as well as practically applicable to real observations. In addition, the theory is generalized to include inversion from limb-darkening curves, single line profiles, or multiplet line intensities. In order to avoid the physically inaccessible observations in King's formulation, it is necessary to introduce new limb-darkening and frequency variables—$(\mu_0 - \mu)/\mu$ and $(\phi_i - \phi_r)/\phi_r$—that differ conceptually from the standard $1/\mu$ and $\phi_r$ variables of observation. These new variables can be interpreted as relative optical depth differences, whereas, the more common $1/\mu$ and $\phi_r$ correspond to optical depth ratios. Zero-point and scaling parameters...
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$(\mu_0$ and $A)$ have been introduced so as to allow better solutions from data limited in $\mu$. The slab solutions with different values of $\mu_0$ and $A$ can be combined to estimate the smooth $S(\tau)$ distribution for a single set of observed intensities.

Calculations on noisy data exhibit the self-limiting property reported by King (1965b). For solar limb-darkening data with a constant rms error of 0.01 $I(\mu = 1)$ we can expect at most two or three slab solutions. It is hoped that the limiting effect of noise can be reduced by using smoothing functions on the data.

I am indebted to Dr. J. T. Jefferies both for directing me to King's valuable work on the Prony algorithm and for his constructive criticism of the content of this paper. I also thank Drs. J. I. F. King and G. D. Finn for useful discussions of the Prony algorithm. This research has been supported in part by contract AF 19(628)5086 for co-operative solar research between Sacramento Peak Observatory and the University of Hawaii.

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———. 1965a, ibid., 22, 96.