Distribution of Neighbors around a Star

Tsutomu SHIMIZU

Department of Astronomy, University of Kyoto, Kyoto
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Abstract

Under the assumption that the stars are distributed at random in space, we derived the theoretical distribution functions of the 1-st, the 2-nd, ... , the 10-th neighbors, and compared the first four ones with the actual distributions for the corresponding orders of the neighbors among the near-by stars. It resulted that the space distribution of the near-by stars appeared likely to be random so long as a binary or multiple star was counted as one star.

Within the near-by binary or multiple stars, however, the total number of the companions included within a sphere of radius \( r \) was nearly proportional to \( \log r \), similar to the planetary system and the satellite ones, but beyond a certain distance the increasing rate of the number decreases gradually. In order to explain such a space distribution, we postulated that existing companions could never have escaped from the primary. If so, the statistical data free from biased selections provide information on the mean random velocity and/or mean systematic one in the parent cloud from which such a physical system was supposedly born. Tentative estimations showed that the assumed motion of a few km/sec might explain the observed distribution in the planetary, the satellite, and the binary systems.

In forthcoming paper we are going to study the random force, on the ground that the stellar distribution in space is quite at random. In this connection this \textit{a priori} ground is previously tested here from the observational side.

1. Distribution of the \( k \)-th Nearest Neighbor

The distribution function of the nearest neighbor in space where particles are scattered randomly in space was originally given by HERTS (1907) and has been used in estimating the mean molecular distance (GANS, 1922, RAMAN, 1924) or in evaluating the random force due to the nearest neighbor (CHANDRASEKHAR, 1943). The notion of the nearest neighbor, on the other hand, may also be applicable to a test of randomness in the stellar distribution in space. But for this purpose it is desirable, in addition to considering the 1-st nearest neighbor, to make use of the 2-nd, the 3-rd, ... ones further. Then we derive here these distribution functions.

We assume a particle to be fixed at the origin or the center of a sphere with an assigned radius \( a \) and \((N-1)\) particles to be scattered quite at random within the sphere, and let \( W_k(r) \) be the probability that the \( k \)-th nearest neighbor lies within a distance \( r \) from the origin. Then, since the probability that the \( k \)-th neighbor occurs at \( r \sim r + dr \) or \( dW_k(r) \) is given by a product of...
three probabilities, such as that the \((k-1)\)-th neighbor is within \(r\) or \(W_{k-1}(r)\),
that the \(k\)-th neighbor does not exists within \(r\) or \((1-W_k(r))\) and that the \(k\)-th neighbor lies in a distance interval \(r \sim r+dr\) or \(N_1 r^3 dr/(4\pi/3(a^3-r^3))\), we obtain
\[
dW_k(r) = W_{k-1}(r) [1-W_k(r)] 3N r^2 dr/(a^3-r^3), \quad W_0(r)=1 \quad (k=1, 2, \cdots),
\]
or simply
\[
dW_k(\xi) = W_{k-1}(\xi) [1-W_k(\xi)] N d\xi/(V-\xi), \quad W_0(\xi)=1, \quad \xi \equiv r^3, \quad V \equiv a^3; \quad (k=1, 2, \cdots)
\]
In view of \(W_0(0)=0\), integration follows at once
\[
W_k(\xi) = 1 - \exp \left[ - \int_0^\xi W_{k-1}(\xi') N d\xi'/(V-\xi') \right], \quad W_0(\xi)=1, \quad (k=1, 2, \cdots) \tag{1.2}
\]
This formula expresses \(W_k(\xi), W_{k\pm}(\xi), \cdots\) in succession, with \(W_k(\xi)=1\). But even if we wish explicit expressions for making numerical evaluation by means of some accessible functional tables, we can hardly derive them except for those only up to the 4-th neighbor. With regard to distribution functions for the remaining neighbors, we are forced to determine them by proceeding with numerical calculations.

The explicit expressions for the first four neighbors are readily obtainable as follows**,
\[
\begin{align*}
W_1(\xi) &= 1 - \frac{(1-\xi/V)^3}{(1-\xi/V)} \\
W_2(\xi) &= 1 - \frac{(1-\xi/V)^3}{(1-\xi/V)} \exp \left[1-(1-\xi/V)^3\right] \\
W_3(\xi) &= 1 - \frac{(1-\xi/V)^3}{(1-\xi/V)} \exp \left[-1 + \exp\left(1-(1-\xi/V)^3\right)\right] \\
W_4(\xi) &= 1 - \frac{(1-\xi/V)^3}{(1-\xi/V)} \exp \left[ \frac{E(\exp(1-(1-\xi/V)^3))}{e-E(1/e)} \right] \\
\end{align*}
\tag{1.3}
\]
In the limiting case where \(a\) or the boundary radius of the particle system tends to infinity by keeping a finite number density or
\[
n_0 = \lim_{a \to \infty} \frac{N}{[(4\pi/3)a^3]=(3/4\pi)} \lim_{V \to \infty} (N/V),
\]
formula (1.2), in virtue of \(\lim_{a \to \infty} \frac{N}{(a^3-r^3)} = \lim_{V \to \infty} N/V = (4\pi/3)n_0\), reduces to
\[
W_k(x) = 1 - \exp \left[ - \int_0^x W_{k-1}(x) dx \right], \quad W_0(x)=1, \quad x \equiv (4\pi n_0/3)r^3, \quad (k=1, 2, \cdots). \tag{1.4}
\]
With the help of (1.4) we can obtain similarly explicit forms of the distribution function up to the 4-th neighbor.
\[
\begin{align*}
W_1(x) &= 1 - e^{-x} \\
W_2(x) &= 1 - \exp(1-x-e^{-x}) \\
W_3(x) &= 1 - \exp(-1+x+\exp(1-e^{-x})) \\
W_4(x) &= 1 - \exp[-x+\{E(\exp(1-e^{-x})-E(1/e)] \tag{1.5}
\end{align*}
\]
Comparing (1.5) with (1.3), we can see that \((1-\xi/V)^3\) in (3) is replaced by \(e^{-x}\), and this clearly ought to be the case because of

* Here the volume occupied by all the particles has been neglected in comparison with \((a^3-r^3)\).
** The explicit expressions for the mean distance or the variance around the mean is easily derivable, at least for the 1-st and the 2-nd neighbors.
\[
\lim_{M \to \infty} (1 - \frac{\xi}{V})^y = \lim_{V \to \infty} (1 - \frac{\xi}{V})^{(V/(1 + \xi \cdot 1/3))} = e^{-\xi}.
\]

Hence, provided that the numerical integration have been carried out by means of formula (1.4) which is simpler than (1.2), the result is readily converted into a result for (1.2) by a mere transformation of the parameter from \(x\) to \((1 - \frac{\xi}{V})^y\). The numerical values of \(W_k(x)\) \((k=1, 2, \cdots, 10)\) are tabulated with the argument of \(x^{1/3}\) in Table 1, and are illustrated also in Figure 1.

2. Statistics on the Neighbors of the Near-by Stars

Let us examine here whether the space distribution of the near-by stars could be regarded as being at random or not by applying the theoretical formulae in the preceding section. With the numerical values given in Allen's table (1963), we have calculated the distances from each star to its neighboring stars and then obtained the distance distributions for the 1-st, the 2-nd, the 3-rd and the 4-th nearest neighbors respectively. In this section a composite star such

<table>
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<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
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<tbody>
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<td>0.0620</td>
<td>0.1943</td>
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<tr>
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<td>0.0234</td>
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<td>0.0063</td>
<td>0.0961</td>
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<td>0.0390</td>
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<td>0.0135</td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x^{1/3})</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9834</td>
<td>0.9971</td>
<td>0.9967</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.9555</td>
<td>0.9920</td>
<td>0.9991</td>
<td>0.9999</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.9113</td>
<td>0.9838</td>
<td>0.9981</td>
<td>0.9999</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.8448</td>
<td>0.9705</td>
<td>0.9966</td>
<td>0.9998</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.7509</td>
<td>0.9499</td>
<td>0.9941</td>
<td>0.9996</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.6824</td>
<td>0.9185</td>
<td>0.9902</td>
<td>0.9993</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.4884</td>
<td>0.8722</td>
<td>0.9843</td>
<td>0.9989</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.3359</td>
<td>0.8065</td>
<td>0.9751</td>
<td>0.9982</td>
<td>0.9999</td>
<td>1.0000</td>
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<td>9</td>
<td>0.2052</td>
<td>0.7177</td>
<td>0.9613</td>
<td>0.9972</td>
<td>0.9999</td>
<td>1.0000</td>
</tr>
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<td>10</td>
<td>0.1090</td>
<td>0.6055</td>
<td>0.9407</td>
<td>0.9956</td>
<td>0.9998</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Fig. 1. Theoretical distributions of the neighbors from the 1-st to the 10-th.

Fig. 2. Observed distributions of the neighbors (a=4.05 pc, b=5.26 pc).
as a binary and a multiple star was counted as a single star since inclusion of these components as individual stars into the whole materials brought about some peculiar aspects, to which other explanations from a quite different view-point seemed necessary as will be taken up in the next section.

The distributions of the respective neighbors were obtained in two cases: for Case I, the reference stars whose neighbors were in question were limited to within 4.05 pc from the sun (total number $N=23$) but their neighbors were looked for in a sphere of radius 5.25 pc around the sun, while for Case II all the stars within 5.25 pc were taken as reference stars ($N=45$) by selecting their neighbors out of the stars nearer than 6.9 pc from the sun.

![Graph showing distributions of neighbors](image)

**Fig. 3.** Observed distributions of the neighbors ($a=4.05$ pc, $b=5.25$ pc)

The resulting distributions are illustrated by step-lines in the diagrams of Figure 2 and Figure 3, respectively. In each diagram, in addition to the observed distribution a pair of the theoretical curves are drawn. The one with a full line was fitted to the corresponding observed distribution in the following way: From the observed distribution of the $k$-th ($k=1, 2, 3$ or 4) nearest neighbor or $W_k(r)$ we read the ordinates $r_1, r_2, \cdots, r_k$ for $W_k(r_1)=0.1$, $W_k(r_2)=0.2, \cdots, W_k(r_k)=0.9$ respectively, while from the corresponding theoretical distribution $W_k(x)$ in Figure 1 we read similarly the ordinates $x_1^{1,3}, x_2^{1,3}, \cdots, x_k^{1,3}$.
for \( W_s(x_1) = 0.1, \ W_s(x_2) = 0.2, \ldots, \ W_s(x_9) = 0.9 \). Since \( \frac{\sum_{i=1}^{9} W_s(x_i)}{\sum_{i=1}^{9} W_s(r_i)} \) gives an average value of the scale factor \( (4\pi n_0/3)^{1/3} \) for conversion from \( r \) to \( x^{1/3} \), the so-called fitting theoretical curve was obtained by applying this scale factor to \( W_s(x) \) in Table 1. The values of the scale factors as well as stellar number densities obtained from various observed distributions are given in Table 2.

### Table 2. The values of the scale factor and the stellar number density.

<table>
<thead>
<tr>
<th>Reference Stars</th>
<th>Neighbors</th>
<th>Nearest Neighbor</th>
<th>( (\frac{4}{3} \pi n_0)^{1/3} )</th>
<th>( n_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case I</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r &lt; 4.05 \text{ pc} ) (Fig. 2)</td>
<td>( r &lt; 5.26 \text{ pc} )</td>
<td>1-st</td>
<td>0.736</td>
<td>0.1061</td>
</tr>
<tr>
<td>( (N = 23) ) ( (N = 45) )</td>
<td>( (N = 45) ) ( (n_0 = 0.0826) ) ( (n_0 = 0.0737) )</td>
<td>2-nd</td>
<td>0.758</td>
<td>0.1040</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3-rd</td>
<td>0.661</td>
<td>0.0690</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4-th</td>
<td>0.611</td>
<td>0.0545</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td></td>
<td></td>
<td>0.698</td>
<td>0.0834</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.707</td>
<td>(0.0812)</td>
</tr>
<tr>
<td><strong>Case II</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r &lt; 5.26 \text{ pc} ) (Fig. 3)</td>
<td>( r &lt; 6.90 \text{ pc} )</td>
<td>1-st</td>
<td>0.689</td>
<td>0.0781</td>
</tr>
<tr>
<td>( (N = 45) ) ( (N = 36) )</td>
<td>( (N = 36) ) ( (n_0 = 0.0737) ) ( (n_0 = 0.0639) )</td>
<td>2-nd</td>
<td>0.606</td>
<td>0.0706</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3-rd</td>
<td>0.623</td>
<td>0.0578</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4-th</td>
<td>0.593</td>
<td>0.0498</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td></td>
<td></td>
<td>0.643</td>
<td>0.0641</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.645</td>
<td>(0.0635)</td>
</tr>
</tbody>
</table>

As regards the other theoretical curve indicated by a broken line, its scale factor for transforming \( W_s(x) \) in Table 1 was so chosen as to be identical with the mean value from all the different neighbor's scale factors, namely \( (4\pi n_0/3)^{1/3} = 0.698 \) for Case I, or \( (4\pi n_0/3)^{1/3} = 0.643 \) for Case II.

In looking at Figures 2 and 3 with reference to Table 2 we notice the following points: (i) The respective fitting theoretical curves represent fairly well the corresponding observed distributions, though the observed frequencies of smaller distances appear somewhat more numerous especially for the 1-st and the 2-nd neighbors. (ii) Deviations of the "mean" theoretical curves from the corresponding observed distributions are rather small except for the 4-th nearest neighbor in particular. (iii) The values of the stellar number density \( n_0 \) used for fitting the theoretical curves decrease with the order of the neighbors. Also, the mean of \( n_0 \)'s for Case I is nearly equal to the mean number density in the sphere including all the reference stars with a radius of 4.05 pc, whereas the mean \( n_0 \) for Case II is less than the mean number density from the corresponding reference stars and is almost equivalent to that derived from all stars including their neighbors within 6.9 pc from the sun.

Now, as to the possible causes of these peculiarities in the observed distributions of the neighbors, the following points may be mentioned: (i) the
boundary sphere of the neighbor has been taken to be too small, (ii) the percentage of stars missed from the observation increases with the distance from the sun, and (iii) the stellar number density is not uniform but has a tendency toward clustering. These will be discussed in succession.

A contribution from the first cause is to be expected because the theoretical distributions have assumed no boundary. To see the circumstance more in detail, we examine what fraction of a theoretical frequency $dW_z(r)$ is really observable. Taking the center at the sun, let $a$ and $b$ be the radius of the boundary sphere for the reference stars and that for the neighbors respectively. Then the boundary sphere for the neighbor cuts off a part of $dW_z(r)$, unless the distance between a reference star and the origin $D(\leq a)$ is less than $(b-r)$ or the radius $r$ is larger than $(a+b)$, in a different way depending on $r$ as well as $D$, so that from geometrical considerations the observable fraction of $dW_z(r)$ or $F(r)$ is readily written as

$$F(r) = \begin{cases} 1 & (\text{for } 0 \leq r \leq b-a), \\ \frac{1}{2} \left( 1 - \frac{3b}{r} \right) \left( \frac{r^3}{a^3} \right) + \frac{1}{16} \left( \frac{r^3}{a^3} - \frac{b^3r}{a^3} + \frac{b^3}{a^3} - \frac{b^3}{a^3r} \right) & (\text{for } b-a \leq r \leq b+a), \\ 0 & (\text{for } r \geq a+b), \end{cases}$$

(2.1)

![Fig. 4. Observed distributions of the neighbors (a=b=5.26 pc).](image)
which reduce, if $a=b$, into

$$F(r) = 1 - \frac{3}{4a} \frac{r}{a} + \frac{1}{16} \frac{r^3}{a^3} \quad \text{for} \ 0 \leq r \leq 2a$$

$$F(r) = 0 \quad \text{for} \ r \geq 2a$$

(2.1')

The reduction factor computed for Case I or Case II is shown in the lowest diagram of Figure 2 or Figure 3 with a dotted line. This may be why the observed distribution for the 4-th neighbor in particular deviates markedly from the mean theoretical curve.

In this connection we have provided another set of observed distributions for the respective neighbors in the special case of $a=b=5.26$ pc, as shown in Figure 4, where the reduction factor $F(r)$ in (2.1') is illustrated in the last diagram with a dotted line (compare Figure 4 with Figure 3). The pair of broken lines for each observed distribution indicate $W_k(r) = \int_0^r F(r) dW_k(r)$ ($k=1, 2, 3$ or $4$) resulting from numerical integrations with $N=45$ and $N=50$, respectively. These reduced theoretical curves, however, are not to be comparable directly with the corresponding observed distribution in which the $k'$-th neighbors may have been measured as the $k$-th ones among all the stars within a boundary radius of $b$. But the comparison between $\sum_{k=1}^{4} W_k(r)$ for the observed distributions and $\sum_{k=1}^{4} W_k(r)$

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Fig. 5. Observed distribution of the neighbors from the 1-st to the 4-th ($a=b=5.26$ pc).
for the reduced theoretical ones should be, so far as the space distribution is random, identical up to a certain distance beyond which the contribution of $\sum_{k=5} W_k(r)$ becomes appreciable. Figure 5 indicates such a comparison. It is seen there that even though a slight excess in the observed number is traceable in the near side to the reference star, the theoretical curve with $N=45$ ($n_0=0.0737$) is in good agreement with the observed $\sum_{k=1}^{4} W_k(r)$ up to a distance of $r=2.1 \text{ pc}$
but beyond $r \approx 2.5\text{pc}$ ($x \approx 1.7$) the effect due to $\sum_{k=5}^{\infty} W_k(r)$ becomes evident. Thus the distance interval between both boundaries of the reference stars and of the neighbors makes the observed distribution of the neighbors change remarkably, unless it covers a large part of $W_4(r)$.

With respect to the second cause, the distribution of the near-by stars around the sun, as shown in Figure 6 with a full step-line, may provide a means of checking if we compare it with the ideal space distributions with different constant densities (for reference’s sake, the distribution of composite stars is also shown by a dotted step line).

In view of Figure 6, we get a general idea that the space density of observed stars decreases gradually as the distance increases, probably due to increasing omission of faint stars from observation as GLIESE (1956) has indicated. This selection effect seems to explain, at least in part, why $n_0$ in Case I is larger than that in Case II, and why the observed $\sum_{k=1}^{4} W_k(r)$ in $r \geq 2$ deviates from $\sum_{k=1}^{4} W_k(r)$ in a systematic way. Moreover, the fact that for Case I, even if $b$ is extended to 6.9pc from 5.26pc by keeping $a=4.05\text{pc}$, the observed distributions for the respective neighbors show little change is also interpreted along the same line.

Finally, as for the clustering tendency of the stars, we suggested that it might explain the excesses at the nearest ends of the observed distributions for the 1-st and the 2-nd neighbors as found in Figures 2, 3, 4 and 5. But this apparent clustering disappears if the distance between two stars in each of seven pairs listed in Table 3 is increased by about 0.4pc, respectively. Hence, whether there exists a clustering tendency or not may depend on whether these seven pairs of stars are brought nearer merely by chance by about 0.4pc than the expectation value or not. Provided that a pair of stars are gravitationally connected, the relative velocity should be less than 1km/sec.. But in examining Table 3, we can hardly find such a pair, though some of the tabulated values are of course yet uncertain. Thus, at the present stage we are inclined to think that the occurrence of these pairs might be merely by chance.

**Table 3. Pairs of stars with small separations**

<table>
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<tr>
<th>Stars</th>
<th>$\alpha$ (1950)</th>
<th>$\delta$ (1950)</th>
<th>$\pi$</th>
<th>$\mu$</th>
<th>$\theta$</th>
<th>$V_\gamma$</th>
</tr>
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<tr>
<td>Ross</td>
<td>42</td>
<td>h 29.4 m</td>
<td>+ 9$^\circ$ 47</td>
<td>0.250</td>
<td>0.30</td>
<td>243$^\circ$</td>
</tr>
<tr>
<td>(+5$^\circ$) 1688</td>
<td>7 24.7</td>
<td>+ 5 29</td>
<td>0.266</td>
<td>3.75</td>
<td>171</td>
<td>26</td>
</tr>
<tr>
<td>+5$^\circ$ 1739</td>
<td>7 36.7</td>
<td>+ 5 21</td>
<td>0.283</td>
<td>1.250</td>
<td>214</td>
<td>3</td>
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<tr>
<td>(-46$^\circ$) 11540</td>
<td>17 24.9</td>
<td>-46 50</td>
<td>0.213</td>
<td>1.04</td>
<td>147</td>
<td>/</td>
</tr>
<tr>
<td>(-44$^\circ$) 11909</td>
<td>17 33.5</td>
<td>-44 16</td>
<td>0.209</td>
<td>1.16</td>
<td>217</td>
<td>/</td>
</tr>
<tr>
<td>L789-6</td>
<td>22 35.7</td>
<td>-15 37</td>
<td>0.300</td>
<td>3.260</td>
<td>45</td>
<td>-60</td>
</tr>
<tr>
<td>(-36$^\circ$) 15693</td>
<td>23 2.6</td>
<td>-36 9</td>
<td>0.273</td>
<td>6.90</td>
<td>79</td>
<td>+10</td>
</tr>
<tr>
<td>(-21$^\circ$) 6267</td>
<td>22 36.0</td>
<td>-20 52</td>
<td>0.219</td>
<td>0.46</td>
<td>99</td>
<td>-8</td>
</tr>
<tr>
<td>(-15$^\circ$) 6290</td>
<td>22 50.5</td>
<td>-14 31</td>
<td>0.206</td>
<td>1.11</td>
<td>123</td>
<td>+9</td>
</tr>
<tr>
<td>Ross</td>
<td>248</td>
<td>23 39.4</td>
<td>+43 55</td>
<td>0.316</td>
<td>1.82</td>
<td>176</td>
</tr>
<tr>
<td>(+4$^\circ$) 44</td>
<td>0 15.5</td>
<td>+43 44</td>
<td>0.278</td>
<td>2.89</td>
<td>82</td>
<td>+12 (14)</td>
</tr>
</tbody>
</table>

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Summarizing the above considerations, we may say that the deviations of
the observed distributions of the neighbors from the corresponding theoretical
ones formulated in the preceding section seem to be ascribable to both effects
of shortening of the boundary radius for the neighbors and of increase in the
missed stars with distance from the sun. Consequently, it may be concluded
that the near-by stars are distributed almost at random in space.

3. Statistics on the Components of the Binary

Here we concern ourselves with the reserved question as to how the com-
ponents are distributed in the composite star system. Among 45 near-by stars
within 5.26 pc ($\pi = 0.190$) from the sun, there are 10 binaries, 4 triplets, 2
quartets and the solar system (c.f. Figure 6). Excepting the solar system and
the unmeasured components, and decomposing multiple stars into sets of binaries,
we obtained 19 binaries available for our statistical materials, so consequently
inaccurate ones were inevitably included. As the component distance from
the primary, if the orbit had been determined the mean distance $r$ in a.u.
was adopted, otherwise the separation distance $d$ in a.u. was taken with a
correction factor of 1.29 corresponding to KUIPER’s empirical correction (KUIPER,
1942).

At first as regards 19 components, the same empirical distance distribution
as in Figure 2 was plotted, but its rapid increase in the neighborhood followed
by its gradual increase up to a great distance more than $10^4$ a.u. could hardly be
duplicated by any theoretical curve predicted in Section 1. Nevertheless, it was
found that this empirical distribution resembled closely that of the planets as
well as of the satellites in the solar system.

In the planetary system, the Titius-Bode law holds, which is written, in a
better approximation, as (Ter HAAR, CAMERON, 1963)

$$r = r_0 m^{N(r)} \quad (r_0, m = 1.89: \text{constants}),$$

where $N(r)$ is the order number of a planet reckoned outwards from Mercury.
In its alternative expression,

$$N(r) = \log r + q \quad (3-1)$$

if $N(r)$ is interpreted as the total number of planets within $r$ from the sun,
then (3-1) is nothing but a relation between the distribution function $N(r)$ and
the distance $r$. Such a trend of linear dependence of $N(r)$ on $\log r$ appears also in
satellite systems as well as in binary system though the gradual decrease of
d$N(r)/dr$ beyond a certain distance is found especially in the case of a binary
system.

The fact that $N(r)$ increases with $\log r$ means that if a spherically symmetric
three-dimensional distribution is assumed, the spatial number density $\rho(r)$ de-
creases with $1/r^3$ or, alternatively, if a circular symmetric two-dimensional
distribution is assumed, the areal number density $\rho'(r)$ varies with $1/r^2$. However,
the sort of mechanism that makes $\rho(r)$ or $\rho'(r)$ more rapidly decrease with
$r$ than $1/r^3$ or $1/r^2$, respectively, must be introduced in order to explain the
gradual decrease of $N(r)$ found in the empirical distribution of the components.
For this requirement we tentatively postulated the random and/or the systematic
motions that might have been shared among original components in the composite
system. This is because almost all of the components that had been moving near the primary have remained captured, while out of those that had been moving far from the primary, some may have made their escapes till now.

According to this idea, let us derive the expected distribution. We assume that (i) the mass of each component \( m \) is equal to or negligible compared with that of the primary \( M \) and (ii) the peculiar velocity distribution of components referred to a local standard of rest is Maxwellian with the dispersion of \( \sigma \). Then the probability that a component at a distance \( r \) from the primary happens to be escaping is given by (Shimizu, the forthcoming paper)

\[
\tau(r) = \frac{2}{\pi} \exp \left( -\frac{V_0^2}{2\sigma^2} \right) \sum_{\nu=0}^{\infty} \frac{(2\nu + 1)!}{(2\nu)!} \Gamma \left( \nu + \frac{3}{2}, \frac{\mu}{2\sigma^2 r} \right),
\]

\[
\mu = 2G(M + m) \quad (G: \text{gravity constant}),
\]

where \( V_0 \) is interpreted as either the primary's peculiar velocity or the rotational velocity of the local standard of rest at \( r \). It can be readily proved that the value of \( \tau(r) \) vanishes at \( r = 0 \) and tends toward 1 as \( r \to \infty \). It is noteworthy that in deriving (3.2) the components with \( V^2 \geq \mu/r \) (\( V \): relative speed) have been regarded as escaping from the primary, so that perturbations due to the interactions among components have been neglected.

Now, using a three-dimensional model, let the spatial number density at \( r \) from the binary be \( \rho(r) = \rho_0/r^3 \) (\( \rho_0 \): constant). Then the total number of the components within \( r \) of the primary should be

\[
N(r) = 4\pi \rho_0 \int_{r_0}^{r} \left[ 1 - \tau(x) \right]/x \, dx = 4\pi \rho_0 \log(x_0/x) - \int_{x}^{x_0} \tau(x)/x \, dx, \quad x = \mu/2\sigma^2 r,
\]

where \( r_0 = \mu/2\sigma^2 x_0 \) is the lower bound of \( r^* \). The dependence of \( N(r)/4\pi \rho_0 \) on \( r \) differs by the value of \( V_0 \) or \( y = 2V_0^2/\sigma^2 \), as illustrated in Figure 7. As for the

![Graph](image)

**Fig. 7.** Curves of \( \int_{x}^{x_0} (1 - \tau(x, y))/x \, dx \).

* The numerical tables for the integrals \( \int_{x}^{x_0} \tau(x)x^n \, dx \) \( (n = 1, 0, -1, \cdots, -4) \) will be given elsewhere together with other tables concerning the random force.
alternative two-dimensional model with the areal number density \( \rho'(r) = \rho_0' / r^2 \) \((\rho_0': \text{const.})\), the expression \( N'(r) \) for the total number of the components within \( r \) comes out to be the same as (3·3) except for a constant factor of \( 2\pi\rho_0' \). The areal density \( \rho'(r) \) is not identical with the projected areal density of the above three-dimensional model or \( (2\rho_0/r^2)(1-r^2/R^2)^{-2} \) \((R: \text{boundary radius})\), but the discussion given hereafter holds equally well with respect to either of the models. By the way, it may be noteworthy that the formulae given above have some resemblance with those for King's empirical distribution law (1962) in star clusters, though in the present case the cut-off effect due to the galactic tidal force is expected to be negligible.

Since the argument of the theoretical distribution function is \( x = \mu/2\sigma^2 r \), the empirical distributions of \( \mu/2r \) for the planets (Ceres is included), the satellites and the binary components were obtained as shown in Figures 8, 9 and 10. In plotting the empirical distribution for satellites, all satellite systems such as those of the earth, Mars, Jupiter etc. were combined together, because the distribution of each satellite system and, accordingly, its gradient against \((M+m)/\sigma)/r(\text{AU})\) appears to be in general accordance except for the Mars’ and earth’s (the moon only!), as illustrated in the lower diagram of Figure 9.

In the case of binary systems, on the other hand, the components whose \( \mu/2r \)'s were known from their predetermined orbital elements were only 9 out of 19, and accordingly for the remainders \((\mu/2) = G(M+m)\) must be evaluated. Our estimation of the stellar mass \( M \) (or \( m \)) was based on two empirical relations, the one between \( M \) and the spectral type and the other between \( M \) and the visual magnitude, given in Allen’s table (1963), their mean being adopted. Whenever an approximate period was noted in the literature, an independent estimate of \( \mu/2r \) was also made by \((\mu/2)^{-1/2}\cdot T^{-2·3} \) (Kepler’s law) for checking error. Unfortunately, for components with small separation angles, the discrepancy of both values of \( \mu/2r \) turns out to be considerable (Figure 10). Although the

![Figure 8](https://example.com/f8.png)

**Fig. 8.** Distribution of planets.
data on binary systems are thus still unsatisfactory, even at the distance of 5 pc, we may find a general tendency that the empirical distribution is likely similar to that of the planetary system as well as that of the satellites.

Now, it was hardly expected to get reliable results owing to the unsatisfactory data on the one hand and the simplified theory on the other hand, but we made a trial fitting of each theoretical curve with $y = 2V_0^2/\sigma^2 = 0.0, 0.5, 2.0$ or $5.0$ to the respective values for $\sigma$ and $V_0$. Of course, it might be possible to deter-
mine both $\sigma^2$ and $y=2V_0^2/\sigma^2$ at the same time by providing more theoretical curves with different $y$'s and by making use of the method of least squares, but at the present stage the gain seemed to be more than offset by the greater labor of calculation.

The fitted results came out as shown in Table 4. With respect to the binary system in particular, two sets of fittings were made, namely, for composite systems with $\pi>0.300$ (these $(M+m)/r$'s are separately indicated in the lower diagram of Figure 10) and for those with $0.300>\pi>0.190$, and the motions calculated from these means were adopted.

<table>
<thead>
<tr>
<th>System</th>
<th>of fitting curve</th>
<th>$(M+m)/(r)_{\text{corresponding to } x=1000}$</th>
<th>$\sigma$</th>
<th>$(y/2)^{1/2} = \sigma^2 + V_0^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planetary</td>
<td>0.0</td>
<td>13.0 (a.u.)</td>
<td>258</td>
<td>3.5 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>11.0 (a.u.)</td>
<td>2200</td>
<td>3.2 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>6.0 (a.u.)</td>
<td>5560</td>
<td>2.4 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>1.1 (a.u.)</td>
<td>1020</td>
<td>1.0 (a.u.)</td>
</tr>
<tr>
<td>Satellite</td>
<td>0.0</td>
<td>2.5 (a.u.)</td>
<td>2320</td>
<td>1.6 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>2.2 (a.u.)</td>
<td>2040</td>
<td>1.4 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.62 (a.u.)</td>
<td>575</td>
<td>0.8 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>0.11 (a.u.)</td>
<td>102</td>
<td>0.3 (a.u.)</td>
</tr>
<tr>
<td>Binary</td>
<td>0.0</td>
<td>5.2 (a.u.)</td>
<td>6400</td>
<td>2.5 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>4.2 (a.u.)</td>
<td>5350</td>
<td>2.3 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>1.5 (a.u.)</td>
<td>1650</td>
<td>1.3 (a.u.)</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>0.17 (a.u.)</td>
<td>245</td>
<td>0.5 (a.u.)</td>
</tr>
</tbody>
</table>

In connection with the above table, it is interesting to refer to Table 5 where both the values of the orbital velocity and of the escape velocity at the apocenter of the outermost object in the planetary system or in each of the satellite systems.

<table>
<thead>
<tr>
<th>System</th>
<th>Object</th>
<th>Orbital Vel. at Apocenter</th>
<th>Escape Vel. at Apocenter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planetary</td>
<td>Pluto</td>
<td>3.7 km/sec</td>
<td>6.0 km/sec</td>
</tr>
<tr>
<td>Satellite</td>
<td>Earth</td>
<td>Moon</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>Mars</td>
<td>Deimos</td>
<td>1.3</td>
</tr>
<tr>
<td></td>
<td>Jupiter</td>
<td>IX</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>Saturn</td>
<td>Phobe</td>
<td>1.4</td>
</tr>
<tr>
<td></td>
<td>Uranus</td>
<td>Oberon</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>Neptune</td>
<td>Nereid</td>
<td>0.41</td>
</tr>
</tbody>
</table>

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The face values of the fitting results indicate that the required motions turn out to be largest for the planetary system, smallest for the satellites, and intermediate for the binaries. However, on reflection concerning our approximation. we should mention only that the motions of the order of a few km/sec. should be assumed in explaining the observed distribution, irrespective of the planetary, the satellite or the binary system. This is not in contradiction with the escape velocities of the outermost objects in both planetary and satellite system in Table 5. When \( V_0 \) is considered as the rotational velocity in favor of the conservation of the angular momentum of the respective system, our assumption of \( V_0 \) = constant is not adequate, but we may regard \( V_0 \), in our order approximation, as a mean rotation velocity of the outer part of the system.

In this connection, Heuvel (1966) inferred recently that the binary systems and the "planetary" systems seemed not to be different in origin. His "planetary systems" except the solar system, however, were included in our binary data, while his binary systems covered the main groups of the visual, spectroscopic and photometric binaries. Nevertheless, so long as our data concern, which contained no close binary, it remains uncertain whether our postulate may be extended for close binary systems or not.

On the other hand, according to Kuiper (1942) the density distribution of the logarithmic separation distance among the near-by binaries (\( \tau > 0.50 \)) is likely represented by a normal curve with the mean of \( 2.07 \pm 0.21 \). The distribution of \( \mu/2\tau \) from his data gave a similar general tendency as ours, but the groupings into some distance sets suggest that his data, especially for distant ones (\( 0.150 < \tau < 0.7050 \)), suffered appreciably from the effect of selection and error, so that we wonder about the reality of such a trend.

In short, our main conclusion is that so far as the composite star is considered as an individual star, the space distribution of near-by stars is likely at random, while the distribution of the components in composite stars, close binaries excluded, seems to obey the same general law.

**References**


