1. Introduction. It has been realized for some time that thermal convection may be inhibited by a magnetic field. Early papers on this subject were by Thompson (1951) and Chandrasekhar (1952). Both of these calculations related to the effect of a magnetic field on the onset of convection in an incompressible fluid*; the first calculations were concerned with the effect of a vertical magnetic field but Chandrasekhar subsequently considered the case of a magnetic field in an arbitrary direction. A full account of this work is given in Chandrasekhar's book (1961).

The main application of the results of this theory has been to sunspots. Thus Biermann (1941) suggested that convection was completely suppressed by the magnetic fields in sunspots, while Hoyle (1949) supposed that the main role of the magnetic field was to direct the convection along the field lines. As the field lines fan out at the surface, the flux would be smaller in the spot than in the surrounding photosphere. It now seems clear that, as pointed out by Cowling (1953), there is some truth in both of these ideas; the magnetic field reduces but does not completely inhibit convection and one of its main effects is a considerable change in flow patterns. It is obvious that a complete theory of the structure of sunspots involves the effect of a magnetic field on fully developed convection but at the

*Or more precisely one in which the Boussinesq approximation is valid.
moment there is no adequate theory of stellar convection even without a magnetic field (see e.g. Kippenhahn 1963). Recently Chitre (1963) and Deinzer (1965) have constructed models of sunspots in which the degree of suppression of convection is taken as a free parameter and they have both deduced plausible values of this parameter from the observed properties of spots.

In this paper we return to the problem of the effect of a magnetic field on the onset of convection. The reason for this is that, whereas most previous criteria have referred to incompressible fluids, the astrophysical fluids in which we are interested are certainly compressible. It seems desirable to obtain criteria for the onset of convection in a compressible fluid.

Criteria for convective instability may be obtained at two levels of sophistication; the rising element of fluid may be supposed to move adiabatically or some account may be taken of the lateral heat flow from the element. In a liquid the first criterion for instability is that there exists any adverse temperature gradient; when lateral heat exchanges are included the temperature gradient has to exceed a certain value and this is known as Rayleigh's criterion for convective instability (Rayleigh 1916). In the case of a compressible fluid the former criterion, assuming adiabatic motion, was first obtained by Schwarzschild (1906). The Rayleigh criterion for compressible fluids is much more difficult to obtain and is still only known approximately. A recent paper on this subject is by Spiegel (1965).

In this paper we consider the effect of a magnetic field on Schwarzschild's criterion in an ideally conducting fluid. Some work on this subject has been done before. Thus Cowley (1961), Newcomb (1961) and Tayler (1961) have shown that a uniform horizontal magnetic field does not affect the Schwarzschild criterion. A similar result has been obtained by Chandrasekhar in the Boussinesq approximation. It should however be made clear that, although the stability criterion is unaltered, the form and growth rate of the instability may be radically changed by the field.

In this paper we consider magnetic fields which have a vertical component and obtain conditions which are sufficient for stability. In fact we obtain one criterion of rather wide applicability. In the simplest case this can be written

\[ \frac{B_v^2}{B_v^2 + \gamma P} > \frac{1}{\gamma} \frac{P}{\rho} \frac{d\rho}{dP}, \]  

(1.1)

where \( B_v \) is the vertical component of the magnetic field, \( P \) pressure, \( \rho \) density and \( \gamma \) ratio of specific heats. This form of criterion holds if both \( \gamma \) and the acceleration due to gravity \( g \) are constants; the modification of criterion (1.1) when \( \gamma \) and \( g \) vary is considered later. If the fluid is a perfect gas with uniform mean molecular weight, the inequality (1.1) can be written

\[ \frac{B_v^2}{B_v^2 + \gamma P} > \nabla - \nabla_{ad}, \]  

(1.2)

where \( \nabla = PdT/TPD \) and \( \nabla_{ad} \) is the value of \( \nabla \) for adiabatic motion \([\gamma - 1]/\gamma\] \( \) The Schwarzschild criterion is

\[ \sigma > \frac{1}{\gamma} \frac{P}{\rho} \frac{d\rho}{dP}, \]  

(1.3)
which is necessary and sufficient for stability. It can be seen that criterion (1.1)
predicts no stabilization if the vertical component of the field vanishes, as was found
previously.

A criterion of the above type is not universally true and there is a very good
reason for this. Although it is generally true that a magnetic field will act to
suppress convection, it is equally true that a magnetic field of arbitrary shape will
introduce hydromagnetic instabilities of various types. In particular it may no
longer be possible unambiguously to label one type of unstable motion as convection.
In this paper we do not wish to become involved in a discussion of hydromagnetic
instabilities and restrict attention to those fields, planar or vacuum, for which the
magnetic effect is purely stabilizing. In so doing, it must be realised that in an
astronomical object the field which does interact with a potential convective motion
may not be of one of these simple forms.

There is one other very important restriction in the present paper; the
electrical conductivity of the fluid is taken to be infinite. The importance of this
restriction is easy to understand. The main effect of the magnetic field is to
inhibit fluid motions across it in comparison with motions along it. An ideally
conducting fluid is tied to the field lines. In a fluid of finite conductivity motion
across the field will be possible at a rate which is governed by the conductivity;
if the conductivity is high and the field strong the motion will be very slow. It
thus seems that a magnetic field will not exert an absolute stabilizing influence on a
fluid of finite conductivity and that criterion (1.1) is a sufficient condition for
growth rates to be substantially reduced and to be governed by fluid diffusion.
First studies suggest that this is true and that the growth rate, \( \omega \), has the asymptotic
behaviour

\[
\omega \propto \eta/B^2
\]

as \( \eta \to 0 \) or \( B \to \infty \), where \( \eta \) is the resistivity of the fluid. It is hoped to follow up this
investigation at a later date.

The remainder of this paper is arranged as follows. The method of obtaining
sufficient conditions for stability is discussed in section 2, stability criteria for two
simple equilibria are obtained in section 3 and a short discussion of the results is
given in the final section.

2. Method of obtaining stability criteria. One of the main properties of a
magnetic field, which influences stability criteria, is that it connects material at
different levels in the fluid. The Schwarzschild criterion (1.3) is a local criterion*
referring to fluid at only one level but, when a magnetic field is introduced, the
necessary and sufficient condition for stability against convection is no longer a
local criterion. However we shall show that, although the necessary and sufficient
condition is global, it is possible in some circumstances to obtain a relatively simple
sufficient condition for stability which is a local criterion.

As mentioned above we have considered only very simple magnetic field
configurations. Thus two cases are studied. In the first the magnetic field is
uniform and inclined to the vertical at an arbitrary angle; in the second the

*It should perhaps be made very clear what is meant by the words 'local criterion'. They
mean that there exists a criterion for stability which must be satisfied separately at all levels in the
fluid and which contains only local values of the physical quantities. They do not mean that
the fluid is stable where the criterion is satisfied and that instabilities can only develop where it is
not satisfied; the latter point should become clear later when it is seen that integration by parts
is used to obtain alternative stability criteria.
magnetic field is vertical but its strength varies in horizontal planes. In both cases it can be shown that criterion (1.1) (or a slight generalization) is a sufficient condition for stability.

The stability criteria are found by use of the energy principle of Bernstein et al. (1958). They showed that, if an equilibrium configuration of an ideally conducting fluid is perturbed in such a way that any element of fluid moves a vector distance $\xi$ from its initial position, the change of potential energy of the system is given by

$$\delta W = \frac{1}{2} \int d\tau [\mathbf{Q}^2 - J \cdot \mathbf{Q} \times \xi + \gamma P \text{div} \xi^2 + \xi \cdot \text{grad} P \text{div} \xi + \xi \cdot \text{grad} \phi \text{div} \rho \xi],$$

(2.1)

where $\mathbf{Q} = \text{curl} (\xi \times \mathbf{B})$, $P$, $\rho$, $\mathbf{B}$ and $\mathbf{j}$ are the equilibrium pressure, density, magnetic field and current density, $\gamma$ is the ratio of specific heats of the fluid and $\phi$ is the external gravitational potential (defined so that the gravitational force is $+\text{grad} \phi$). In expression (2.1) the electromagnetic units used are rationalized Gaussian units with velocity of light put equal to unity.

The system is unstable to the perturbation $\xi$ if $\delta W$ is negative; if the system is stable, $\delta W$ must be positive for all perturbations which satisfy certain boundary conditions. Necessary and sufficient conditions for stability can only be obtained by minimizing $\delta W$ with respect to all allowable perturbations. However a sufficient condition for stability is obtained if it can be shown that $\delta W$ is certainly positive for any class of perturbations $\xi$ which includes all of the allowable ones.

In the derivation of equation (2.1) given by Bernstein et al. it was assumed that the ratio of specific heats was constant. This restriction is not necessary provided that $\gamma$ is defined by the equation

$$\frac{dP}{dt} + \gamma P \text{div} \mathbf{v} = 0,$$

(2.2)

where $d/dt$ denotes differentiation following the fluid element and $\mathbf{v}$ is the fluid velocity. If $u(P, v)$ is the internal energy per unit mass, where $v(=1/\rho)$ is the volume per unit mass, the equation for adiabatic changes can be written

$$\frac{du}{dt} + P \frac{dv}{dt} = 0.$$

(2.3)

Combination of equations (2.2), (2.3) and the equation of continuity, shows that

$$\gamma = \frac{v[P + \partial u/\partial v]/P \partial u/\partial P}{P}.$$

(2.4)

This is the standard definition for $\gamma$ in the ionization zone of a star and it has been used in a study of stellar stability by Dyson (1962).

In expression (2.1) the integral is over all space if the fluid fills all space. The same expression holds if the fluid is surrounded by perfectly conducting walls and the equilibrium magnetic field does not penetrate the boundary. The expression can be generalized if space is divided into regions filled with ideally conducting fluid and vacua. However, in the present problem we use only expression (2.1) and obtain sufficient conditions for stability against perturbations that are confined to the conducting fluid. Thus we suppose there are upper and
lower boundaries outside which the perturbations vanish. Our reasons for doing this are two-fold. Firstly, in a typical stellar convection problem, we expect the potentially unstable region to be layered between two potentially stable regions so that the effect of distant boundaries is minimized. Secondly it is probably unrealistic to talk about the effects of boundaries while neglecting both field curvature and finite resistivity.

As mentioned above, necessary and sufficient conditions for stability can only be obtained if $\delta W$ is minimized with respect to all allowable perturbations and, even for a simple equilibrium configuration, this may involve much arduous computation. However, sufficient conditions for stability can be obtained if $\delta W$ can be expressed as a sum of terms such that all but one are positive definite for all allowable perturbations and the remaining one is also positive if the local values of the physical quantities satisfy some inequality. In some cases $\delta W$ may take this form directly; in other cases part of the minimization procedure may be followed first. In any case the technique involves showing that $\delta W$ is greater than some quantity which is itself obviously positive provided one further condition is satisfied.

Once one sufficient condition for stability has been obtained it is often possible to obtain many more and one of these may be more useful than the one first obtained. In the simplest case this has been discussed by Suydam (1960). Thus suppose

$$\delta W = \int_{z_1}^{z_0} [A\xi'^2 + C\xi'^2] dz,$$  \hspace{1cm} (2.5)

where the prime denotes differentiation with respect to $z$, $C$ is positive definite and a sufficient condition for stability is

$$A > 0.$$  \hspace{1cm} (2.6)

Introduce a function $\Phi$ which is at present arbitrary and use the relation

$$\int_{z_1}^{z_0} 2\Phi \xi' dz = -\int_{z_1}^{z_0} \Phi' \xi dz,$$  \hspace{1cm} (2.7)

which holds since $\xi$ vanishes at $z = z_0$ and $z_1$. Then it is easy to see that equation (2.5) can be rewritten

$$\delta W = \int_{z_1}^{z_0} \left[ (A - \Phi' - \frac{\Phi^2}{C}) \xi'^2 + C \left( \xi' - \frac{\Phi}{C} \xi \right)^2 \right] dz$$  \hspace{1cm} (2.8)

so that a new sufficient condition for stability is

$$A > \Phi' + \frac{\Phi^2}{C}.$$  \hspace{1cm} (2.9)

Conditions of the form of inequality (2.9) will be obtained in what follows.

It is possible to obtain a further set of sufficient conditions by putting $\xi = \psi \eta$, where $\psi$ is an arbitrary positive definite function, and by performing further integrations by parts, but we do not obtain criteria by this method in the present paper.
3. Derivation of stability criteria

(a) Straight magnetic field. Consider an equilibrium configuration in which

\[ P = P_0(z), \tag{3.1} \]
\[ \rho = \rho_0(z), \tag{3.2} \]
\[ \text{grad } \phi = (\sigma, \sigma, g_0(z)) \tag{3.3} \]
\[ \gamma = \gamma_0(z) \tag{3.4} \]

and

\[ B = B_0(l, m, n). \tag{3.5} \]

\( P_0, \rho_0 \) and \( g_0 \) are connected by

\[ \frac{dP_0}{dz} = \rho_0 g_0. \tag{3.6} \]

Perturbations are considered which depend on \( x \) and \( z \) but not on \( y \). This means that, for a given inclination of the magnetic field to the vertical, \( n \) is fixed but that all values of \( l \) and \( m \) satisfying \( l^2 + m^2 = 1 - n^2 \) must be considered.*

For this equilibrium configuration

\[ \delta W = \frac{1}{2} \int d\tau \left[ Q_x^2 + \gamma P(\text{div } \xi)^2 + 2\rho g \xi_x \text{div } \xi + g \frac{dp}{dz} \xi_z^2 \right], \tag{3.7} \]

where

\[ Q_x = -IB \text{ div } \xi + \left( IB \frac{\partial}{\partial x} + nB \frac{\partial}{\partial z} \right) \xi_x, \]
\[ Q_y = -mB \text{ div } \xi + \left( IB \frac{\partial}{\partial x} + nB \frac{\partial}{\partial z} \right) \xi_y, \tag{3.8} \]
\[ Q_z = -nB \text{ div } \xi + \left( IB \frac{\partial}{\partial x} + nB \frac{\partial}{\partial z} \right) \xi_z. \]

It can be seen that \( m \) and \( \xi_y \) enter \( \delta W \) only through \( Q_y \) and the term involving \( Q_y \) positive definite. It may be possible to obtain a \( \xi_y \) satisfying the required boundary conditions by demanding that \( Q_y \) vanishes; if such a \( \xi_y \) exists it is the minimizing \( \xi_y \). In any case, if we neglect the term in \( Q_y \), we obtain sufficient conditions for stability by demanding that the remainder of \( \delta W \) is positive for all perturbations. The term in \( Q_x \), which is also positive definite, proves difficult to handle and \( \delta W \) is written

\[ \delta W = \frac{1}{2} \int d\tau \left( Q_x^2 + Q_y^2 + \delta W' \right), \tag{3.9} \]

where

\[ \delta W' = \frac{1}{2} \int d\tau \left\{ \left( IB \frac{\partial \xi_x}{\partial x} - nB \frac{\partial \xi_x}{\partial z} \right)^2 + \gamma P(\text{div } \xi)^2 + 2\rho g \xi_x \text{div } \xi + g \frac{dp}{dz} \xi_z^2 \right\}, \tag{3.10} \]

and the system is certainly stable if \( \delta W' \) is positive for all perturbations†.

* In what follows the zero suffixes are omitted from \( B_0, P_0, \) etc. as this leads to no ambiguity.
† It should be noted that it is obvious from equations (3.7) and (3.10) that the field exerts stabilizing influence.
It is now possible to go some way towards minimizing $\delta W'$. The perturbation $\xi$ can be Fourier analysed in the $x$ direction and each mode can be considered separately. Thus write

\[
\xi_x = \mathcal{R} (\eta e^{ikx}) = \mathcal{R} (\eta_R + i\eta_I) e^{ikx} \\
\xi_z = \mathcal{R} (-i\zeta e^{ikx}) = \mathcal{R} (-i\zeta_R + \zeta_I) e^{ikx}
\]  

(3.11)

If these expressions are substituted into equation (3.10), the $x$ and $y$ integrations can be performed and there results

\[
\delta W' \propto \int dz [l^2 B^2 k^2 (\xi_R^2 + \xi_I^2) + 2lnB^2 k^2 (\eta_I \zeta_R - \eta_R \zeta_I) + n^2 B^2 k^2 (\eta_R^2 + \eta_I^2) \\
+ \gamma P (\xi_R'^2 + \xi_I'^2) + k^2 \eta_R^2 + k^2 \eta_I^2 - 2k \eta_R \zeta_R' - 2k \eta_I \zeta_I'] \\
+ 2\rho g (\zeta_R \xi_R' + \xi_I \zeta_I' - k\eta_R \xi_R - k\eta_I \xi_I') + g^2 \eta_R \xi_R' + \xi_I \zeta_I'] \\
+ \frac{I}{(\gamma P + n^2 B^2)} \{\eta_R'^2 + \eta_I'^2 - \frac{I}{(\gamma P + n^2 B^2)} \} \\
(3.12)
\]

The $\eta_R$ and $\eta_I$ Euler equations are now algebraic and minimizing; this is a simple consequence of the fact that all the squared terms in $\eta_R$, $\eta_I$ are positive definite. These Euler equations give

\[
\eta_R = \frac{I}{k(\gamma P + n^2 B^2)} [\gamma P \zeta_R' + \rho g \zeta_R + lnB^2 k \xi_R'] \]  

(3.13)

\[
\eta_I = \frac{I}{k(\gamma P + n^2 B^2)} [\gamma P \zeta_I' + \rho g \zeta_I - lnB^2 k \xi_R] \]  

When the expressions (3.13) are substituted into (3.12) it becomes

\[
\delta W' \propto \int dz \left[ l^2 B^2 k^2 (\xi_R^2 + \xi_I^2) + \gamma P (\xi_R'^2 + \xi_I'^2) + 2\rho g (\zeta_R \xi_R' + \xi_I \zeta_I') \\
+ g^2 (\zeta_R^2 + \xi_I^2) - \frac{I}{(\gamma P + n^2 B^2)} \{\gamma P \zeta_R' + \rho g \zeta_R + lnB^2 k \xi_R'\}^2 \\
- \frac{I}{(\gamma P + n^2 B^2)} \{\gamma P \zeta_I' + \rho g \zeta_I - lnB^2 k \xi_R\}^2 \right]. 
\]  

(3.14)

This expression can be rewritten

\[
\delta W' \propto \int dz \left[ \frac{B^2 \gamma P}{\gamma P + n^2 B^2} \{(li_{\xi_I} - n_{\xi_R})^2 + (li_{\xi_I} + n_{\xi_I})^2\} \\
+ \left\{\left(\frac{\rho g \gamma P}{\gamma P + n^2 B^2}\right)' - \frac{\rho^2 g^2}{\gamma P + n^2 B^2} - \rho g'\right\} (\xi_R^2 + \xi_I^2) \right]. 
\]  

(3.15)

In obtaining this result some terms have been integrated by parts and the integrated part vanishes provided the perturbation is confined to a finite region. The first term in expression (3.15) is never negative and the system is certainly stable if the second term is positive. Thus a sufficient condition for stability is

\[
\left(\frac{\rho g \gamma P}{\gamma P + n^2 B^2}\right)' - \frac{\rho^2 g^2}{\gamma P + n^2 B^2} - \rho g' > 0. 
\]  

(3.16)

In obtaining this criterion we have successively discarded three positive terms in $\delta W$; it remains to be seen whether we can discard so much and still obtain a useful
criterion. Criterion (3.16) can be considerably simplified. Differentiation of
the first term and multiplication by the positive quantity \( \gamma P + n^2 B^2 \) gives
\[
\rho' g \gamma P - \rho g' n^2 B^2 + \rho g (\gamma P' + \gamma' P) - \frac{\rho g' \gamma P (\gamma' P + \gamma' P)}{\gamma P + n^2 B^2} - \rho^2 g^2 > 0. \tag{3.17}
\]
Use of equation (3.6) and the combination of the third and fourth terms in
inequality (3.17) then gives
\[
\frac{n^2 B^2 P' (\gamma' P' + \gamma' P)}{\gamma P + n^2 B^2} - \rho g' n^2 B^2 > P'^2 - \frac{\gamma P P' \rho'}{\rho}. \tag{3.18}
\]
Division by the positive quantity \( \gamma P^2 \) and further use of equation (3.6) then gives
\[
\frac{n^2 B^2}{\gamma P + n^2 B^2} \left( 1 + \frac{\gamma' P}{\gamma P'} \right) - \frac{n^2 B^2}{\gamma P} \frac{g' P}{g P'} > \frac{1}{\gamma} - \frac{P P'}{\rho P'}. \tag{3.19}
\]
In the case of a constant gravitational field and in a fluid of constant ratio of specific
heats, this reduces to
\[
\frac{B_v^2}{B_v^2 + \gamma P} > \frac{1}{\gamma} - \frac{P}{\rho} \frac{dP}{dP} = \nabla - \nabla_{\text{ad}}, \tag{3.20}
\]
which is criterion (1.1).

Consider now the more general criterion (3.19). As \( g \) is supposed to be an
external gravitational field it can of course consistently only vary linearly with \( z \); in fact, if the self gravitation of the convective layer can be neglected, it is probably
reasonable to neglect the variation in the gravitation field. However, although the
scale height of the gravitational field can usually be considered very large compared
with the pressure scale height, it is not necessarily true that the scale height of \( \gamma \) is
also very large. In fact, in the ionization zone of an abundant element,
\[
|d \log \gamma / d \log P|
\]
can become large enough to make criterion (3.19) useless; this happens for
example if \( g' \) vanishes and \( d \log \gamma / d \log P < -1 \). In such a case criterion (3.19) is
more stringent than that obtained in the absence of the field. As mentioned
earlier, the field does exert a stabilizing influence but the neglect of the three
positive definite terms in \( \delta W \) means that this is no longer apparent in the stability
criterion. It is still possible to obtain a useful sufficient condition by using the
 technique described in equations (2.7–2.9). This will be discussed after the
second equilibrium configuration has been studied.

(b) Vertical magnetic field varying in horizontal plane. Consider next the
following equilibrium configuration
\[
\mathbf{B} = (0, 0, B_0(x)), \tag{3.21}
\]
\[
P = P_0(x, z), \tag{3.22}
\]
\[
\rho = \rho_0(z) \tag{3.23}
\]
and
\[
\text{grad } \phi = (0, 0, g_0(z)), \tag{3.24}
\]
where \( P_0, \rho_0, g_0 \) and \( B_0 \) are related by
\[
\frac{\partial P_0}{\partial z} = \rho_0 g_0 \tag{3.25}
\]
The influence of a magnetic field

and

\[ \frac{\partial P_0}{\partial x} + B_0 \frac{dB_0}{dx} = 0. \]  

(3.26)

It can also be seen from equation (2.4) that in general

\[ \gamma = \gamma_0(x, z)^*. \]  

(3.27)

Using equation (3.21), it can be seen that

\[ \Phi = \left( B \frac{\partial \xi_x}{\partial x}, B \frac{\partial \xi_y}{\partial y}, B \frac{\partial \xi_z}{\partial z} - B \text{div } \xi - \xi_x \frac{dB}{dx} \right) \]  

(3.28)

and

\[ j \cdot \Phi = \frac{dB}{dx} \xi_x \left[ B \frac{\partial \xi_z}{\partial z} - B \text{div } \xi - \xi_x \frac{dB}{dx} \right] + B \frac{dB}{dx} \xi_z \frac{\partial \xi_x}{\partial z}. \]  

(3.29)

Thus

\[ \delta W = \frac{1}{2} \int d\tau \left\{ B^2 \left[ \left( \frac{\partial \xi_x}{\partial x} \right)^2 + \left( \frac{\partial \xi_y}{\partial y} \right)^2 \right] + \left[ B \frac{\partial \xi_z}{\partial z} - B \text{div } \xi - \xi_x \frac{dB}{dx} \right]^2 \right\} 
+ \frac{dB}{dx} \left( \xi_x^2 + \frac{\gamma P}{\xi} \text{div } \xi \right)^2 + \left( \xi_x \frac{\partial P}{\partial x} + \xi_z \frac{\partial P}{\partial z} \right) \text{div } \xi 
+ \xi_z \left( \rho \text{div } \xi + \xi_z \frac{d\rho}{dz} \right) \right\}. \]  

(3.30)

This can be rewritten

\[ \delta W = \frac{1}{2} \int B^2 \left( \frac{\partial \xi_y}{\partial y} \right)^2 d\tau + \delta W' \]  

(3.31)

and a sufficient condition for stability can be obtained by demanding that \( \delta W' \) be positive.

\( \delta W' \) can be considered to be a functional of \( \text{div } \xi, \xi_x \) and \( \xi_z \) and it can be minimized algebraically with respect to \( \text{div } \xi \). \( \delta W' \) can be rewritten using equations (3.25) and (3.26)

\[ \delta W' = \frac{1}{2} \int d\tau \left\{ B^2 \left( \frac{\partial \xi_x}{\partial x} \right)^2 + \left( B \frac{\partial \xi_z}{\partial z} - \xi_x \frac{dB}{dx} \right)^2 + B \frac{dB}{dx} \left( \frac{\partial \xi_z}{\partial z} \xi_x - \xi_z \frac{\partial \xi_x}{\partial x} \right) 
- \left( \frac{dB}{dx} \right)^2 \xi_x^2 + \xi_z^2 \frac{d\rho}{dz} + 2\rho g \xi_z - 2B^2 \frac{\partial \xi_z}{\partial z} \right) \text{div } \xi + (B^2 + \gamma P)(\text{div } \xi)^2 \right\}. \]  

(3.32)

The \( \text{div } \xi \) Euler equation is

\[ (B^2 + \gamma P) \text{div } \xi = B^2 \frac{\partial \xi_z}{\partial z} - \rho g \xi_z \]  

(3.33)

and this is obviously minimizing. When equation (3.33) is substituted into

* Once again the suffix zero is omitted in what follows.
(3.32), there results

\[
\delta W' = \frac{1}{2} \int d\tau \left\{ B^2 \left( \frac{\partial \xi_z}{\partial x} \right)^2 + B^2 \left( \frac{\partial \xi_z}{\partial z} \right)^2 - B \frac{dB}{dx} \xi_z \frac{\partial \xi_z}{\partial z} - B \frac{dB}{dx} \xi_z \frac{\partial \xi_z}{\partial z} \\
+ \xi_z^2 g \frac{dp}{dz} - \frac{(\rho g \xi_z - B \frac{\partial \xi_z}{\partial z})^2}{B^2 + \gamma P} \right\}. \tag{3.34}
\]

In expression (3.34) it can be seen that \( \delta W' \) does not depend on horizontal derivatives of \( \xi \); thus a separate stability criterion can be found on each field line. Furthermore, for perturbations confined to a finite region, the terms

\[
B \frac{dB}{dx} \frac{\partial}{\partial z} (\xi_z \xi_z^z)
\]

integrate out to zero and a sufficient condition for stability can be found if one more positive term is omitted,

\[
\int d\tau B^2 \left( \frac{\partial \xi_z}{\partial z} \right)^2.
\]

Thus consider

\[
\delta W'' = \frac{1}{2} \int dz \left\{ B^2 \left( \frac{\partial \xi_z}{\partial z} \right)^2 + \xi_z^2 g \frac{dp}{dz} - \frac{(\rho g \xi_z - B \frac{\partial \xi_z}{\partial z})^2}{B^2 + \gamma P} \right\}. \tag{3.35}
\]

This can be rewritten

\[
\delta W'' = \frac{1}{2} \int dz \left\{ \gamma PB^2 \left( \frac{\partial \xi_z}{\partial z} \right)^2 + \left[ g \frac{dp}{dz} - \frac{\rho^2 g^2}{B^2 + \gamma P} - \left( \frac{\rho g B^2}{B^2 + \gamma P} \right) \right] \xi_z^2 \right\}. \tag{3.36}
\]

A sufficient condition for stability is then

\[
gp' - \frac{\rho^2 g^2}{B^2 + \gamma P} - \left( \frac{\rho g B^2}{B^2 + \gamma P} \right)' > 0. \tag{3.37}
\]

After some manipulation this can be rewritten

\[
\frac{B^2}{B^2 + \gamma P} \left[ \frac{1 + \gamma' P}{\gamma P} \right] - \frac{B^2}{\gamma P} \frac{g' P}{g P} > \frac{1}{\gamma} - \frac{P\rho'}{\rho P'}, \tag{3.38}
\]

which is the same as the criterion found earlier since in this case the field is entirely vertical.

(c) Derivation of alternative stability criteria. What has been obtained above could well be a very useful stability criterion in a situation in which \( \gamma \) and \( g \) are effectively constant but this is not so in the ionization zone in the outer layers of a star. In this case it would be useful to find an alternative stability criterion which always shows some stabilizing effect of the magnetic field. It is suggestive that such a criterion should not contain \( \gamma' \) or \( g' \) as they give rise to the terms of uncertain sign. In what follows, to avoid algebraic complexity, we will assume that \( g \) is constant as it is the effect of \( \gamma' \) which is most important.

Consider expressions (3.15) and (3.36). The second of these is precisely of the form (2.5). Although (3.15) is not of this form, it is easy to show that the
procedure described in equations (2.7)-(2.9) can also be used in this case. We consider the second case in detail. According to inequality (2.9) a new sufficient condition for stability is

$$g_P' - \frac{g^2 \rho^2}{\gamma P + B^2} - \left( \frac{g_P B^2}{\gamma P + B^2} \right)' > \Phi' + \frac{\Phi^2(B^2 + \gamma P)}{\gamma PB^2}, \quad (3.39)$$

where $\Phi$ is an arbitrary function. Alternatively this criterion can be written

$$\frac{\gamma P^2 g^2}{\gamma P + B^2} \left[ \frac{B^2}{\gamma P + B^2} \left( 1 + \frac{\gamma' P}{\gamma P'} \right) + \frac{P \rho'}{P'} - \frac{1}{\gamma} \right] > \Phi' + \frac{\Phi^2(B^2 + \gamma P)}{\gamma PB^2}. \quad (3.40)$$

It is hoped that a useful criterion can be found which does not involve derivatives of $\gamma$. This will be achieved if

$$\frac{\partial \Phi}{\partial \gamma} = -\frac{g_P PB^2}{(\gamma P + B^2)^2}, \quad (3.41)$$

or

$$\Phi = -\frac{g_P PB^2}{\gamma P + B^2} + \Psi. \quad (3.42)$$

If this expression for $\Phi$ is substituted into equation (3.40) the criterion can be rewritten

$$\frac{P dP}{\rho} - \frac{1}{\gamma} - \frac{B^2}{\gamma P + B^2} > \frac{P \psi''}{\gamma (\gamma P + B^2)} + \frac{B^2}{\gamma (\gamma P + B^2)} \left( 1 - \frac{(\gamma P + B^2)}{g_P B^2} \Psi \right)^2. \quad (3.43)$$

This can be further rewritten

$$\frac{P dP}{\rho} - \frac{1}{\gamma} + \frac{B^2}{\gamma (\gamma P + B^2)} > \frac{P \psi''}{\gamma (\gamma P + B^2)} + \frac{B^2}{\gamma (\gamma P + B^2)} \left( 1 - \frac{(\gamma P + B^2)}{g_P B^2} \Psi \right)^2. \quad (3.44)$$

In this expression, by putting $\Psi = 0$, we recover the criterion which holds in the absence of the field. We now wish to find a non-zero $\Psi$ which will give a better criterion.

For example, we can try

$$\Psi = g_P B^2/(\alpha P + B^2), \quad (3.45)$$

where $\alpha$ is a constant greater than zero. Then inequality (3.44) becomes

$$\frac{P dP}{\rho} - \frac{1}{\gamma} - \frac{B^2}{\gamma (\gamma P + B^2)} > \frac{P dP}{\rho} \frac{B^2}{\rho dP (\alpha P + B^2)} - \frac{\alpha PB^2}{(\alpha P + B^2)^2} + \frac{B^2(\alpha - \gamma)^2 P^2}{\gamma (\gamma P + B^2)(\alpha P + B^2)^2}. \quad (3.46)$$

After some rearrangement this can be written

$$\frac{\alpha \gamma + \alpha - \gamma}{\alpha \gamma} \frac{B^2}{\alpha P + B^2} + \frac{1}{\gamma} - \frac{P dP}{\rho} = \nabla - \nabla ad. \quad (3.47)$$

Provided only that

$$\alpha > \gamma/(\gamma + 1), \quad (3.48)$$

criterion (3.47) is a sufficient condition for stability which shows some of the
stabilizing effect of the magnetic field. A variety of stability criteria can clearly be obtained. For example

\[ \alpha = \frac{B^2}{\gamma(P + B^2)} > \nabla - \nabla_{ad}. \] (3.49)

(i) \( \alpha = \frac{1}{\gamma} \): \( \frac{B^2}{\gamma(P + B^2)} > \nabla - \nabla_{ad}. \)

(ii) \( \alpha = \gamma_{\text{max}} \) (the maximum value of \( \gamma \) in the layer):

\[ \left( 1 + \frac{1}{\gamma} - \frac{1}{\gamma_{\text{max}}} \right) \frac{B^2}{(\gamma_{\text{max}}P + B^2)} > \nabla - \nabla_{ad}. \] (3.50)

As mentioned earlier exactly similar criteria apply in the case of the uniform field inclined at an arbitrary angle to the vertical, provided only that \( B_v \) replaces \( B \) in inequality (3.47).

A simple consequence of criterion (3.49) is that a strong enough magnetic field will stabilize any atmosphere in which \( P \) and \( \rho \) decrease together. For inequality (3.49) can also be written

\[ \frac{B^2}{\gamma(B^2 + P)} > \frac{1 - \rho}{\gamma} \frac{dP}{d\rho} \] (3.51)

and the left-hand side approaches \( 1/\gamma \) as \( B \to \infty \) while the right-hand side is less than \( 1/\gamma \) if \( P \) and \( \rho \) decrease together. The criterion also has the virtue that, when \( \gamma \) is small (the minimum value of \( \gamma \) in a convective zone is perhaps \( \gamma \)) and the right-hand side becomes large, the left-hand side also becomes large; it is thus a criterion that it should be possible to satisfy throughout a convective zone. It should however be noted that, when the variations in \( \gamma \) are slight, criterion (3.49) may be considerably less useful than criterion (1.1) slightly modified by the \( \gamma \) variation.

It is perhaps worth pointing out that this method does not yield a useful sufficient condition in the case of an incompressible fluid. For in that case equations (3.9) and (3.10) can be written

\[ \delta W \geq \delta W = \frac{1}{2} \int d\tau \left\{ \left( IB \frac{\partial \xi_z}{\partial x} - nB \frac{\partial \xi_x}{\partial x} \right)^2 + g \frac{d\rho}{d\xi} \xi_z^2 \right\}. \] (3.52)

In equation (3.52), the magnetic term can be made as small as is desired by demanding that the scale of the \( x \) variation of \( \xi \) is sufficiently large and the sufficient condition for stability is exactly the usual criterion in the absence of a field. For an incompressible fluid the term

\[ \left( IB \frac{\partial \xi_z}{\partial x} - nB \frac{\partial \xi_x}{\partial x} \right)^2 \]

must be included and any criterion including the effect of the field must depend on the distance between the levels at which \( \xi_z \) vanishes. The crucial difference between compressible and incompressible fluids is that in the latter stabilization occurs only when fluid elements try to cross the field, whereas a compressible element expands on rising and this in itself is resisted by the field.

4. Discussion. In the previous section we have obtained sufficient conditions for stability against convection for a simple magnetic field configuration in an ideally conducting fluid. This idealized problem differs very considerably from the situation in, for example, the solar atmosphere. The true magnetic field
configuration may introduce further instabilities and the effect of finite transport processes would have to be considered. In particular the major factor influencing the structure of a sunspot is the amount of energy carried by fully developed convection when the magnetic field is not strong enough to suppress convection completely.

Although the transport of energy by convection even in the absence of a magnetic field is still not fully understood, there is beginning to be increasing support (cf. Howard 1963) for the suggestion of Malkus (1954) that the fluid flow which actually occurs in fully developed convection is the one which maximizes the heat transport. If the magnetic field is strong enough it certainly constrains the flow patterns and therefore probably reduces the maximum possible heat flux. We therefore suggest that, if the relevant criterion of (1.1) and (3.49) is close to being satisfied, there will be a substantial reduction in the transport of energy by convection.

Table I

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(T)</th>
<th>(P_{gas})</th>
<th>(B)</th>
<th>(B^2/(B^2 + 4\pi P))</th>
<th>(\nabla)</th>
<th>(\gamma_{max}\nabla)</th>
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</thead>
<tbody>
<tr>
<td>0.075</td>
<td>4131</td>
<td>0.958</td>
<td>(\sim 2500)</td>
<td>0.85</td>
<td>0.39</td>
<td>0.66</td>
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<tr>
<td>6891</td>
<td>3.507</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>8468</td>
<td>4.008</td>
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<td></td>
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</tr>
<tr>
<td>10850</td>
<td>4.209</td>
<td>(\sim 2750)</td>
<td>0.60</td>
<td>6.68</td>
<td>11.1</td>
<td></td>
</tr>
<tr>
<td>13645</td>
<td>4.310</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>0.16</td>
<td>13300</td>
<td>5.009</td>
<td>(\sim 3000)</td>
<td>0.58</td>
<td>0.15</td>
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</tr>
<tr>
<td>16695</td>
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<tr>
<td>0.30</td>
<td>17675</td>
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</tr>
<tr>
<td>18596</td>
<td>24.345</td>
<td>(\sim 5000)</td>
<td>0.45</td>
<td>0.16</td>
<td>0.27</td>
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<tr>
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<tr>
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<td>(\sim 7000)</td>
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<tr>
<td>25914</td>
<td>130.64</td>
<td>(\sim 9500)</td>
<td>0.35</td>
<td>0.21</td>
<td>0.35</td>
<td></td>
</tr>
</tbody>
</table>

The first three columns of this table are taken directly from Chitre (1963); the other values are estimated or deduced from information in his paper.

Both Chitre (1963) and Deinzer (1965) have obtained semi-empirical expressions for the degree of suppression of convection in their sunspot models but it is difficult to extract enough information from their papers to evaluate the two sides of criterion (3.49). In particular, although Chitre gives a temperature-pressure profile for his model, he only gives approximate values for the magnetic field and the ratio of specific heats is not tabulated.

The results given by Chitre are shown in Table I; at intermediate points in the spot the value of the magnetic field may be uncertain by as much as 50% and the values of various other quantities in the table are only approximate. To compare Chitre’s results with criterion (3.49); it is convenient to write the criterion in the form

\[
\frac{B^2}{B^2 + 4\pi P} > \gamma [\nabla - \nabla_{ad}],
\]

(4.1)

where the \(4\pi\) has been introduced to agree with the units used by Chitre. In Table I \(T\) is measured in \(^{°}\)K, \(P_{gas}\) in \(10^5\) dyn/cm\(^2\) and the radiation pressure may be neglected, \(B\) in Gauss, \(\gamma_{max}\) is \(5/3\) the value for a fully ionized gas and \(\beta\) is the convective efficiency factor. \(\beta\) is defined so that, if \(F\) is the flux due to convection in the absence of the magnetic field, \(\beta F\) is the flux in the presence of the field.
As $\nabla_{ad}$ is positive, it is clear from criterion (4.1) that the system would certainly be stable in our approximation if

$$\frac{B^2}{B^2 + 4\pi P} > \gamma_{\text{max}} \nabla.$$  \hspace{1cm} (4.2)

It can be seen from the table that this is true for almost the whole of Chitre’s sunspot model. There is however one region where all of the sufficient criteria are violated. This is the neighbourhood of a temperature of 10,000 °K where the ionization is just becoming important and where there is a strong density inversion in Chitre’s model. It is difficult to say how much of this density inversion is genuine and how much is due to the particular theory of convection used (Faulkner, Griffiths & Hoyle 1965).

On the basis of this comparison between our sufficient condition for stability and Chitre’s sunspot model, it seems reasonable to suppose that the magnetic field should cause a serious reduction of convective efficiency. However it is difficult to make any more precise comments because of all of the factors that are omitted from the simple theory; finite transport processes, field curvature and boundary conditions.

Acknowledgments. One of us (D. O. G.) is grateful to the S.R.C. for the award of a research studentship.

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1965 August.

Note added in proof. The replacement of $1/\gamma - P d\rho/d\rho$ by $\nabla - \nabla_{ad}$ in the stability criteria with variable $\gamma$ incorrectly assumes that $\rho$ does not vary when $\gamma$ does. However, the criteria in terms of $P d\rho/d\rho$ are all correct. The comparison with the sun spot model of Chitre is altered in some minor details but the general results remain as stated.

References

Rayleigh, Lord, 1916. Phil. Mag., 32, 520 (see also Collected Papers, 6, 432).
Thompson, W. B., 1951. Phil. Mag., 42, 1417.