SCATTERING OF LIGHT IN AN ATMOSPHERE
OF FINITE OPTICAL THICKNESS

V. V. Ivanov

Astronomical Observatory, Leningrad State University
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The X and Y functions for isotropic scattering [Ambartsumyan's functions \( \phi(x, \tau_0), \psi(x, \tau_0) \)] are investigated. Their asymptotic expressions for \( \tau_0 \gg 1 \) and \( z \geq 0 \) are found. One feature in the behavior of these functions facilitating their tabulation is noted. The asymptotic properties of the X and Y functions for completely noncoherent scattering are also examined.

The exact solution of the problem of the scattering of light in an atmosphere of finite optical thickness is closely associated with the theory of a special class of functions called the X and Y functions. For the particular case of isotropic scattering without change of frequency, these functions were introduced by V. A. Ambartsumyan [1]. However, the detailed investigation of them for the general case (arbitrary scattering law) was made by S. Chandrasekhar [2]. Analogous functions can also be introduced when scattering is accompanied by a complete redistribution of the frequencies [3, 4].

Up to the present time, the X and Y functions have been investigated in considerable detail. However, several gaps remain and, in particular, there is an absence of tables of these functions for a sufficiently wide interval of the parameters. It can be hoped that this gap will be filled in the near future. However, since the tabulation of the X and Y functions is very laborious, it is desirable to determine the properties of these functions that would simplify tabulation. The most important of these properties is the asymptotic behavior.

The first two sections of the present paper are devoted to the simplest kinds of X and Y functions, namely those for isotropic scattering without frequency redistribution. In order to emphasize this situation, the notation used is that originally proposed by V. A. Ambartsumyan: \( \phi(x, \tau_0) \) and \( \psi(x, \tau_0) \). The behavior of these functions is studied for all positive values of \( z \). Asymptotic formulas for \( \phi(x, \tau_0) \) and \( \psi(x, \tau_0) \) are derived for large \( \tau_0 \) and \( 0 \leq z \leq \infty \). Some of the expressions for \( z = 1 \) known from previous work are special cases of the more general formulas derived below.

Although the first two sections of the present paper deal with the case of scattering without a change of frequency, the results are also of interest for the case of scattering with a complete frequency redistribution since they allow us to establish certain important features of the behavior of the X and Y functions in the latter case. The questions associated with this are discussed in the third section.

I. The functions \( \phi(x, \tau_0) \) and \( \psi(x, \tau_0) \) are defined by the following system of integral equations:

\[
\begin{align*}
\varphi(x, \tau_0) &= 1 \\
+ \frac{\lambda}{2} \int_0^x \frac{\psi(x, \tau_0)\varphi(x', \tau_0) - \varphi(x', \tau_0)\psi(x, \tau_0)}{x' - z} \, dx' \\
+ \frac{\lambda}{2} \int_0^x \frac{\psi(x, \tau_0)\psi(x', \tau_0) - \psi(x', \tau_0)\psi(x, \tau_0)}{x' + z} \, dx',
\end{align*}
\]

(1)

\[
\psi(x, \tau_0) = \frac{e}{\tau_0}
\]

\[
+ \frac{\lambda}{2} \int_0^x \frac{\varphi(x, \tau_0)\varphi(x', \tau_0) - \varphi(x', \tau_0)\varphi(x, \tau_0)}{x' - z} \, dx' \\
+ \frac{\lambda}{2} \int_0^x \frac{\psi(x, \tau_0)\psi(x', \tau_0) - \psi(x', \tau_0)\psi(x, \tau_0)}{x' + z} \, dx',
\]
where $\lambda \leq 1$ is the probability for a quantum to survive an act of scattering, and $\tau_0$ is the optical thickness of the layer ($0 \leq \tau_0 \leq \infty$). As is known, [5], these equations have a one-parametric family of solutions. We will consider the solution of (1) that is regular for $|x| > 0$. Such a solution is unique. It is directly related to the function $p(\tau, z, \tau_0)$ defined by

$$p(\tau, z, \tau_0) = \frac{\lambda}{2} \int_{0}^{\infty} E_1(\tau - \tau') p(\tau', z, \tau_0) d\tau' + \frac{\lambda}{4\pi} e^{-\frac{\tau}{z}},$$

(2)

where $E_1(t)$ is the integral exponential function of the first order. The relationship is the following:

$$\varphi(z, \tau_0) = \frac{4\pi}{\lambda} p(0, z, \tau_0),$$

$$\psi(z, \tau_0) = \frac{4\pi}{\lambda} p(\tau, z, \tau_0).$$

In most papers, the functions $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ were only considered for $0 \leq z \leq 1$. For these values of $z$ the functions $p(\tau, z, \tau_0)$ and, consequently, $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ have a simple probabilistic interpretation [3]: $p(\tau, z, \tau_0) d\omega$ is the probability that the quantum absorbed at an optical depth $\tau$ in a plane parallel layer of thickness $\tau_0$ will emerge out of it through the surface $\tau = 0$ at an angle $\cos z$ within solid angle $d\omega$.

However, there are a number of problems for whose solution it is also necessary to obtain $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ for $z > 1$. As an example, we can mention the problem of fluorescent scattering in planetary atmospheres considered by Sobouti [6]. The general investigation of the $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ functions (more accurately, even for X and Y functions) was carried out by Busbridge [7] for arbitrary complex $z$. However, the question of the asymptotic behavior of these functions for $\tau_0 \gg 1$ and $z > 1$, important from a practical point of view, was not investigated.

V. V. Sobolev [8, 9] has obtained asymptotic expressions for $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ correct for $z \leq 1$. They have the following form:

$$\varphi(z, \tau_0) = \varphi(z) - C \frac{z \varphi(z)}{1 - \frac{2k}{e^{-\frac{\tau}{z}}}} e^{-\frac{2kz}{\tau}},$$

$$\psi(z, \tau_0) = C_1 \frac{z \varphi(z)}{1 - \frac{2k}{e^{-\frac{\tau}{z}}}} e^{-\frac{2kz}{\tau}}.$$  

(4)

Here $\varphi(z) \equiv \varphi(z, \infty)$ is the Ambartsumyan function for a semi-infinite medium defined by

$$\varphi(z) = 1 + \frac{\lambda}{2} z \varphi(z) \int_{0}^{\infty} \frac{\varphi(\tau')}{\tau' + z} d\tau',$$

(5)

$k$ is the positive root of the characteristic equation

$$\frac{\lambda}{2k} \ln \frac{1 + k}{1 - k} = 1,$$

(6)

and $C$ and $C_1$ are constants such that

$$C_1 \int_{0}^{1} \frac{\eta \varphi(\eta)}{(1 - k\eta)^2} d\eta = 2k \int_{0}^{1} \frac{\eta \varphi(\eta)}{1 - k\eta^2} d\eta,$$

$$2kC = \epsilon z.$$  

(7)

(8)

It should be noted that $0 \leq k \leq 1$ and $k \to 0$ as $\lambda \to 1$.

If the main reason for the loss of quanta from the process of multiple scattering is true absorption, then $k\tau_0 \gg 1$ and instead of (4), we have

$$\varphi(z, \tau_0) = \varphi(z) - \frac{z \varphi(z)}{1 - \frac{2k}{e^{-\frac{\tau}{z}}} e^{-\frac{2kz}{\tau}}},$$

(9)

$$\psi(z, \tau_0) = C \frac{z \varphi(z)}{1 - \frac{2k}{e^{-\frac{\tau}{z}}} e^{-\frac{2kz}{\tau}}},$$

(10)

where $\gamma = 2q(\infty) = 1.4209$; here $q(\tau)$ is the Hopf function.

In the derivation of formulas (4), quantities of order $e^{-\tau_0/z}$ were considered to be small by comparison with $e^{-k\tau_0}$ and were neglected. However, if it is assumed that $z$ can assume any positive value, then it is no longer possible to do this. In order to obtain asymptotic formulas for $\varphi(z, \tau_0)$ and $\psi(z, \tau_0)$ correct for $z > 1$, we make use of the following procedure. It is clear that the simplification that appears for large $\tau_0$ is intimately related with the simplification in the structure of the radiation field that arises deep inside a semi-infinite medium. Therefore, we will initially study the asymptotic behavior of the function $p(\tau, z, \tau_0) \equiv p(\tau, z, \infty)$, defined by the equation

$$p(\tau, z) = \frac{\lambda}{2} \int_{0}^{\infty} E_1(\tau - \tau') p(\tau', z) d\tau' + \frac{\lambda}{4\pi} e^{-\frac{\tau}{z}}.$$  

(11)
It can be shown (see [3], chapter 6) that
\[ p(\tau, z) = \frac{\lambda}{4\pi} \varphi(z) \left( e^{-\frac{\tau}{z}} + \int_0^\tau e^{-\frac{s-z\tau}{z}} \Phi(s) \mathrm{d}s \right), \tag{12} \]
where
\[ \Phi(\tau) = 2\pi \int_0^1 \rho(\tau, \eta) \frac{\mathrm{d}n}{\eta}, \tag{13} \]
while
\[ \int_0^\infty e^{-\eta} \Phi(\tau) \mathrm{d}\eta = \varphi\left(\frac{1}{\tau}\right) - 1. \tag{14} \]

Applying the Laplace transformation with respect to \( \tau \) to (12) we obtain
\[ \bar{p}(s, z) = \frac{\lambda}{4\pi} \frac{z\varphi(z)}{1 + sz}, \tag{15} \]
where \( \bar{p}(s, z) \) is the Laplace transform of \( p(\tau, z) \):
\[ \bar{p}(s, z) = \int_0^\infty e^{-st} p(\tau, z) \mathrm{d}\tau. \tag{16} \]

The problem now consists in carrying out the inverse transformation,
\[ p(\tau, z) = \frac{\lambda}{4\pi} z\varphi(z) \frac{1}{2\pi i} \int_{\gamma - \infty}^{\gamma + \infty} e^{s(1/z)} \frac{\varphi(1/s)}{1 + sz} \mathrm{d}s. \tag{17} \]

For the calculation of the integral (17) we note that the function \([\varphi(1/s)]/(1 + sz)\) has the following singularities in the left half-plane: a pole at \( s = -1/z \) (because of the factor \( 1/(1 + sz) \)), a pole at \( s = -k \), and a branch line \((\infty, -1)\) on account of the function \( \varphi(1/s) \) (for example, see [7]). Let us deform the contour of integration in (17) in such a manner that it encompasses all of the singularities of the integrand lying in the left half-plane (see Fig. 1). Since we are only interested in the asymptotic behavior of \( p(\tau, z) \) as \( \tau \to \infty \), it is sufficient to consider the contribution to the integral (17) from the two rightmost singularities, i.e., the poles \( s = -1/z \) and \( s = -k \). For large \( \tau \) the integral along the cut will be small by comparison with the contributions from these poles. In this way we find that for \( \tau \gg 1 \) and \( z > 1 \)
\[ p(\tau, z) = A \frac{z\varphi(z)}{1 - k^2} e^{-\kappa z} + \frac{\lambda}{4\pi} z\varphi(z) e^{-\omega z}, \tag{18} \]
where
\[ 2\pi A \int_0^1 \frac{\eta\varphi(\eta)}{(1 + k\eta)^2} \mathrm{d}\eta = 1. \tag{19} \]

But, as is known (for example, see [7]),
\[ \varphi(z)\varphi(-z) = \frac{1}{a(z)^2}, \tag{20} \]
where
\[ a(z) = 1 + \frac{\lambda}{2} \ln \left| \frac{1 + z}{1 - z} \right|. \tag{21} \]

Therefore, we finally find that
\[ p(\tau, z) = A \frac{z\varphi(z)}{1 - k^2} e^{-\kappa z} + \frac{\lambda}{4\pi} \frac{z}{a(z)} \varphi(z) e^{-\omega z}. \tag{22} \]

It should be noted that as \( z \to 1/k \) the two terms in (22) tend to infinity, although their sum remains finite. Since \( \tau \gg 1 \), for \( ks \ll 1 \) the second term on the right-hand side of (22) can be neglected by comparison with the first. The expression obtained is formally identical with the expression for the probability that a quantum emerges from deep within a semi-infinite medium (see [3], chapter 6), however the latter is valid for \( z \ll 1 \). Thus, formula (22) describes the asymptotic behavior of \( p(\tau, z) \) for any \( z \gg 0 \).

We can now proceed directly to the derivation of the asymptotic formulas for \( \varphi(z, \tau_0) \) and \( \psi(z, \tau_0) \). From (2) and (11) it is not difficult to obtain
\[ p(\tau, z, \tau_0) = p(\tau, z) - 2\pi \int_0^\tau p(\tau_0 + t, z) \, dt \]
\[ \times \left( e^{-\frac{1}{\kappa z}} \right) \frac{\kappa \tau}{\nu(z) \kappa_0} \, \frac{1}{\eta + z} \, d\eta. \quad (23) \]

For \( \tau_0 \gg 1 \), we find from this, using (22), that
\[ p(\tau, z, \tau_0) = p(\tau, z) - A \frac{z}{1 - k\kappa} D(\tau_0 - \tau, \eta) e^{-\kappa_0 z} \]
\[ - \frac{\kappa \tau}{2} \frac{\tau_0}{\kappa_0} \frac{e^{-\kappa_0 z}}{\eta + z} \, \, d\eta, \quad (24) \]

where
\[ D(\tau_0, \eta) = 2\pi \int_0^{\tau_0} p(\tau, \eta, \tau_0) \, d\eta. \quad (25) \]

Let us take \( \tau = 0 \) and \( \tau = \tau_0 \) in (24) in succession.

In view of the fact that for large \( \tau_0 \) we have (see [8])
\[ D(0, \tau_0) = 1 - \frac{1}{\kappa} \left( \frac{1}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \right), \]
\[ D(\tau_0, \tau_0) = \frac{C_0 e^{-\kappa_0 \tau_0}}{2\pi \kappa} \left( \frac{1}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \right), \quad (26) \]

after simple transformations and the use of (3) and (22) we finally obtain
\[ \varphi(z, \tau_0) = \varphi(z) - \frac{z}{1 - k\kappa} \frac{e^{-2\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \]
\[ - \frac{z}{1 + k\kappa} \frac{C_1}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \]
\[ + \frac{e^{-\kappa_0 \tau_0}}{a(z)\varphi(z)} \left( 1 + \frac{z}{1 + k\kappa} \frac{C_1}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \right). \quad (27) \]

These expressions generalize the asymptotic formulas (4) to the case of arbitrary positive \( z \).

It should be noted that we frequently have \( k\tau_0 \gg 1 \) or \( k\tau_0 \ll 1 \). In these limiting cases formulas (27) take the simpler form:

for \( k\tau_0 \gg 1 \)
\[ \varphi(z, \tau_0) = \varphi(z) - \frac{z}{1 + k\kappa} \frac{C_0}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \]
\[ \psi(z, \tau_0) = \psi(z) - \frac{z}{1 + k\kappa} \frac{C_1}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \]
\[ + \frac{e^{-\kappa_0 \tau_0}}{a(z)\varphi(z)} \left( 1 + \frac{z}{1 + k\kappa} \frac{C_1}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 - \frac{1}{2\kappa} C_0 e^{-2\kappa_0 \tau_0}} \right). \quad (28) \]

for \( k\tau_0 \ll 1 \),
\[ \varphi(z, \tau_0) = \varphi(z) - \frac{z}{\tau_0 + \gamma} \frac{C_0}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 + k\kappa}, \]
\[ \psi(z, \tau_0) = \psi(z) - \frac{z}{\tau_0 + \gamma} \frac{C_1}{a(z)\varphi(z)} \frac{e^{-\kappa_0 \tau_0}}{1 + k\kappa}. \quad (29) \]

Examples of the use of the above formulas will be given below.

II. Let us now discuss briefly the general character of the behavior of the functions \( \varphi(z, \tau_0) \) and \( \psi(z, \tau_0) \). In order to make this discussion clearer, we give in Fig. 2 the curves of \( \psi(z, \tau_0) \) as a function of \( z \) for \( \tau_0 = 1 \) and \( \tau_0 = 2.5 \). The curves refer to the case of pure scattering \( (\lambda = 1) \), and are based on the values of \( \psi(z, \tau_0) \) calculated by Sobouti [10].

The most characteristic feature of the curves shown in Fig. 2 is that the function \( \psi(z, \tau_0) \) at first varies comparatively rapidly and then, when \( z \) becomes appreciably larger than \( \tau_0 \), the curves become almost parallel to the axis of abscissas.

The same behavior is also observed in the case of \(\varphi(z, \tau_0)\) for \( \lambda = 1 \).

This suggests that if the arguments of the functions \( \varphi(z, \tau_0) \) and \( \psi(z, \tau_0) \) are taken to be \( \tau = \tau_0 / z \), then the form of the curves should be appreciably
simplified. This choice of variable is also suggested by the fact that in the basic equation (2) for the function \( \varphi(\tau, z, \tau_0) \) the variable \( z \) appears only in the combination \( \tau/z \).

In view of these considerations we have plotted \( \psi(\tau_0/t, \tau_0) \) as a function of \( t \) in Fig. 3. The figure immediately brings out an important property which had not been noticed up to the present, that in such coordinates the function \( \psi \) behaves much more smoothly. Thus, the problem of tabulation and interpolation is simplified. It is interesting to note that already at \( \tau_0 = 1 \) the curve \( \psi(\tau_0/t, \tau_0) \) differs little from the limiting value attained when \( \tau_0 = \infty \). The limiting curve was obtained in the following manner.

Using (1), it is easy to show that for pure scattering \( \varphi(z, \tau_0) = \psi(\infty, \tau_0) = \beta \psi(0, \tau_0) \), where

\[
\beta_0 = \int_0^1 \psi(\eta, \tau_0) d\eta.
\]  

(30)

With the help of (10) we therefore find that for large \( \tau_0 \)

\[
\psi(\infty, \tau_0) = \psi(\infty, \tau_0) = \frac{\sqrt{3}}{2}(\tau_0 + \gamma).
\]  

(31)

This expression can also be derived directly from (29). Let us note further that for \( \lambda = 1 \) and \( z \to \infty \)

\[
\varphi(z) = \sqrt{3}(z + \eta(z) + \ldots),
\]  

(32)

\[
a(z) = -\frac{1}{3z^2} - \frac{1}{5z^4} - \ldots
\]  

(33)

This follows from (5) and (21).

From (29), together with (31)–(33), we find that

\[
F_1(t) = \lim_{\tau_0 \to \infty} \psi(\tau_0/t, \tau_0) = \frac{2}{\sqrt{\pi}} (t - 1 + e^{-t}),
\]  

(34)

\[
F_2(t) = \lim_{\tau_0 \to \infty} \psi(\infty, \tau_0) = \frac{2}{\sqrt{\pi}} [1 - (1 + t)e^{-t}].
\]  

The second of these formulas was used for obtaining the curve in Fig. 3 corresponding to \( \tau_0 = \infty \).

The same analysis can be carried out when \( \lambda < 1 \). With the help of (28) it is easy to show that in this case

\[
F_1(t) = 1,
\]  

\[
F_2(t) = e^{-t}.
\]  

(35)

Thus, we can assert that the tabulation of the functions \( \varphi \) and \( \psi \) is most conveniently carried out in terms of the argument \( \tau = \tau_0/z \) and not simply \( z \). This assertion remains in force for the general \( X \) and \( Y \) functions.

III. In the study of the \( X \) and \( Y \) functions for completely incoherent scattering [4], values of the argument \( z \) greater than unity occur as an integral part of the problem. In this case there must also be limiting expressions analogous to (34) and (35). We will now derive them for the case when the coefficient of absorption in the line is a purely Doppler one, while absorption in the continuous spectrum is negligibly small.

As is known [4], the \( X \) and \( Y \) functions in this case satisfy the following system of equations, which are analogous to (1):

\[
X(z, \tau_0) = 1 + \lambda \frac{z}{\sqrt{\pi}} \int_0^z X(z', \tau_0) Y(z', \tau_0) \frac{Y'(z', \tau_0) G(z') dz'}{z'},
\]  

(36)

\[
Y(z, \tau_0) = e^{-\tau_0 z} \int_0^z X(z', \tau_0) Y(z', \tau_0) \frac{Y(z', \tau_0) G(z') dz'}{z'} + \frac{\lambda}{\sqrt{\pi}} \int_0^z X(z', \tau_0) Y(z', \tau_0) \frac{Y(z', \tau_0) G(z') dz'}{z'}
\]  

where

\[
G(z) = \sqrt{\frac{\pi}{2}} \left( 1 - \frac{z}{\sqrt{\pi}} \int_0^z e^{y^2} dy \right)
\]  

(37)

for \( z > 1 \) and \( G(z) = (2z)^{-1/2} \) for \( z \leq 1 \). In Eqs. (36) \( \tau_0 \) is the optical thickness at the center of the line. The physical meaning of the variable \( z \) can be seen from its definition: \( z = \eta/\alpha(x) \), where \( \alpha \) is the angle between the direction of propagation of radiation and the outward normal to the layer and \( \alpha(x) \) is the variation of the normalized absorption coefficient with frequency, i.e., the ratio of the absorption coefficient at frequency \( x \) to its value at the center of the line. The frequency \( x \) is measured to either side of the line center in units of the Doppler width, so that in the case under consideration of a purely Doppler absorption coefficient we have \( \alpha(x) = e^{-x^2} \).

The functions \( X(z, \tau_0) \) and \( Y(z, \tau_0) \) have a simple probabilistic interpretation; the quantities

\[
\lambda \int_0^1 e^{-x^2} X(\eta, \tau_0) d\eta dx \quad \text{and} \quad \lambda \int_0^1 e^{-x^2} Y(\eta, \tau_0) d\eta dx
\]

represent the probabilities that a quantum absorbed on the upper and lower boundaries respectively.
will emerge through the upper boundary at angle \( \arccos \eta \) to the normal within a solid angle \( d \omega \) having a frequency between \( x \) and \( x + dx \).

In addition to system (36), the functions \( X(z, \tau_0) \) and \( Y(z, \tau_0) \) also satisfy the following equations [4]:

\[
\frac{\partial X(z, \tau_0)}{\partial \tau_0} = Y(z, \tau_0) \Phi(\tau_0, \tau_0),
\]

\[
\frac{\partial Y(z, \tau_0)}{\partial \tau_0} = -\frac{1}{z} Y(z, \tau_0) + X(z, \tau_0) \Phi(\tau_0, \tau_0),
\]

\[
\Phi(\tau_0, \tau_0) = \frac{\lambda}{2} \int_0^\infty Y(z', \tau_0) G(z') dz'.
\]

and the boundary conditions

\[
X(z, 0) = 1,
\]

\[
Y(z, 0) = \begin{cases} 
0 & \text{for } z = 0, \\
1 & \text{for } z > 0.
\end{cases}
\]

Let us use these equations to obtain expressions of the type of (34) and (35). Let us take

\[
\frac{X(z, \tau_0) - 1}{X(\infty, \tau_0)} = f_1 \left( \frac{\tau_0}{z} \right) + S(z, \tau_0),
\]

\[
\frac{Y(z, \tau_0) - 1}{Y(\infty, \tau_0)} = f_2 \left( \frac{\tau_0}{z} \right) + R(z, \tau_0).
\]

When \( \tau_0 \) and \( z \) tend to infinity simultaneously in such a way that their ratio \( t = \tau_0 / z \) remains constant, the functions \( S(z, \tau_0) \) and \( R(z, \tau_0) \) must tend to zero. Our aim is therefore to obtain \( f_1(t) \) and \( f_2(t) \).

Let us first of all consider the more interesting case of pure scattering (\( \lambda = 1 \)). As can be easily seen from (38), in this case

\[
\Phi(\tau_0, \tau_0) = \frac{Y_0(\tau_0)}{Y_0(\tau_0)},
\]

\[
Y_0(\tau_0) = \frac{1}{X(\infty, \tau_0)} = \frac{1}{Y(\infty, \tau_0)} = \int_0^\infty Y(z', \tau_0) G(z') dz'.
\]

Using this, we find from (38) and (41) that

\[
\frac{1}{z} f_1 \left( \frac{\tau_0}{z} \right) (1 - Y_0(\tau_0)) + \frac{\delta S(z, \tau_0)}{\delta \tau_0} (1 - Y_0(\tau_0))
\]

\[
+ S(z, \tau_0) \Phi(\tau_0, \tau_0) = f_2 \left( \frac{\tau_0}{z} \right) \Phi(\tau_0, \tau_0) + R(z, \tau_0) \Phi(\tau_0, \tau_0),
\]

\[
\frac{1}{z} f_1' \left( \frac{\tau_0}{z} \right) + f_1 \left( \frac{\tau_0}{z} \right) \Phi(\tau_0, \tau_0) = -\frac{1}{z} f_2' \left( \frac{\tau_0}{z} \right) - \frac{1}{z} R(z, \tau_0)
\]

\[
+ f_1 \left( \frac{\tau_0}{z} \right) (1 - Y_0(\tau_0)) \Phi(\tau_0, \tau_0)
\]

\[
+ S(z, \tau_0) (1 - Y_0(\tau_0)) \Phi(\tau_0, \tau_0).
\]

If we now make use of the asymptotic expression for \( \Phi(\tau_0, \tau_0) \) valid for large \( \tau_0 \) obtained earlier [4],

\[
\Phi(\tau_0, \tau_0) = \frac{1}{z \tau_0}
\]

introduce the variable \( t = \tau_0 / z \) instead of \( z \) in (44), and then pass to the limit \( \tau_0 \to \infty \), then after simple transformations we obtain the following system of equations for the determination of \( f_1(t) \) and \( f_2(t) \):

\[
2t f_1'(t) + f_1(t) - f_2(t) = 0,
\]

\[
2t f_2'(t) + (1 + 2t) f_2(t) - f_1(t) = 0.
\]

From (41) it follows that

\[
f_1(0) = f_2(0) = 1.
\]

The solution of equations (46) with the boundary conditions (47) has the form

\[
f_1(t) = e^{-2t} \left[ I_0 \left( \frac{t}{2} \right) + I_1 \left( \frac{t}{2} \right) \right],
\]

\[
f_2(t) = e^{-2t} \left[ I_0 \left( \frac{t}{2} \right) - I_1 \left( \frac{t}{2} \right) \right],
\]

where \( I_n(x) \) are Bessel functions of an imaginary argument.

In the same way it is easy to show that for \( \lambda < 1 \)

\[
f_1(t) = 1, \quad f_2(t) = e^{-t}.
\]

It should be noted that the functions \( F_1(t) \) and \( F_2(t) \) given by formulas (34) satisfy the following system of differential equations:

\[
t F_1'(t) + F_1(t) - F_2(t) = 0,
\]

\[
t F_2'(t) + (1 + t) F_2(t) - F_1(t) = 0.
\]

This can be obtained in the same way as we have just derived Eqs. (46).

Formulas (48) and (49) are useful for the numerical solution of the equations governing the \( X \) and \( Y \) functions for isotropic scattering with a
complete frequency redistribution. However, they are also of independent interest. As an illustration of the use of formulas (48), we will consider the problem of incoherent scattering of light in a plane-parallel layer containing uniformly distributed radiation sources and $\lambda = 1$. Let us assume for definiteness that each unit volume receives an amount of energy $C$ from the sources and that the distribution with frequency of this energy is proportional to $\alpha(x)$.

It can be shown (see [3], chapter 6), that in this case the intensity of emerging radiation is given by

$$I(0, \eta, x) = I(t_0, \eta, x)$$

$$= \frac{C}{Y} x^{(\eta - 1)} X(\eta \epsilon, \eta \epsilon) - Y(\eta \epsilon, \eta \epsilon).$$

(51)

The investigation of the behavior of the functions $X$ and $Y$ for large $\tau_0$ [4] and the results of a direct numerical solution of the transport equation [11, 12] show that for $\tau_0 \gg 1$ the profile of the line has a characteristic double-humped shape. The positions of the intensity maxima are given by the condition

$$t = \frac{\tau_0 \epsilon}{\eta} \sim 1.$$

It should be noted that the quantity $t$ represents the optical thickness of the layer for quanta of frequency $x$ propagating along the direction $\cos^{-1} \eta$, the optical thickness being measured along the direction of propagation.

The results obtained above allow us to study the shape of the line for a layer of large optical thickness. Let us make $\tau_0$ and $x$ in (49) tend to infinity in such a way that the quantity $t = \tau_0 \epsilon x^2 / \eta$ remains constant. Using (41) and (48), we then find that

$$I(t) = \lim_{\eta \tau_0 \rightarrow \infty} \frac{\gamma \Pi I(0, \eta, x)}{C x^2 \tau_0}$$

$$= f_1(t) - f_2(t) = 2e^{-\alpha t} I(t) \left( \frac{t}{2} \right).$$

(52)

The function $I(t)$ is the limit approached by the relative distribution with frequency and angle of the emergent radiation as the thickness of the layer tends to infinity. The graph of $I(t)$ is given in Fig. 4. It can be seen from (52) that the value of $t$ corresponding to the maximum of $I(t)$ is given by the root of the equation

$$I_0 \left( \frac{t}{2} \right) - \left( 1 + \frac{2}{t} \right) I_1 \left( \frac{t}{2} \right) = 0.$$  

(53)

Calculations yield $t_{\text{max}} = 3.090$.

For small $t$, the function $I(t)$ is proportional to $t$. This result has the following significance. Inasmuch as the results obtained refer to the limiting case of large $\tau_0$, the quantity $t = \tau_0 \epsilon x^2 / \eta$, where $0 \leq \eta \leq 1$, will be small only for large $|x|$, i.e., in the wings of the line. Therefore, the behavior of $I(t)$ as $t \rightarrow 0$ reflects the shape of the line wings and the proportionality between $I(t)$ and $t$ just noted is physically obvious and signifies that the intensity in the line wings is proportional to the absorption coefficient. The decrease in $I(t)$ for $t > t_{\text{max}}$ reflects the presence of a depression at the center of the line.

It is clear that for finite values of $\tau_0$ the variation of the intensity of emerging radiation with the parameter $t$ (i.e., the line profile) will be somewhat different from that given by $I(t)$. However, except for a region close to the line center, this difference will be small.

An analogous analysis can be carried out in all cases when the intensity of emerging radiation is expressed in terms of the $X$ and $Y$ functions. This is the case when the distribution of radiation sources with depth is given by the product of a polynomial and an exponential function.

**LITERATURE CITED**

1. V. A. Ambartsumyan, Dokl. AN SSSR, 38, 257 (1943) (see also Collected Works [in Russian], 1, 232, Izd. AN ArmSSR, Erevan, 1960).


8. V. V. Sobolev, Dokl. AN SSSR, 155, 316 (1964) [Soviet Physics – Doklady, Vol. 9, p. 222].

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. Some or all of this periodical literature may well be available in English translation. A complete list of the cover-to-cover English translations appears at the back of this issue.