LIGHT SCATTERING IN A PLANE LAYER

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The problem of isotropic light scattering in a plane layer is considered. The probabilistic approach is used. It is shown that the solution of the problem of light scattering in a half-space can be expressed in terms of the Ambartsumyan's function \( \varphi(\eta) \) and the solution of the problem of a point source in an infinite medium \( B_\infty(\tau) \). The exact solution of the integral equation describing light scattering in a layer of finite optical thickness \( \tau_0 \) is found. Its resolvent is expressed in terms of \( B_\infty(\tau) \) and the Ambartsumyan's functions \( \varphi(\eta, \tau_0) \) and \( \psi(\eta, \tau_0) \). The results are easily extended to include the case of isotropic scattering with complete frequency redistribution.

\section*{Introduction}

It is well known that the problem of the multiple scattering of light in a plane layer in the case of a spherical indicatrix of scattering essentially involves solution of the following integral equation for the source function \( B(\tau) \):

\[
B(\tau) = \frac{\lambda}{2} \int_0^\infty e^{-\frac{\tau}{\tau_0}} \frac{d\tau'}{\tau'} \int B(\tau') d\tau' + g(\tau),
\]  

(1)

where

\[
E_i(t) = \int_0^\infty \frac{e^{-\frac{\tau}{\tau_0}}}{\tau} d\tau,
\]  

(2)

\( \tau \) is the optical depth read along the normal from the boundary, \( \tau_0 \) is the optical thickness of the layer, \( \lambda \leq 1 \) is the probability of "survival" of a quantum during scattering and \( g(\tau) \) characterizes the distribution of sources in the layer.

A large number of studies have been devoted to investigation of Eq. (1). An exposition of the results can be found in any monograph on the theory of multiple scattering of radiation or neutrons (for example, see [1–4]). Solution of equation (1) has been found at this time only for the case \( \tau_0 = \infty \).

The Ambartsumyan function \( \varphi(\eta) \) enters into this solution. This paper gives a precise solution of equation (1) for the case of finite \( \tau_0 \). The Ambartsumyan functions \( \varphi(\eta, \tau_0) \) and \( \psi(\eta, \tau_0) \) [5] are used in the solution.

The content of the paper is as follows. We first (Section 1) cite known relations applying to an infinite medium. A simple method is proposed thereafter for finding a precise solution of the problem of light scattering in a semi-infinite medium (Section 2). This method makes it easy to find the resolvent of equation (1) for finite \( \tau_0 \). This is done in Section 3. At the end of the article the results are applied to the case of completely incoherent scattering.

\section{1. Infinite Medium}

The radiation field of an isotropic point source, situated in an infinite homogeneous medium, is known (for example, see [4]). Its computation essentially involves solution of the following integral equation for the source function \( B_\infty(\tau) \):

\[
B_\infty(\tau) = \frac{\lambda}{4\pi} \int B_\infty(\tau') e^{-\frac{\tau}{\tau_0}} \frac{d\tau'}{|\tau - \tau'|^2} + \frac{\lambda \sigma^2}{4\pi} \frac{e^{-\tau}}{\tau^2},
\]  

(3)

where \( \sigma \) is the absorption coefficient, \( \tau = |\tau| \) is the optical radius vector, \( \tau = \sigma \tau \) is the optical distance from the source. It is assumed that the source has the intensity \( 4\pi \) and is situated at the origin of coordinates. The integration in (3) is extended to all space.

Solution of equation (3) is found easily using a Fourier transform. It is found that
\[ B_\infty(\tau) = \frac{\sigma^2}{4\pi \tau} \left\{ 2kA(k)e^{-k\tau} + \frac{\lambda}{4\pi} F(x)e^{-x\tau} dx \right\} \] (4)

where \( k \) is the positive root of the equation

\[ \frac{\lambda}{2k} \ln \frac{1 + k}{1 - k} = 4 \] (5)

and

\[ A(k) = \frac{k(1 - k^2)}{\lambda + k^2 - 1} \] (6)

\[ F(x) = 4x \left[ (\lambda x)^2 + \left( 2x + \lambda \ln \frac{x - 1}{x + 1} \right)^2 \right]^{-1}. \] (7)

In the case of an arbitrary distribution of radiation sources in an infinite homogeneous medium the source function can be obtained from \( B_\infty(\tau) \) by simple integration. Later we will require a case in which the sources are distributed uniformly along a certain plane. Assume their intensity per unit area is \( 4\pi \). Then the corresponding source function \( \Phi_\infty(\tau) \) is derived from \( B_\infty(\tau) \) by integration along this plane (\( \tau \) is the distance from the considered point to the plane)

\[ \Phi(\tau) = 2\pi \int_0^\infty B_\infty(\tau') r dr, \] (8)

where

\[ \tau^2 = \tau^2 + \sigma_0 \tau. \] (9)

Thus,

\[ \Phi_\infty(\tau) = 2\pi \int_0^\infty B_\infty(\tau') \tau' d\tau'. \] (10)

It is easy to find from (3) and (10) that \( \Phi_\infty(\tau) \) should satisfy the equation

\[ \Phi_\infty(\tau) = \frac{\lambda}{2} \int_{-\infty}^{\infty} E_i(|\tau - \tau'|) \Phi_\infty(\tau') d\tau' + \frac{\lambda}{2} E_i(|\tau|). \] (11)

This equation directly follows also from the physical sense of \( \Phi_\infty(\tau) \).

By substituting (4) into (10), we obtain the precise solution (11)

\[ \Phi_\infty(\tau) = A(k)e^{-k\tau} + \frac{\lambda}{2} \int_1^\infty F(x)e^{-x\tau} dx. \] (12)

We denote the resolvent of equation (11) by \( \Gamma_\infty(\tau, \tau') \), that is, the solution of the equation is

\[ \Gamma_\infty(\tau, \tau') = \frac{\lambda}{2} \int_{-\infty}^{\infty} E_i(|\tau - t|) \Gamma_\infty(t, \tau') dt + \frac{\lambda}{2} E_i(|\tau - \tau'|). \] (13)

It is clear that

\[ \Gamma_\infty(\tau, \tau') = \Gamma_\infty(\tau', \tau) \] (14)

and

\[ \Gamma_\infty(\tau + x, \tau' + x) = \Gamma_\infty(\tau, \tau'). \] (15)

Therefore

\[ \Gamma_\infty(\tau, \tau') = \Gamma_\infty(0, \tau - \tau') = \Phi_\infty(\tau - \tau'). \] (16)

The relation (16), together with formula (12), gives an explicit expression for \( \Gamma_\infty(\tau, \tau') \).

The results cited in this section are needed below.

2. Semiinfinite Medium

The full determination of the radiation field in a semiinfinite layer with an arbitrary (dependent only on \( \tau \)) distribution of radiation sources requires determination of the resolvent \( \Gamma(\tau, \tau') \) of Eq. (1) in the case \( \tau_0 = \infty \), that is, solution of the equation

\[ \Gamma(\tau, \tau') = \frac{\lambda}{2} \int_{-\infty}^{\infty} E_i(|\tau - t|) \Gamma(t, \tau') dt + \frac{\lambda}{2} E_i(|\tau - \tau'|). \] (17)

When the resolvent \( \Gamma(\tau, \tau') \) is known, the source function is found by simple integration:

\[ B(\tau) = g(\tau) + \int_{-\infty}^{\infty} g(t) \Gamma(t, \tau') dt. \] (18)

It has been demonstrated ([4, p. 107], [3, p. 225], [6]) that in order to obtain \( \Gamma(\tau, \tau') \) it is sufficient to find the function \( \Phi(\tau) = \Gamma(0, \tau) = \Gamma(\tau, 0) \), since

\[ \Gamma(\tau, \tau') = \frac{\lambda}{2} \int_{-\infty}^{\infty} E_i(|\tau - t|) \Phi(\tau') dt + \frac{\lambda}{2} E_i(|\tau - \tau'|). \] (19)
The following equation was derived for $\Phi(\tau)$:

$$
\Phi(\tau) = N(\tau) + \int_0^\tau N(\tau - \tau') \Phi(\tau') d\tau',
$$

(20)

where

$$
N(\tau) = \frac{\lambda}{2} \int_0^\tau e^{-\frac{\tau}{\eta}} \varphi(z) \frac{dz}{z},
$$

(21)

and $\varphi(\eta)$ is the Ambartsumyan function, being the solution of the equation

$$
\varphi(\eta) = 1 + \frac{\lambda}{2} \eta \varphi(\eta) \int_0^\eta \frac{\varphi(z)}{\eta + z} dz.
$$

(22)

Equation (20) can be solved with a Laplace transform. The following expression was derived [7] for $\Phi(\tau)$ in this manner:

$$
\Phi(\tau) = C(k) e^{-k\tau} + \frac{\lambda}{2} \int_0^\infty F(x) e^{-x} \frac{dx}{\varphi\left(\frac{1}{x}\right)},
$$

(23)

where

$$
C(k) = \left[\frac{\lambda}{2} \int_0^1 \frac{\eta \varphi(\eta)}{(1-k\eta)^2} d\eta\right]^{-1},
$$

(24)

$k$ is the positive root of the equation (5), and $F(x)$ is given by formula (7). The function $\varphi(\eta)$ entering into (23) can be considered known, since it was derived in explicit form (for example, see [2, Chapter 6], [3, Chapter 4]) and tabulated in detail [8]. Thus, the problem of isotropic scattering of light in a semiinfinite medium for any distribution of radiation sources is solved completely.

In the solution method discussed the most difficult step is the inversion of the Laplace transform, giving the function $\Phi(\tau)$. However, when the solution of the problem of a point source in an infinite medium is known, this step can be avoided easily. Simple reasoning makes it possible to express $\Phi(\tau)$ directly through $\Phi_{\infty}(\tau)$ and $\varphi(\eta)$. This is possible because the entire physics of the problem is contained in $\Phi_{\infty}(\tau)$ and it is necessary to take into account only the difference in the geometry between the infinite and semiinfinite media.

We will use the probabilistic concepts introduced into the light scattering theory by V. V. Sobolev [9, 3]. The values $\Gamma_{\infty}(\tau, \tau') d\tau$ and $\Gamma(\tau, \tau') d\tau'$ represent the probabilities that a quantum, absorbed at the depth $\tau$ in infinite and semiinfinite media respectively, will be absorbed again between the depths $\tau + d\tau$ and $\tau' + d\tau'$ after multiple scatterings. The physical sense of the function $\Phi(\tau)$ is clear from the fact that $\Phi(\tau) = \Gamma(0, \tau) = \Gamma(\tau, 0)$. We will denote by $p(\tau, \eta) d\omega$ the probability that a quantum, absorbed at the depth $\tau$ in a semiinfinite medium, will emerge from it at the angle $\arccos \eta$ to the normal within the solid angle $d\omega$ (both directly and after scattering). The function $p(\tau, \eta)$ satisfies the equation

$$
p(\tau, \eta) = \frac{\lambda}{2} \int_0^\infty E_1(\tau - \tau') p(\tau', \eta) d\tau' + \frac{\lambda}{4\pi} \frac{1}{\eta}.
$$

(25)

It can be demonstrated easily that in the case $\tau, \tau' \geq 0$ the following relation applies:

$$
\Gamma_{\infty}(\tau, \tau') = \int_0^\infty \int_0^\infty p(\tau, \eta) d\eta \int_0^\infty e^{-\frac{t}{\eta}} \Gamma_{\infty}(\tau', \tau') dt.
$$

(26)

In actuality, in an infinite medium a quantum can pass from the $\tau \geq 0$ to the level $\tau' \geq 0$ in two ways. First, it can diffuse in some way in the half-space $\tau \geq 0$ without intersecting the plane $\tau = 0$. This gives the term $\Gamma_{\tau}(\tau, \tau')$ in the right-hand side of (26). Second, the quantum can describe a path intersecting the plane $\tau = 0$. To do so it should emerge from the depth $\tau$ from the half-space $\tau > 0$ at a certain angle $\eta < \pi/2$, travel some path in the half space $\tau < 0$ without scattering, be absorbed at some level $\tau = -t < 0$ and then pass from this level into the level $\tau' \geq 0$ (generally speaking, again after diffusion in the entire space). These considerations make it possible to write a second term in the right-hand side of (26).

We now assume $\tau = 0$ in (26), first noting that

$$
p(0, \eta) = \frac{\lambda}{4\pi} \varphi(\eta).
$$

We obtain

$$
\Phi_{\infty}(\tau') = \Phi(\tau') + \frac{\lambda}{2} \int_0^\tau \varphi(\eta) d\eta \int_0^\infty e^{-\frac{t}{\eta}} \Gamma_{\infty}(\tau', -t) dt.
$$

(27)

Using (16) and replacing the notation of the independent variable $\tau'$ by $\tau$, we find finally

$$
\Phi(\tau) = \frac{\lambda}{2} \int_0^\tau \Phi_{\infty}(\tau + t) dt \int_0^\infty e^{-\frac{t}{\eta}} \varphi(\eta) d\eta.
$$

(28)
This is the required relation expressing $\Phi(\tau)$ through $\Phi_{\infty}(\tau)$ and $\varphi(\eta)$.

Formula (23) is a direct corollary of (23). In order to obtain (23) it is necessary to substitute into (23) the explicit expression for $\Phi_{\infty}(\tau)$ from (12) and integrate for $t$. Using (22) and taking into account that

$$A(k) \left( 1 - \frac{\lambda}{2} \sum_{n=1}^{\infty} \frac{\Phi(\eta) d\eta}{1 + \lambda \eta} \right) = C(k),$$

(29)

we arrive at (23). The relation (29) can be derived using (22) [7].

Thus, in accordance with (23), by determining the radiation field of a point source in an infinite medium and determining the function $\varphi(\eta)$, we obtain a solution of the problem of light scattering in a seminfinite medium.

We note that considerations similar to those used in derivation of formula (26) also make it possible to write the following relation:

$$\Gamma(\tau, \tau') = \Gamma_{\infty}(\tau, \tau') - \int_{0}^{\infty} \Phi(\tau + t) \Phi(\tau' + t) dt.$$ (30)

This relation expresses the fact that in a semi-infinite medium, in contrast to a finite medium, the paths of quanta intersecting the plane $\tau = 0$ are "forbidden." The value $\Phi(\tau + t) \Phi(\tau' + t) dt d\tau'$ is the probability that a quantum, absorbed at the depth $\tau \geq 0$ in an infinite medium, subsequent to diffusion will be absorbed between $\tau'$ and $\tau' + d\tau'$ ($\tau' \leq 0$), experiencing scattering between $-t$ and $-t + dt$ ($t > 0$) and describing in the medium a path not entering into the half-space $\tau < -t$. The integral term in (30) therefore gives the probability of the transition of a quantum from the level $\tau$ to $\tau'$ in an infinite medium along those paths which intersect the plane $\tau = 0$.

It is interesting that formula (19) also can be written at once on the basis of similar considerations. The obvious advantage of (30) over (19) is the symmetry of this expression.

Relations (28) and (30) were derived earlier by Case [10] in a different manner. The approach used in our article made it possible to derive them more simply. The simple physical sense of these relations has become clear at the same time. Most importantly, this approach makes it possible to solve fully the problem of light scattering in a plane layer of finite thickness. The next section is devoted to this solution.

3. Layer of Finite Thickness

We will denote the resolvent of equation (1) by $\Gamma(\tau, \tau'; \tau_0)$. It can be shown [11] that to find the resolvent it is sufficient to determine the function $\Phi(\tau; \tau_0) = \Gamma(0, \tau; \tau_0) = \Gamma(\tau, 0; \tau_0)$, since

$$\Gamma(\tau, \tau'; \tau_0) =$$

(31)

$$\Phi(|\tau - \tau'|; \tau_0) + \int_{0}^{\min(\tau, \tau')} \Phi(x + |\tau - \tau'|; \tau_0) \Phi(x; \tau_0) dx - \Phi(\tau_0 - x - |\tau - \tau'|; \tau_0) \Phi(\tau_0 - x; \tau_0) dx.$$ (32)

Until now the function $\Phi(\tau, \tau_0)$ has not been found in explicit form. We now will express it through $\Phi_{\infty}(\tau)$ and the Ambartsumyan functions $\varphi(\eta; \tau_0)$ and $\psi(\eta; \tau_0)$.

Using the probabilistic sense of the resolvent, it is easy to show that when $0 \leq \tau, \tau' \leq \tau_0$ the following relation should apply:

$$\Gamma_{\infty}(\tau, \tau') = \Gamma(\tau, \tau'; \tau_0)$$

$$+ 2\pi \int_{0}^{\infty} \rho(\tau, \eta; \tau_0) \int_{0}^{\infty} e^{-\eta t} \Gamma_{\infty}(-t, \tau') dt$$

$$+ 2\pi \int_{0}^{\infty} \rho(\tau_0 - \tau, \eta; \tau_0) \int_{0}^{\infty} e^{-\eta \tau_0 + t} \Gamma_{\infty}(\tau_0 + t, \tau') dt.$$ (32)

Here $\rho(\tau, \eta, \tau_0)$ is the probability of emergence of a quantum from a layer of the thickness $\tau_0$, that is, the solution of the equation is

$$p(\tau, \eta; \tau_0) =$$

$$\frac{\lambda}{2} \int_{0}^{\tau_0} E_1(|\tau - \tau'|) p(\tau', \eta; \tau_0) d\tau' + \frac{\lambda}{4\pi} e^{-\eta \tau}.$$ (33)

The first term in (32) gives the probability of transition of a quantum from level $\tau$ to $\tau'$ in an infinite medium along paths not intersecting the planes $\tau = 0$ and $\tau = \tau_0$. The second term takes into account paths which intersect the plane $\tau = 0$ at least once; the third term takes into account paths intersecting the plane $\tau = \tau_0$.

Assuming in (32) that $\tau = 0$ and replacing the variable $\tau'$ by $\tau$, we obtain
The results obtained in the preceding sections in many respects apply to the case of isotropic scattering with complete frequency redistribution. If there is no absorption in the continuous spectrum, instead of (1) for the source function we will have the following equation (for example, see [17, 18]):

$$B(\tau) = \frac{\lambda}{2} \int \frac{d\tau'}{\tau - \tau'} B(\tau') d\tau' + g(\tau), \quad (37)$$

where

$$K(t) = A \int_{-\infty}^{+\infty} a^2(x)E_1(a(x)t) dx, \quad (38)$$

\(\tau\) is the optical depth at the center of the line, \(\alpha(x)\) is the contour of the absorption coefficient, that is, the ratio of the absorption coefficient at the frequency \(x\) to the absorption coefficient at the center of the line, \(\alpha(x)\) is the distance from the center of the line. The nucleus (38) of Eq. (37) can be reduced [16] to the form

$$K(t) = \int_{-\infty}^{+\infty} e^{-\frac{t}{x^2}} G(x') dx', \quad (40)$$

where

$$G(z) = 2A \int_{0}^{z} a^2(t) dt, \quad (41)$$

with \(x(z) = 0\) when \(z \leq 1\) and \(\alpha(x(z)) = 1/2\) when \(z \geq 1\).

In particular, for the Doppler absorption coefficient \(\alpha(x) = e^{-x^2}\), we have

$$G(z) = \frac{4}{\sqrt{\pi}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2} du \right), \quad (42)$$

when \(z \geq 1\) and \(G(z) = 1/2\) when \(z \leq 1\). If the absorption coefficient is determined fully by attenuation, \(\alpha(x) = 1/(1 + x^2)\) (in this case \(x\) denotes the distance from the center of the line, expressed in widths, caused by attenuation). For this absorption coefficient

$$G(z) = \frac{1}{2} - \frac{1}{4} \frac{1}{\pi} \arctg \frac{z-1}{z} - \frac{4}{\pi} \frac{\sqrt{z-1}}{z} \quad (43)$$

when \(z \geq 1\) and \(G(z) = 1/2\) when \(z \leq 1\).
Equation (37) can be investigated in the very same way as Eq. (1) was analyzed above. The first step is to find the radiation field of a point source in an infinite homogeneous medium scattering quanta with complete frequency redistribution. This requires solution of an equation similar to (3):

\[ B_{\infty}(\tau) = \frac{\lambda A}{4\pi} \int \frac{B_{\infty}(\tau')}{r^2} d\tau' + \frac{\lambda A \sigma}{2} \frac{1}{r^2} \int_{-\infty}^{+\infty} a^2(x) e^{-\alpha(x)|r-r'|} dx, \quad (44) \]

where \( \sigma \) is the absorption coefficient at the line center; integration for \( r' \) extends to the entire space.

If \( B_{\infty}(\tau) \) is determined, in order to obtain the resolvent of Eq. (37) when \( \tau = \infty \) the only necessary additional step is to find the function \( H(z) \) satisfying the equation

\[ H(z) = 1 + \frac{\lambda}{2} z H(z) \int_{0}^{\infty} \frac{H(z')}{z + z'} G(z') dz'. \quad (45) \]

This function has the same role in this equation as \( \varphi(\eta) \) in Eq. (1).

In order to obtain a precise solution of (37) for finite \( \tau_0 \), it is necessary to know \( B_{\infty}(\tau) \) and have the functions \( X(z; \tau_0) \) and \( Y(z; \tau_0) \), similar to the Ambartsumyan functions \( \psi(\eta; \tau_0), \psi(\eta; \tau_0) \).

Equation (37) was studied earlier by the author in [18–20]. A detailed investigation of the function \( H(z) \) was made for absorption coefficients of several special types (Doppler, caused only by attenuation, and Voigt). \( H(z) \) was tabulated for the Doppler absorption coefficient. The functions \( X(z; \tau_0) \) and \( Y(z; \tau_0) \) also were studied. In particular, asymptotic expressions for these functions were obtained for large \( \tau_0 \). Work now is being done on tabulation of these functions [for \( \alpha(x) = e^{-x^2} \)]. Thus, the full solution of equation (37) requires more than a knowledge of the function \( B_{\infty}(\tau) \), determined by Eq. (44). It would appear that the solution of this equation [at least for \( \alpha(x) = 1/(1 + x^2) \)] involves no particular difficulty. The author intends to consider this problem in a future paper.

The relations derived in this study on the basis of simple physical considerations also can be derived in a purely formal manner.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. Some or all of this periodical literature may well be available in English translation. A complete list of the cover-to-cover English translations appears at the back of this issue.