NON-RADIAL OSCILLATIONS OF GASEOUS MASSES

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ABSTRACT

In this paper the fundamental frequencies of non-radial oscillations of polytropic gas spheres, belonging to spherical harmonics of orders \( l = 1 \) and \( 2 \), are evaluated, in a “second approximation,” by a variational method. Also, the value of the ratio of the specific heats \( \gamma \) is determined for which an accidental degeneracy occurs between the fundamental modes of radial oscillation and non-radial oscillation belonging to \( l = 2 \); it is found that this value of \( \gamma \) varies from 1.6 for a homogeneous compressible sphere to 1.5719 for a polytrope of index \( n = 3 \).

I. INTRODUCTION

In a recent paper (Chandrasekhar and Lebovitz 1963; this paper will be referred to hereafter as “Paper I”) the non-radial oscillations of gaseous masses belonging to the spherical harmonics of orders \( l = 1 \) (in a “second approximation”) and \( l = 3 \) (in a “first approximation”) were considered on the basis of the virial equations of orders 1 and 3 and trial functions of suitable forms for the Lagrangian displacement. The modes of oscillation belonging to \( l = 2 \) (in a first approximation) had been considered earlier (Chandrasekhar and Lebovitz 1962a, c); since their evaluation in a second approximation by the method of Paper I would have required the use of the virial equations of order 4, as well, it was not attempted. However, soon afterward, a general variational principle governing the non-radial oscillations of gaseous masses and belonging to the spherical harmonics of the different orders was established (Chandrasekhar 1964; this paper will be referred to hereafter as “Paper II”). In this paper the variational principle will be used to evaluate the modes belonging to \( l = 2 \) in a second approximation with the principal object of determining, more precisely than hitherto, the value of the ratio of the specific heats \( \gamma \) for which an accidental degeneracy occurs between the fundamental modes of radial oscillation and non-radial oscillation belonging to \( l = 2 \). The facts that such a degeneracy must occur and, further, that it may lie at the base of the beat phenomenon exhibited by the \( \beta \) Canis Majoris stars have been pointed out (Chandrasekhar and Lebovitz 1962a, b, and d); and the value \( \gamma = 1.6 \) (independently of the structure of the configuration) for the occurrence of such a degeneracy was deduced by equating the first approximations to the characteristic frequencies given by the two formulae (Ledoux 1945; Chandrasekhar and Lebovitz 1962a)

\[
\sigma_R = \frac{3(\gamma - 4)}{5} \frac{|\Phi|}{I} \quad \text{and} \quad \sigma^2 = \frac{4}{5} \frac{|\Phi|}{I},
\]

where \( \Phi \) denotes the gravitational potential energy and \( I \) the moment of inertia of the configuration. By restricting ourselves to polytropic gas spheres and evaluating \( \sigma_R^2 \) and \( \sigma_s^2 \) in a second approximation, we shall determine the dependence of \( \gamma \) (for which \( \sigma_R^2 = \sigma_s^2 \)) on the density distribution in the configuration.

II. THE CHARACTERISTIC EQUATION IN THE SECOND APPROXIMATION

As shown in Paper II (Sec. III) the non-radial oscillations of gaseous masses, belonging to spherical harmonics of a particular order \( l \), are determined by two radial functions

\[ \sigma_R^2 = 0.054251 \] (in place of 0.054834); the last of the equations in (87) should read \( \sigma^4 - 0.12097 (\gamma - 1.04641)\sigma^2 + 0.0011820 (\gamma - 1.2784) = 0 \); and finally in Table 1 on p. 199 the entry opposite \( \psi d \xi \) in the last column should read 160.393 (in place of 158.689).
ψ and χ. And the variational principle, which expresses σ² as a functional of ψ and χ, requires that σ² be stationary with respect to arbitrary, infinitesimal variations of ψ and χ compatible with the boundary conditions.

Now it can be deduced from Pekeris' (1939) exact treatment of the non-radial oscillations of a homogeneous compressible sphere that the proper solutions for ψ and χ, in that case, are of the forms

\[ \psi = Ar^{l+1} + Br^{l+3} \quad \text{and} \quad \chi = Ar^{l+1} + Cr^{l+3}. \] (2)

We shall accordingly assume for ψ and χ these forms and treat the constants A, B, and C, as variational parameters.

It will be observed that we have made the coefficients of \( r^{l+1} \) equal in the assumed forms of ψ and χ. This equality in the coefficients follows, as it has been pointed out to us by Professor Paul H. Roberts, from the requirement that div χ, as given by equation (33) of Paper II, should behave like \( r^l \) at the origin—a fact which becomes apparent when we examine the behavior at the origin of the expression for \( \delta \rho \) given by equation (35) of Paper II.

By substituting for ψ and χ the chosen forms in Paper II, equation (41), we find that the characteristic equation for \( \sigma^2 \) becomes

\[ |M| = 0, \] (3)

where \( M \) is a symmetric matrix whose elements are

\[ M_{11} = \sigma^2 \frac{2l+1}{l} \rho_{2l} - 2(2l-1)(l+1) \rho_{2l-2} - V_{2l+2} \]
\[ + 4\pi G \int_0^R r^{2l} [F_{l+1} - (l+1) f]^2 \, dr, \]

\[ M_{12} = \sigma^2 \rho_{2l+2} - (2l+1)(l+1) \rho_{2l} - V_{2l+4} \]
\[ + 4\pi G \int_0^R r^{2l} F_{l+3} [F_{l+1} - (l+1) f] \, dr, \]

\[ M_{13} = \sigma^2 \frac{l+3}{l} \rho_{2l+2} - (2l+1)(l+3) \rho_{2l} \]
\[ - 4\pi G (l+3) \int_0^R r^{2l} g [F_{l+1} - (l+1) f] \, dr, \]

\[ M_{22} = \sigma^2 \rho_{2l+4} - \gamma (l+3)^2 \rho_{2l+2} - V_{2l+6} + 4\pi G \int_0^R r^{2l} F_{l+3}^2 \, dr, \]

\[ M_{23} = (l+3) [ (l+3) \gamma - (2l+3) ] \rho_{2l+2} - 4\pi G (l+3) \int_0^R r^{2l} F_{l+3} g \, dr, \]

\[ M_{33} = \sigma^2 \frac{(l+3)^2}{l(l+1)} \rho_{2l+4} - \gamma (l+3)^2 \rho_{2l+2} + 4\pi G (l+3)^2 \int_0^R r^{2l} g^2 \, dr, \]

Footnote: This circumstance, that \( \chi \) must behave like \( r^{l+3} \) as \( r \to 0 \), makes the reduction of the variational principle given by eq. (41) of Paper II to the form of a determinant of a 2 × 2 matrix (by varying ψ and χ independently of each other) strictly incorrect. However, it appears that the determinantal form retains a certain usefulness in providing an adequate approximation (see n. 4 on p. 1522) and clarifying the origin of the two physically distinct classes of modes.
and the following abbreviations have been used:

\[ \rho_m = \int_0^R \rho r^m dr, \quad \rho_m = \int_0^R \rho r^m dr, \]

\[ V_m = \int_0^R \frac{1}{\rho} \frac{d\rho}{dr} \frac{d}{dr} (\rho r^m) \frac{dr}{r^2} = \int_0^R \frac{d\rho}{dr} \frac{d}{dr} (\rho r^m) \frac{dr}{r^2} \]

\[ f(r) = \int_r^R \frac{ds}{s} \rho(s), \quad g(r) = \int_r^R ds s \rho(s), \]

and

\[ F_{l+1}(r) = \int_r^R \frac{ds}{s^{l+1}} \frac{d}{ds} (\rho s^{l+1}) = - \rho + (l+1) f(r), \]

\[ F_{l+3}(r) = \int_r^R \frac{ds}{s^{l+3}} \frac{d}{ds} (\rho s^{l+3}) = - \rho r^2 + (l+1) g(r). \]

The integrals which are explicitly written out in equations (4) can be reduced by integrations by parts and expressed in terms of the moments,

\[ f_m = \int_0^R f \rho r^m dr, \quad g_m = \int_0^R g \rho r^m dr. \]

We find

\[ \int_0^R r^{2l} f_2^2 dr = \frac{2}{2l+1} f_{2l}, \]

\[ \int_0^R r^{2l} f g dr = \frac{1}{2l+1} (f_{2l+2} + g_{2l+2}), \]

\[ \int_0^R r^{2l} g^2 dr = \frac{2}{2l+1} g_{2l+2}, \]

\[ \int_0^R r^{2l} F_{l+1} f dr = \frac{1}{2l+1} f_{2l}, \]

\[ \int_0^R r^{2l} F_{l+1} g dr = \frac{1}{2l+1} [(l+1) f_{2l+2} - l g_{2l}], \]

\[ \int_0^R r^{2l} F_{l+3} f dr = \frac{1}{2l+1} [(l+1) g_{2l} - l f_{2l+2}], \]

\[ \int_0^R r^{2l} F_{l+3} g dr = \frac{1}{2l+1} g_{2l+2}, \]

\[ \int_0^R r^{2l} F_{2l+1} dr = \int_0^R r^{2l} \rho^2 dr - \frac{2l(l+1)}{2l+1} f_{2l}, \]

\[ \int_0^R r^{2l} F_{l+1} F_{l+3} dr = \int_0^R r^{2l+2} \rho^2 dr - \frac{l(l+1)}{2l+1} (f_{2l+2} + g_{2l}), \]

\[ \int_0^R r^{2l} F_{2l+3} dr = \int_0^R r^{2l+4} \rho^2 dr - \frac{2l(l+1)}{2l+1} g_{2l+2}. \]
The integral defining $V_m$ (eq. [6]) can also be reduced by integrations by parts and by making use of Poisson's equation and the equation governing equilibrium. Thus,

$$V_m = \int_0^R \frac{1}{r^4} \left( \frac{dN}{dr} \frac{d}{dr} (\rho r^m) \right) dr$$

$$= - \int_0^R \rho r^m \left[ \frac{1}{r^4} \frac{d}{dr} (\frac{dN}{dr} \frac{d}{dr} (\rho r^m)) - \frac{4}{r^3} \frac{dN}{dr} \right] dr$$

$$= \int_0^R \rho r^m \left( \frac{4\pi G m}{r^2} + \frac{4}{r^3} \frac{d}{dr} (\rho) \right) dr$$

$$= 4\pi G \int_0^R \rho^2 r^{m-2} dr - 4(m - 3) \rho_{m-4} \quad (m \geq 4).$$

Using equations (10) and (11), we find that the expressions for the matrix elements of $M$ become

$$M_{11} = \sigma^2 \frac{2l+1}{l} \rho_{2l} - 2(2l-1)(l-1) \rho_{2l-2},$$

$$M_{12} = \sigma^2 \rho_{2l+2} - (2l+1)(l-3) \rho_{2l} - 4\pi G (l+1) g_{2l},$$

$$M_{13} = \sigma^2 \frac{l+3}{l} \rho_{2l+2} - (2l+1)(l+3) \rho_{2l+1} + 4\pi G (l+3) g_{2l},$$

$$M_{22} = \sigma^2 \rho_{2l+4} - [\gamma(l+3)^2 - 4(2l+3)] \rho_{2l+3} - 8\pi G \frac{l(l+1)}{2l+1} g_{2l+2},$$

$$M_{23} = (l+3)[(l+3)\gamma - (2l+3)] \rho_{2l+3} - 4\pi G \frac{l+3}{2l+1} g_{2l+2},$$

$$M_{33} = \sigma^2 \frac{(l+3)^2}{l(l+1)} \rho_{2l+4} - \gamma(l+3)^2 \rho_{2l+4} + 8\pi G \frac{(l+3)^2}{2l+1} g_{2l+2}. $$

The approximation considered by Cowling (1942) and others of neglecting the variations in the gravitational potential during the oscillations is equivalent, in the present context, to ignoring the integrals over $F_{l+1}, F_{l+2}, f$, and $g$ in the expressions for the matrix elements of $M$ given in equations (4). Ignoring, then, these terms and substituting for the remaining terms in $V_m$ in accordance with equation (11), we obtain the simpler expressions

$$M_{11} = \sigma^2 \frac{2l+1}{l} \rho_{2l} - 2(2l-1)(l-1) \rho_{2l-2} - 4\pi G \int_0^R \rho^2 r^{2l} dr,$$

$$M_{12} = \sigma^2 \rho_{2l+2} - (2l+1)(l-3) \rho_{2l} - 4\pi G \int_0^R \rho^2 r^{2l+2} dr,$$

$$M_{13} = \sigma^2 \frac{l+3}{l} \rho_{2l+2} - (2l+1)(l+3) \rho_{2l+1} + 4\pi G (l+3) g_{2l},$$

$$M_{22} = \sigma^2 \rho_{2l+4} - [\gamma(l+3)^2 - 4(2l+3)] \rho_{2l+3} - 8\pi G \frac{l(l+1)}{2l+1} g_{2l+2},$$

$$M_{23} = (l+3)[(l+3)\gamma - (2l+3)] \rho_{2l+3} - 4\pi G \frac{l+3}{2l+1} g_{2l+2},$$

$$M_{33} = \sigma^2 \frac{(l+3)^2}{l(l+1)} \rho_{2l+4} - \gamma(l+3)^2 \rho_{2l+4} + 8\pi G \frac{(l+3)^2}{2l+1} g_{2l+2}. $$
The Characteristic Equation for the Radial Mode of Oscillation in the Second Approximation

The variational expression for the characteristic frequencies of radial oscillation is given in Paper II (eq. [49]). With the trial function

\[ \psi = ar^3 + br^5 \]  

(14)

with the two variational parameters \( a \) and \( b \), we obtain the characteristic equation

\[
\begin{pmatrix}
\sigma^2 \rho_4 - 3(3\gamma - 4) \rho_2 & \sigma^2 \rho_6 - 5(3\gamma - 4) \rho_4 \\
\sigma^2 \rho_8 - 5(3\gamma - 4) \rho_4 & \sigma^2 \rho_6 - (25\gamma - 28) \rho_6
\end{pmatrix} = 0,
\]  

(15)

where \( \rho_m \) and \( \rho_n \) continue to have the same meanings as in equations (4). It is known (cf. Ledoux andWalraven 1958) that this "second approximation" gives the frequency of the fundamental mode of radial oscillation to well within 5 per cent in most cases of physical interest.

III. APPLICATION TO POLYTROPES

We shall use the characteristic equations derived in the preceding section to evaluate the frequencies of the fundamental modes of the radial and the non-radial oscillations of the polytropes.

When \( r, \rho, \) and \( p \) are expressed in terms of the usual Emden variables \( \xi \) and \( \theta \) (cf. Chandrasekhar and Lebovitz 1962c, eq. [8]) and \( \sigma^2 \) is measured in the unit \( 4\pi G\rho_0/(n + 1) \) (where \( n \) is the polytropic index), it can be readily verified that the elements of the secular matrix continue to be given by equations (12) if we replace \( 4\pi G \), wherever it occurs, by \( n + 1 \) and define \( \rho_m, \rho_m, \) and \( g_m \) in terms of the corresponding dimensionless variables as follows:

\[
\rho_m = \int_0^\xi \theta^n \xi^m d\xi,
\]

\[
p_m = \int_0^\xi \theta^{n+1} \xi^m d\xi,
\]

and

\[
g_m = \int_0^\xi g \theta^n \xi^m d\xi = \int_0^\xi d\xi \xi^n \xi^m \int_0^\xi d\eta \eta \theta^n(\eta),
\]  

(16)

where \( \xi_1 \) is the first zero of \( \theta \).

In the case under consideration, \( g_m \) can be expressed directly in terms of \( \rho_m \) and \( p_m \); thus,

\[
g(\xi) = \int_0^\xi \theta^n(\eta) \eta d\eta = -\int_0^\xi \left( \eta \frac{d^2\theta}{d\eta^2} + 2 \frac{d\theta}{d\eta} \right) d\eta = \xi \frac{d\theta}{d\xi} + \theta + \xi_1 | \theta_1', |
\]  

(17)

where \( \theta_1' \) is the value of \( d\theta/d\xi \) at \( \xi_1 \), and

\[
g_m = \xi_1 | \theta_1' | \rho_m + p_m + \int_0^{\xi_1} \theta^n \frac{d\theta}{d\xi} \xi^{n+1} d\xi = \xi_1 | \theta_1' | \rho_m + \left( \frac{n - m}{n + 1} \right) p_m.
\]

(18)

Also, it may be noted that

\[
\rho_2 = \xi_1 | \theta_1' | \quad \text{and} \quad p_2 = \frac{n + 1}{5 - n} \xi_1^3 | \theta_1' | ^2.
\]

(19)

In Table 1, we list the values of the various integrals which are needed for the determination (in the second approximation) of the lowest modes belonging to \( l = 0, 1, \) and 2.

a) The Convective Instability of the Polytropes for \( \gamma < 1 + 1/n \) by Modes Belonging to \( l = 1 \)

We have already shown in Paper I, by an application of the virial equations, that the manifestation of the convective instability of the polytropes for \( \gamma < 1 + 1/n \), by modes
belonging to \( l = 1 \), can be explicitly demonstrated: the critical value of \( \gamma \) for marginal stability predicted by the (approximate) theory differs from \( 1 + \frac{1}{n} \) by less than 0.7 per cent for \( n \leq 3.5 \). It is evident that the present characteristic equation \( |M| = 0 \) for \( l = 1 \) must predict the same critical values of \( \gamma \) (for the different polytropes) as were derived in Paper I: for the trial function assumed for the Lagrangian displacement in both the treatments are the same; the characteristic equations which follow cannot, therefore, be different. And the fact that the present characteristic equation \( |M| = 0 \) for \( l = 1 \) leads to the same critical values for \( \gamma \) as were derived in Paper I can, indeed, be verified directly.

### Table 1

<table>
<thead>
<tr>
<th>Integral</th>
<th>( n = 1 )</th>
<th>( n = 1.5 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 3.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_0 )</td>
<td>3 14159</td>
<td>2 71406</td>
<td>2 41105</td>
<td>2 01824</td>
<td>1 89056</td>
</tr>
<tr>
<td>( p_1 )</td>
<td>12 1567</td>
<td>11 1927</td>
<td>10 6110</td>
<td>10 8516</td>
<td>11 7454</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>62 8853</td>
<td>64 9770</td>
<td>71 7372</td>
<td>109 748</td>
<td>160 39</td>
</tr>
<tr>
<td>( p_8 )</td>
<td>379 112</td>
<td>460 821</td>
<td>617 802</td>
<td>1625 30</td>
<td>3585 89</td>
</tr>
<tr>
<td>( p_9 )</td>
<td>1 41815</td>
<td>1 34001</td>
<td>1 27421</td>
<td>1 16855</td>
<td>1 12515</td>
</tr>
<tr>
<td>( p_{10} )</td>
<td>1 57080</td>
<td>1 44002</td>
<td>1 33547</td>
<td>1 18120</td>
<td>1 12446</td>
</tr>
<tr>
<td>( p_{11} )</td>
<td>4 38231</td>
<td>4 23148</td>
<td>4 17017</td>
<td>3 31761</td>
<td>4 56828</td>
</tr>
<tr>
<td>( p_{12} )</td>
<td>17 4550</td>
<td>18 7887</td>
<td>21 0149</td>
<td>30 4548</td>
<td>40 9726</td>
</tr>
<tr>
<td>( p_{13} )</td>
<td>2 35619</td>
<td>1 72803</td>
<td>1 33547</td>
<td>0 885900</td>
<td>0 74639</td>
</tr>
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<td>( g_1 )</td>
<td>5 83825</td>
<td>4 02839</td>
<td>3 00731</td>
<td>2 20611</td>
<td>1 02105</td>
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<tr>
<td>( g_2 )</td>
<td>19 2477</td>
<td>14 4461</td>
<td>11 7151</td>
<td>9 2748</td>
<td>9 0368</td>
</tr>
</tbody>
</table>

b) The "Kelvin Modes" Belonging to \( l = 2 \); and the Modes Exhibiting Convective Instability

Using the same matrix equation \( |M| = 0 \) for \( l = 2 \), we have determined, with the aid of the integrals listed in Table 1, the lowest characteristic values for polytropes with the indices 1, 1.5, 2, 3, and 3.5. The calculated characteristic values together with the constants of the corresponding characteristic vector \((1, B/A, C/A)\) are listed in Table 2.

We notice the remarkably slight dependence of \( \sigma^2 \) on \( \gamma \) for \( n \leq 2 \). We also observe that for moderate central condensations \((n \leq 1.5)\) the approximation \( \psi \approx \chi \approx \gamma^{1.11} \).

The fact that the two methods, the variational and the virial, must lead to the same characteristic equation (albeit by different routes) can be seen as follows: in the variational method, the characteristic equation for \( \sigma^2 \) appropriate to a trial function of the form

\[
\xi_i = L_{ij}kx_jx_k + L_i
\]

(assumed in Paper I) follows from inserting this expression for \( \xi \) in a certain Hermitian form \((\xi, H\xi)\), where \( H\xi \) gives the time-dependence of \( \xi \) for small departures from equilibrium in accordance with the linearized equations of motion, and making the result stationary for small variations of the constants \( L_{ij}k \) and \( L_i \); whereas, in the virial method, we start with the three zero-order and the eighteen second-order \( x_jx_k \)-moments of the exact non-linear equations of motion, linearize these moment equations for small departures from equilibrium, and then make the same assumption for the Lagrangian displacement. The two sets of homogeneous equations which follow for the constants \( L_{ij}k \) and \( L_i \), clearly, cannot differ (cf. Clement [1964] where the analysis exhibiting this equivalence is set out in full in another context).

We find, for example, that the equation \( |M| = 0 \) for \( l = 1 \) leads to the values \( \gamma = 1.996, 1.662, 1.494, 1.327, \) and 1.279 for the marginal stability of the polytropes \( n = 1, 1.5, 2, 3, \) and 3.5, respectively; and these values should be compared with 1.995, 1.661, 1.498, 1.326, and 1.278 derived in Paper I (eq [88]).
provides a satisfactory representation. As we have seen in Paper II (Sec. IVb) the assumption \( \psi = \chi = r^{p+1} \) leads to an expression for \( \sigma^2 \) which is exactly analogous to Kelvin's formula for the non-radial modes of an incompressible sphere. On this account, it would seem proper to describe the modes obtained in this section as the "Kelvin modes."

### TABLE 2

The Squares of the Characteristic Frequencies and Related Constants Belonging to the Fundamental Modes of Radial and Non-radial \((l = 2)\) Oscillations

\((\sigma^2 \text{ is listed in the unit } 4\pi G \rho_c/[n + 1])\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\gamma)</th>
<th>(\sigma^2)</th>
<th>(a^2/a_1)</th>
<th>(\sigma^2)</th>
<th>(B/A)</th>
<th>(C/A)</th>
</tr>
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<tr>
<td>1 0</td>
<td>1.55</td>
<td>25092</td>
<td>0.1594</td>
<td>30305</td>
<td>0.04938</td>
<td>0.02537</td>
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<td></td>
<td>1.60</td>
<td>30726</td>
<td>0.1928</td>
<td>30332</td>
<td>0.04604</td>
<td>0.02287</td>
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<td></td>
<td>1.65</td>
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<td>30357</td>
<td>0.04293</td>
<td>0.02055</td>
</tr>
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<td>0.2357</td>
<td>30365</td>
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<td>1.55</td>
<td>24711</td>
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<td>29302</td>
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<td></td>
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<td>29390</td>
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</tr>
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<td>0.3790</td>
<td>29475</td>
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To test how good the approximation is of ignoring the variations in the gravitational potential during the oscillation, we have evaluated \(\sigma^2\) from the simplified equation \(|M^{(a)}| = 0\) (cf. eq. [13]) for the case \(n = 3, l = 2, \text{ and } \gamma = 1.6\) and \(\frac{5}{3}\). We find

\[
\sigma^2 = \begin{cases} 
0.2264 & (\gamma = 1.6), \\
0.2398 & (\gamma = \frac{5}{3})
\end{cases}
\]

and these values should be compared with \(\sigma^2 = 0.1884\) and 0.1972 listed in Table 2. It would not appear from this comparison that the approximation is a very good one under the circumstances considered.

In addition to the Kelvin modes we have considered, there exist also modes which exhibit the convective instability of the polytropes for \(\gamma < 1 + 1/n\). Indeed, we find

\[\frac{\partial M^{(a)}}{\partial r} = 0\]

If we had varied \(\psi\) and \(\chi\) independently of each other, then for the same assumed form of the trial functions we should have found \(\sigma^2 = 0.1800, 0.1868, 0.1935,\) and 0.1957 for the case \(n = 3\) and \(\gamma = 1.55, 1.60, 1.65,\) and \(\frac{5}{3}\), respectively; and these values should be contrasted with the values \(\sigma^2 = 0.1818, 0.1885, 0.1950,\) and 0.1972 listed in Table 2 derived after satisfying the proper boundary condition, \(\psi - \chi \to \text{constant}\) as \(r \to 0\).

\(\sigma^2\) is listed in the unit \(4\pi G \rho_c/[n + 1]\)
from the same secular equation \(|M| = 0\) and \(l = 2\) that a neutral mode occurs when (cf. Paper I, eq. [88]; also n. 3 on p. 1522)

\[
\gamma = \begin{cases} 
1.9946 & \text{for } n = 1.0, \\
1.6588 & \text{for } n = 1.5, \\
1.4899 & \text{for } n = 2.0, \\
1.3188 & \text{for } n = 3.0, \\
1.2686 & \text{for } n = 3.5.
\end{cases}
\] (21)

The departures of these values from \(1 + 1/n\) is a measure of the accuracy attained by the present manner of application of the variational method.

c) The Radial Modes

The characteristic equation (15) governing the radial modes has also been solved for the same values of \(\gamma\) and \(n\) for which the Kelvin modes for \(l = 2\) have been determined. The results of the calculations are included in Table 2.

**TABLE 3**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\gamma)</th>
<th>Radial (\sigma^2)</th>
<th>Non-radial (\mu^2)</th>
<th>(a_1/a_{\ell})</th>
<th>(b/a_{\ell})</th>
<th>(c/a_{\ell})</th>
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<td>2619</td>
<td>12591</td>
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</table>

d) The Value of \(\gamma\) for Which the Accidental Degeneracy Occurs

By interpolating among the values of \(\sigma^2\) listed in Table 2, the values of \(\gamma\), for which \(\sigma^2\) for the radial mode and the non-radial mode belonging to \(l = 2\) are equal, were determined. The results of this interpolation are given in Table 3.

We observe that, in contrast to what follows from the solutions in the first approximation, the value of \(\gamma\) for which the accidental degeneracy occurs depends on the density distribution in the configuration. However, the dependence is not very pronounced.

For the polytrope \(n = 3\), the accidental degeneracy occurs for \(\gamma = 1.572\). While this value of \(\gamma\) differs from 1.6 by only 0.028, the stellar requirements for the coincidence of the two frequencies are appreciably different: on the assumption that differences in \(\gamma\) arise from different admixtures of a monatomic gas and radiation, the change in the effective ratio of the specific heats from 1.6 to 1.572 means that \(1 - \beta = (\text{radiation pressure/total pressure})\) changes from 0.0532 to 0.0868; and this change in \((1 - \beta)\) implies, on the standard model, a change in \((M \mu^2/\Sigma)\) from 4.62 to 6.36. It would therefore appear that the suggested interpretation of the beat phenomenon exhibited by the \(\beta\) Canis Majoris stars will not be found inadequate on this account: its validity will have to be tested on other grounds.

We are greatly indebted to Miss Donna D. Elbert for her assistance with all the numerical work in connection with the preparation of this paper.
APPENDIX

AN ALTERNATIVE FORMULATION OF THE VARIATIONAL PRINCIPLE

In deriving the variational principle in Paper II, the assumption was made that the density vanishes on the boundary; and this assumption was explicitly used in the further reductions. Nevertheless, it was found, after suitable transformations of the basic equations, that the variational principle gives correctly the characteristic frequencies of the Kelvin modes of an incompressible sphere (cf. Paper II, Sec. IVb, eq. [53]). The fact that such "suitable transformations" are possible suggests that there is an alternative formulation of the variational principle which is valid without the assumption that the density vanishes on the surface. We shall show that such an alternative formulation exists which, moreover, permits the density to be discontinuous in the interior.

It is convenient for our present purposes to adopt the Lagrangian, instead of the Eulerian, formulation of the perturbation equations.

If $\Delta F(x)$ and $\delta F(x)$ denote, respectively, the Lagrangian and the Eulerian changes in a variable $F(x)$, then (cf. Lebovitz 1961, eq. [21])

$$\Delta F(x) = \delta F(x) + \xi \cdot \text{grad} F(x), \tag{A.1}$$

where $\xi(x)$ is the Lagrangian displacement. Also, it may be verified that the Lagrangian change in the gradient of $F(x)$ is given by

$$\Delta \left( \frac{\partial F}{\partial x_i} \right) = \frac{\partial \Delta F}{\partial x_i} - \frac{\partial F}{\partial x_k} \frac{\partial \xi_k}{\partial x_i}. \tag{A.2}$$

Using the foregoing equations, we find that the Lagrangian form of the equations governing the perturbations are (cf. Paper I, eqs. [9]–[13])

$$\sigma^2 \rho \xi_i = \frac{\partial \Delta \rho}{\partial x_i} - \Delta \rho \frac{\partial \mathfrak{B}}{\partial x_i} - \rho \frac{\partial \Delta \mathfrak{B}}{\partial x_i}, \tag{A.3}$$

$$\Delta \rho = - \gamma \rho \text{div} \xi, \quad \Delta \varphi = - \gamma \rho \text{div} \xi, \tag{A.4}$$

and

$$\Delta \mathfrak{B} = \delta \mathfrak{B} + \xi \cdot \text{grad} \mathfrak{B}, \tag{A.5}$$

where

$$\delta \mathfrak{B}(x) = G \int_V \rho(x') \xi_i(x') \frac{1}{|x-x'|} d x', \tag{A.6}$$

and the boundary condition is that $\Delta \varphi$ vanish on the boundary.

Multiplying equation (A.3) by $\xi_i$, contracting, and integrating over the volume occupied by the fluid, we obtain

$$\sigma^2 \int_V \rho |\xi|^2 d x = \int_V \gamma \rho (\text{div} \xi)^2 d x + \int_V \xi_i \frac{\partial \rho}{\partial x_i} \text{div} \xi d x$$

$$- \int_V \rho \xi_i \frac{\partial}{\partial x_i} (\xi_k \frac{\partial \mathfrak{B}}{\partial x_k}) d x - \int_V \rho \xi_i \frac{\partial \delta \mathfrak{B}}{\partial x_i} d x, \tag{A.7}$$

where we have performed an integration by parts in the first term on the right-hand side of equation (A.7) and have used equations (A.4) and (A.5) as well as the equilibrium condition $\text{grad} \varphi = \rho \text{grad} \mathfrak{B}$.
It can be verified that equation (A.7) provides a variational base for determining the characteristic frequencies. The verification is sufficiently similar to that given in Paper II (Sec. II) that it is omitted here.

We shall now specialize equation (A.7) to the case when the equilibrium configuration is spherically symmetric. One can, without loss of generality, restrict consideration to a single spherical harmonic \( Y_l^m(\theta, \varphi) \). The Lagrangian displacement may then be written (cf. Paper II, eq. [31]; see also eqs. [32] and [33])

\[
\xi_r = \frac{\psi(r)}{r^2} Y_l^m(\theta, \varphi), \quad \xi_\theta = \frac{1}{l(l+1) r} \frac{d\chi(r)}{dr} \frac{\partial Y_l^m(\theta, \varphi)}{\partial \theta},
\]

and

\[
\xi_\varphi = \frac{1}{l(l+1) r \sin \theta} \frac{d\chi(r)}{dr} \frac{\partial Y_l^m(\theta, \varphi)}{\partial \varphi}.
\]

These expressions for the components of \( \xi \) must be substituted into equation (A.7) and the integrations over the angles performed. This reduction has already been performed for the left-hand side and the first two terms on the right-hand side of equation (A.7) (Paper II, eq. [38]) with the result

\[
\sigma^2 \int_0^R \left[ \frac{\psi^2}{r^2} + \frac{1}{l(l+1)} \left( \frac{d\chi}{dr} \right)^2 \right] dr = \int_0^R \gamma \rho \left[ \frac{d\psi}{dr} \right]^2 \frac{dr}{r^2}
\]

\[
+ \int_0^R \frac{d\rho}{dr} \psi \frac{d\chi}{dr} \frac{dr}{r^2} - \int_0^R \rho \xi_i \frac{\partial}{\partial x_i} \left( \xi_k \frac{\partial \psi}{\partial x_k} \right) dx - \int_0^R \rho \xi_i \frac{\partial}{\partial x_i} \frac{\partial \psi}{\partial r} dx,
\]

where we have suppressed a factor \( N_{lm} = 4\pi (1 + |m|)!/(2l + 1) (l - |m|)! \) since it will ultimately appear in every term of equation (A.10) and hence may be canceled.

To facilitate evaluating the final two terms of equation (A.9), we state the following lemma: Let \( \xi \) be given by equation (A.8), \( F(r) \) be arbitrary, and let \( S_1 \) denote the unit sphere and \( dS_1 \) the associated element of area. Then

\[
\int_{S_1} \xi \cdot \text{grad} [F(r) Y_l^m(\theta, \varphi)] dS_1 = \frac{N_{lm}}{r^2} \left( \psi \frac{dF}{dr} + F \frac{d\chi}{dr} \right) \delta \rho \delta_{lm}.
\]

The proof is omitted, since it is an easy consequence of a known result (cf. Chandrasekhar 1961, p. 625).

Turning now to the third term on the right-hand side of equation (A.10), we find

\[
- \int_0^R \rho \xi \cdot \text{grad} \left( \psi \frac{d\psi}{dr} \right) dx = - \int_0^R drr \rho(r) \int_{S_1} \xi \cdot \text{grad} \left( \psi \frac{d\psi}{dr} Y_l^m \right)
\]

\[
= - N_{lm} \int_0^R \rho(r) \left[ \psi \frac{d}{dr} \left( \frac{\psi}{r^4} r^2 \frac{d\psi}{dr} \right) + \psi \frac{d}{r^2} \frac{d\psi}{dr} \frac{d\psi}{dr} \right] dr
\]

\[
= - N_{lm} \int_0^R \rho(r) \left( - \frac{4 \psi^2}{r^2} \frac{d\psi}{dr} + \psi \frac{d\psi}{r^2} \frac{d\psi}{dr} - 4\pi G \rho \frac{\psi^2}{r^2} + \psi \frac{d\psi}{r^2} \frac{d\psi}{dr} \right) dr
\]

\[
= - N_{lm} \int_0^R \left[ - \frac{4 \psi^2}{r^2} \frac{d\rho}{dr} + \psi \frac{d\psi}{r^2} \frac{d\psi}{dr} (\psi + \chi) \frac{d\psi}{dr} - 4\pi G \rho^2 \psi^2 \right] dr.
\]
To reduce the final term in equation (A.9) we first find $\delta \mathcal{B}$ with the aid of equation (A.7) and the known expansion of $|x-x'|^{-1}$ in spherical harmonics,

$$\frac{1}{|x-x'|} = \sum_{l=0}^{\infty} f_l(r, r') \sum_{m=-l}^{l} \frac{(l+|m|)!}{(l-|m|)!} Y_l^m(\vartheta, \varphi) Y_l^m(\vartheta', \varphi'), \quad (A.12)$$

where

$$f_l(r, r') = \begin{cases} r^{l'/r^{l+1}} & \text{if } r' < r, \\ r^{l'/r^{l+1}+1} & \text{if } r' > r. \end{cases} \quad (A.13)$$

Applying the lemma (eq [A.10]), we find

$$\delta \mathcal{B} (x) = G \int_{V} \rho(x') \xi_i(x') \frac{\partial}{\partial x_i'} \frac{1}{|x-x'|} \, dx'$$

$$= G \int_{0}^{R} d r' r'^2 \rho(r') \int_{S_1} \, d S_1 \xi_i(x') \frac{\partial}{\partial x_i'} \frac{1}{|x-x'|}$$

$$= G N_{l m} \frac{(l-|m|)!}{(l+|m|)!} Y_l^m(\vartheta, \varphi) \int_{0}^{R} \rho(r') \left[ \psi(r') \frac{\partial f_l(r, r')}{\partial r} \right]$$

$$+ \frac{d \psi(r')}{d r} f_l(r, r') \right] d r' = \frac{4 \pi G}{2l+1} Y_l^m(\vartheta, \varphi) \left[ \frac{J_l(r)}{r^{l+1}} - r^{l+1} K_l(r) \right], \quad (A.14)$$

where, in virtue of equation (A.13),

$$J_l(r) = \int_{0}^{r} \rho(s) s^l \left[ \frac{l \psi(s)}{s} + \frac{d \psi(s)}{d s} \right] d s$$

and

$$K_l(r) = \int_{r}^{R} \rho(s) s^{l+1} \left[ (l+1) \frac{l \psi(s)}{s} - \frac{d \psi(s)}{d s} \right] d s. \quad (A.15)$$

We cannot eliminate both $J_l$ and $K_l$ from the final formula, but we can eliminate one of them, say $J_l$. For this purpose we need the following formula, which is an easy consequence of equations (A.15):

$$\frac{d}{d r} \left( \frac{J_l}{r^{l+1}} - r^{l+1} K_l \right) = - \frac{l+1}{r^{l+2}} J_l - l r^{l-1} K_l + (2l+1) \frac{\rho \psi}{r^2}. \quad (A.16)$$

Using equations (A.14) and (A.16) and the lemma (A.10), we now find

$$\int \rho \mathbf{\xi} \cdot \nabla \delta \mathcal{B} d x = \frac{4 \pi G}{2l+1} \int_{0}^{R} d r \rho r^2 \int_{S_1} \, d S_1 \xi_i \cdot \nabla \left[ \left( \frac{J_l}{r^{l+1}} - r^{l+1} K_l \right) Y_l^m \right]$$

$$= \frac{4 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho \left[ \psi \frac{d}{d r} \left( \frac{J_l}{r^{l+1}} - r^{l+1} K_l \right) + \frac{d \psi}{d r} \left( \frac{J_l}{r^{l+1}} - r^{l+1} K_l \right) \right] d r$$

$$= \frac{4 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho \left[ \frac{\psi}{r^2} \frac{d J_l}{d r} K_l - K_l \frac{d J_l}{d r} \right] d r$$

$$= \frac{4 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho \left[ \frac{\psi}{r^2} \frac{d J_l}{d r} - \frac{8 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho K_l \frac{d J_l}{d r} d r \right]$$

$$= \frac{4 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho \frac{\psi}{r^2} d r - \frac{8 \pi G N_{l m}}{2l+1} \int_{0}^{R} \rho r K_l \left( \frac{l \psi}{r} + \frac{d \psi}{d r} \right) d r. \quad (A.17)$$
On account of equations (A.11) and (A.17), equation (A.9) becomes

\[ \sigma^2 \int_0^R \rho \left[ \frac{\psi^2}{r^2} + \frac{1}{l(l+1)} \left( \frac{d\chi}{dr} \right)^2 \right] dr = \int_0^R \gamma \left[ \frac{d}{dr} (\psi - \chi) \right]^2 dr \]

(A.18)

\[ + 2 \int_0^R \frac{dp}{dr} \frac{\psi}{r^2} \left( \frac{2}{r} - \frac{d\psi}{dr} \right) dr + \frac{8\pi G}{2l+1} \int_0^R \rho K_l r^l \left( l \frac{\psi}{r} + \frac{d\chi}{dr} \right) dr , \]

which is the desired formula.

It may be useful to note that the formula (A.18) can be deduced also from equation (41) of Paper II, if one continues to assume that \( \rho(R) = 0 \). Straightforward manipulations involving integration by parts and explicit use of the condition \( \rho(R) = 0 \) alter two of the terms of that equation as follows:

\[ \int_0^R \frac{d}{dr} \frac{\psi^2}{r^2} \frac{dr}{r^2} = \int_0^R \frac{d\psi}{dr} \frac{d}{dr} (\rho \psi^2) \frac{dr}{r^2} \]

(A.19)

\[ = 4 \int_0^R \frac{d}{dr} \frac{\psi^2}{r^2} dr + 4\pi G \int_0^R \rho \frac{\psi^2}{r^2} dr \]

and

\[ \int_0^R r^{2l} \left[ \int_0^R \frac{d}{dr} \frac{\psi^2}{s^{l-1}} ds \right] dr = \int_0^R \rho \frac{\psi^2}{r^2} dr + \int_0^R r^{2l} K_l \left( K_l - \frac{2}{r^{l+1}} \right) dr \]

(A.20)

\[ = \int_0^R \rho \frac{\psi^2}{r^2} dr - \frac{2}{2l+1} \int_0^R \rho K_l r^l \left( l \frac{\psi}{r} + \frac{d\chi}{dr} \right) dr , \]

where

\[ \delta \rho(r) = \frac{1}{r^3} \left[ \frac{d}{dr} (\rho \psi) - \rho \frac{d\chi}{dr} \right] , \]

and we have made the observation that

\[ \int_0^R \frac{\delta \rho(s)}{s^{l-1}} ds = -\frac{\rho \psi}{r^{l+1}} + K_l(r) . \]

(A.21)

Using equations (A.19) and (A.20) in equation (41) of Paper II, we recover equation (A.18).

REFERENCES


———. 1962c, *ibid.*, 136, 1082.


