DIFFUSION OF RESONANCE RADIATION IN STELLAR
AND NEBULAR ATMOSPHERES. II. A LAYER
OF FINITE THICKNESS

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This article treats the problem of diffusion of resonance radiation in a plane-parallel layer of
finite optical thickness \( \tau_0 \). It is assumed that scattering is completely incoherent. Functions
\( X(z; \tau_0) \) and \( Y(z; \tau_0) \), similar to the Ambartsumyan functions \( \varphi(\eta; \tau_0) \) and \( \psi(\eta; \tau_0) \), are introduced and investigated. In particular, asymptotic formulas are found for \( X(z; \tau_0) \) and \( Y(z; \tau_0) \) for large \( \tau_0 \). An expression is derived for the average number of scattering events in a
uniform distribution of sources throughout the layer and at \( \tau_0 \gg 1 \) (for a Doppler absorption coefficient).

The first portion of the work described in this article,[1], considered the problem of diffusion of radiation with total redistribution in frequency in a seminfinite medium. We shall now extend some of the results obtained earlier to the case of a medium of finite optical thickness. Particular attention will be bestowed on the study of that case where the optical thickness of the medium is large.

Most of the relationships cited in Section 1 are the direct analog of the expressions obtained some 20 years ago by V. A. Ambartsumyan [2] in the problem of the diffusion of radiation with no change in frequency. They constitute a particular case of the more general relationship found recently by V. V. Sobolev [3] for one class of integral equations. In the sequel, a number of new results relating to a "thick" layer will be derived on the basis of the formulas in Section 1 and in [1]. The application of these results to astrophysical problems will be attended to later on.

We here utilize the same notation as in the first part of this article,[1].

1. Fundamental Equations

The calculation of the radiation field in a plane-parallel slab in the presence of fully incoherent isotropic scattering reduces to the solution of the following transport equation:

\[
-\frac{\lambda}{2} \int_{-\infty}^{\infty} I(\tau, z') G(z')dz' = A g(\tau),
\]

where

\[
z = \frac{\eta}{a(s)},
\]

\[
G(t) = 2A \int_{x(t)}^{\infty} a^2(y) dy,
\]

in which

\[
x(t) = 0 \quad \text{at} \quad t \ll 1,
\]

\[
a[x(t)] = \frac{1}{t} \quad \text{at} \quad t \gg 1.
\]

Here \( \eta \) is the cosine of the angle formed by the direction of flight of the quantum and the normal to the slab; \( x \) is the dimensionless frequency constituting the distance from the center of the line as expressed in Doppler widths; \( \tau \) is the optical depth at the center of the line (at frequency \( x = 0 \)); \( \alpha(x) \) is the ratio of the absorption coefficient at frequency \( x \) to the absorption coefficient at the line center; \( A \) is a normalizing constant defined by the condition

\[
A \int_{-\infty}^{\infty} \alpha(x')dx' = 1;
\]

\( \lambda \leq 1 \) being the ratio of the scattering coefficient to the sum of the scattering coefficient plus the true absorption.
coefficient (probability that a quantum will survive on scattering); \( I(\tau, z) \) is the emission intensity at frequency \( \nu \) for radiation propagating at an angle \( \cos^{-1} \eta \) to the normal, with \( \frac{\eta}{\alpha} = z \); \( 4\pi g(\tau) \) is the power of the radiation sources (it being assumed that the energy emitted by these sources is frequency-dependent, just as is the absorption coefficient).

Equation (1) is derived under the assumption that absorption is absent in the spectral continuum. It must be solved under the boundary conditions

\[
I(0, z) = 0 \quad \text{at} \quad z < 0, \quad \quad I(\tau_0, z) = 0 \quad \text{at} \quad z < 0, \quad \quad (5)
\]

where \( \tau_0 \) is the optical thickness of the slab. These conditions signify that radiation does not impinge from outside at the boundary of the medium.

We denote the source function as \( B(\tau; \tau_0) \):

\[
B(\tau; \tau_0) = \frac{\lambda}{2a} \int_{-\infty}^{\infty} I(\tau, z') G(z') \, dz' + g(\tau). \quad (6)
\]

We infer from (1) and (5) that \( B(\tau; \tau_0) \) satisfies the following equation:

\[
B(\tau; \tau_0) = \frac{\lambda}{2} \int_{0}^{\infty} K(|\tau - \tau'|) B(\tau'; \tau_0) \, d\tau' + g(\tau), \quad (7)
\]

where

\[
K(\tau) = \int_{0}^{\infty} e^{-\tau z} G(z') \, dz'. \quad (8)
\]

Clearly, the solution of Eq. (7) is equivalent to the solution of the transport equation (1) at the boundary conditions (5).

The investigative technique developed by V. V. Sobolev [3] for equations of type (7) having kernels of the form

\[
b \int_{a}^{b} e^{-\pi t} A(x) \, dx
\]

may be applied directly to Eq. (7). We shall skip the details here, and confine ourselves to the general trend of thought.

The resolvent \( \Gamma(\tau, \tau'; \tau_0) \) of Eq. (7) may be expressed in terms of the function \( \Phi(\tau; \tau_0) = \Gamma(0, \tau; \tau_0) \), specified by equation

\[
\Phi(\tau; \tau_0) = \frac{\lambda}{2a} \int_{0}^{\infty} K(|\tau - \tau'|) \Phi(\tau'; \tau_0) \, d\tau'. \quad (9)
\]

Now we introduce any auxiliary function \( P(\tau, z; \tau_0) \) such that

\[
P(\tau, z; \tau_0)
\]

\[
= \frac{\lambda}{2} \int_{0}^{\infty} K(|\tau - \tau'|) P(\tau', z; \tau_0) \, d\tau' + \frac{\lambda}{4\pi} e^{-\frac{\tau}{2}} \quad (10)
\]

This function has a simple physical sense; the quantity

\[
A(\tau) P(\tau, \frac{\eta}{\alpha}(z); \tau_0) \frac{d\omega}{d\sigma} = P(\tau, \eta, x; \tau_0) \frac{d\omega}{d\sigma}
\]

is the probability that a quantum absorbed at depth \( \tau \) will emerge from the medium through the boundary \( \tau = 0 \) at an angle \( \cos^{-1} \eta \) to the outward normal within the solid angle \( d\omega \), and in the frequency interval \( \lambda \lambda_0 \).

Comparing (9) and (10) while taking (8) into account we perceive that

\[
\Phi(\tau; \tau_0) = 2\pi \int_{0}^{\infty} P(\tau, z; \tau_0) G(z') \frac{dz'}{z'}.
\]

In making use of Eq. (10), we may demonstrate that \( P(\tau, z; \tau_0) \) indeed satisfies the equation

\[
\frac{\partial P(\tau, z; \tau_0)}{\partial \tau}
\]

\[
= - \frac{1}{4} P(\tau, z; \tau_0) + \frac{\lambda}{4\pi} X(z; \tau_0) \Phi(\tau; \tau_0)
\]

\[
- \frac{\lambda}{4\pi} Y(z; \tau_0) \Phi(\tau_0 - \tau; \tau_0),
\]

where

\[
X(z; \tau_0) = \frac{4\pi}{\lambda} P(0, z; \tau_0),
\]

\[
Y(z; \tau_0) = \frac{4\pi}{\lambda} P(\tau_0, z; \tau_0).
\]

From Eq. (12), we find

\[
P(\tau, z; \tau_0)
\]

\[
= \frac{\lambda}{4\pi} X(z; \tau_0) \left[ e^{-\frac{\tau}{z}} + \int_{0}^{\tau} e^{-\frac{\tau-t}{z}} \Phi(\tau', \tau_0) \, d\tau' \right]
\]

\[
- \frac{\lambda}{4\pi} Y(z; \tau_0) \int_{0}^{\tau} e^{-\frac{\tau-t}{z}} \Phi(\tau_0 - \tau'; \tau_0) \, d\tau'.
\]

Substitution of Eq. (14) into (11) yields

\[
\Phi(\tau; \tau_0) = \Phi(\tau; \tau_0) + \frac{\tau}{4\pi} \int_{0}^{\tau} \left( \Phi(\tau'; \tau_0) N(\tau - \tau'; \tau_0) \right)
\]

\[
+ \frac{\lambda}{2} K(\tau).
\]
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\[ -\Phi(t_0, \tau'; t_0, t_0) M(\tau, \tau'; t_0) \, dt', \quad (15) \]

where

\[ N(\tau, t_0) = \frac{\lambda}{2} \int_0^\infty e^{-\frac{\tau}{2}} X(z'; t_0) G(z') \frac{dz'}{z'}, \]

\[ M(\tau, t_0) = \frac{\lambda}{2} \int_0^\infty e^{-\frac{\tau}{2}} Y(z'; t_0) G(z') \frac{dz'}{z'}. \quad (16) \]

As for our functions X and Y, three systems of equations may be obtained for their determination, by the same method as employed in the case of diffusion, with no attendant change in frequency [4]. These are similar to the equations derived earlier by V. V. Sobolev [5] in the problem of diffusion of radiation with complete redistribution in frequency throughout a one-dimensional medium. The equations in question exhibit the following form:

I.

\[ X(z; t_0) = 1 + \frac{\lambda}{2} \int_0^z e^{-\frac{\tau}{2}} X(z'; t_0) X(z'; t_0) - Y(z; t_0) Y(z; t_0) \frac{dz'}{z'}, \quad (17) \]

\[ Y(z; t_0) = e^{-\frac{\tau}{2}} + \frac{\lambda}{2} \int_0^z e^{-\frac{\tau}{2}} X(z'; t_0) X(z'; t_0) - X(z; t_0) X(z; t_0) \frac{dz'}{z'-z}. \]

II.

\[ X(z; t_0) = 1 + \frac{\lambda}{2} \int_0^z Y(z; t' \tau; t') \, dt' Y(z'; t' \tau; t') G(z') \frac{dz'}{z'}, \quad (18) \]

\[ Y(z; t_0) = e^{-\frac{\tau}{2}} + \frac{\lambda}{2} \int_e^z e^{-\frac{\tau}{2}} X(z'; t' \tau; t') G(z') \frac{dz'}{z'-z}. \]

III.

\[ X(z; t_0) = 1 + \frac{\lambda}{2} \int_0^z e^{-\frac{\tau}{2}} (z' - z) \left[ (\frac{1}{z'} + \frac{1}{z}) \right] X(z'; t' \tau; t') G(z') \frac{dz'}{z'}, \quad (19) \]

\[ Y(z; t_0) = e^{-\frac{\tau}{2}} + \frac{\lambda}{2} \int_0^z e^{-\frac{\tau}{2}} X(z'; t' \tau; t') Y(z'; t' \tau; t') G(z') \frac{dz'}{z'-z}. \]

Consequently, the complete solution of the problem boils down to finding X and Y from any of the systems of equations (17) to (19) and subsequent solution of Eq. (15). We could hardly expect to obtain the solution of the problem in explicit form. A numerical solution of the equations would require enormous computational labor. It is therefore quite a natural expedient to consider the case where certain simplifying assumptions may be justified by one or another consideration. A plausible example is the problem of finding the intensity of emergent radiation in a uniform distribution of sources inside a slab. Its solution does not require a knowledge of the function \( \Phi(\tau; \tau_0) \), but only that X and Y be known (on this point and the reason behind it, cf. the book by V. V. Sobolev [ref. 6, p. 209 in the original Russian edition; English translation available]). Solutions of several other problems are likewise expressed directly in terms of X and Y. This is precisely the motivation for the attention to be given in what follows to the investigation of these two functions. Assuming the optical thickness \( \tau_0 \) of the medium to be quite large, we encounter some additional simplifications. However, before proceeding to examine these simplifications, let us run through some of the general relationships which X and Y satisfy.

By definition,

\[ X_0(t_0) = \int_0^\infty X(z; t_0) G(z) \, dz, \]

\[ Y_0(t_0) = \int_0^\infty Y(z; t_0) G(z) \, dz. \quad (20) \]

Using Eq. (17), we may readily show the validity of

\[ X_0 = 1 + \frac{\lambda}{2} (X_0^2 - Y_0^2). \quad (21) \]

At \( \lambda = 1 \), this formula transforms to the following:

\[ X_0 + Y_0 = 2. \quad (22) \]

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As \( z \) increases from 0 to \( \infty \), functions \( X \) and \( Y \) increase, respectively, from 1 and from 0 to
\[
X(\infty; \tau_0) = Y(\infty; \tau_0) = 1
\]
\[
1 - \frac{\lambda}{2} [X_0(\tau_0) - Y_0(\tau_0)].
\]  
(23)

The latter formula is derived with ease from either of Eqs. (17). In the pure scattering case (\( \lambda = 1 \)), Eq. (23) takes on the form
\[
X(\infty; \tau_0) = Y(\infty; \tau_0) = Y^{-1}(\tau_0) \tag{24}
\]
once Eq. (22) is taken into account.

In the pure scattering case, the solution of the equations for \( X \) and \( Y \) is not unique. It is readily shown that functions \( X \) and \( Y \), having the necessary physical significance [as determined by Eqs. (13)], must in this case satisfy the integral equation
\[
\int_0^\infty [X(z; \tau_0) - Y(z; \tau_0)] zG(z) dz = \tau_0 Y_0(\tau_0). \tag{25}
\]

It makes it possible to remove the ambiguity in the choice of \( X \) and \( Y \) ([6], p. 207).

Note the additional crucial circumstance which follows. Putting \( \tau = \tau_0 \) in Eq. (11), and resorting to (13), we find
\[
\Phi(\tau_0; \tau_0) = \frac{\lambda}{2} \int_0^\infty Y(z; \tau_0) G(z) \frac{dz'}{z'}. \tag{26}
\]

Bearing this in mind, and using the second equation in (19) instead of the first equation in (18), we have, correspondingly,
\[
\frac{dX(z; \tau_0)}{d\tau_0} = Y(z; \tau_0) \Phi(\tau_0; \tau_0), \\
\frac{dY(z; \tau_0)}{d\tau_0} = -\frac{1}{z} Y(z; \tau_0) + X(z; \tau_0) \Phi(\tau_0; \tau_0). \tag{27}
\]

Hence, in order to find functions \( X \) and \( Y \), it is sufficient to first determine only the function \( \Phi(\tau_0; \tau_0) \). We shall exploit this fact in the future in investigating \( X(z; \tau_0) \) and \( Y(z; \tau_0) \) at \( \tau_0 \gg 1 \).

We find from Eq. (27), bearing (20) and (26) in mind, that
\[
\frac{dX_0(\tau_0)}{d\tau_0} = \Phi(\tau_0; \tau_0) Y_0(\tau_0), \\
\frac{dY_0(\tau_0)}{d\tau_0} = \left( -\frac{2}{\lambda} + X_0(\tau_0) \right) \Phi(\tau_0; \tau_0). \tag{28}
\]

Equation (28) will also come in handy later on.

With the intention of considering in greater detail the case of a medium of large optical thickness, we ad-

duce the important relation associating \( P(\tau, z; \tau_0) \) and \( P(\tau, z) \equiv P(\tau, z; \infty) \). This may be derived from physical considerations, if we assume that a slab of thickness \( \tau_0 \) is superposed on a semi-infinite medium. The relationship in question has the form
\[
P(\tau, z; \tau_0) = P(\tau, z) - 2\pi \int_0^\infty P(\tau_0 + t, z) dt \\
	imes \int_0^\tau e^{-\frac{t}{\tau_0}} P(\tau_0, z) G(z) \frac{dz'}{z'}. \tag{29}
\]

Formally, it is derived as follows. Clearly, \( P(\tau, z) \) is defined by Eq. (10) at \( \tau_0 = \infty \); for \( \tau = \tau_0 \) this equation may be recast in the form
\[
P(\tau, z) = \frac{\lambda}{2} \int_0^\tau K(\tau - \tau') P(\tau', z) d\tau' + \frac{\lambda}{4\pi} e^{-\frac{\tau}{\tau_0}} \\
+ \frac{\lambda}{2} \int_0^\infty P(\tau_0 + t, z) dt \times \int_0^\tau e^{-\frac{t}{\tau_0}} e^{-\frac{\tau_0}{z'}} G(z) \frac{dz'}{z'}. \tag{30}
\]

Comparing Eqs. (30) and (10), we see that Eq. (30) differs from (10) only in the presence of the third term on the right-hand side. But this term constitutes a superposition of free terms belonging to Eq. (10). On this basis, we conclude that Eq. (29) must be valid.

Substituting Eq. (30) into (11) and setting \( \tau = \tau_0 \), we obtain
\[
\Phi(\tau_0; \tau_0) = \Phi(\tau_0) \\
- \frac{\lambda}{2} \int_0^\infty \Phi(\tau_0 + t) dt \int_0^\tau e^{-\frac{t}{\tau_0}} X(z; \tau_0) G(z) \frac{dz'}{z'}. \tag{31}
\]

where \( \Phi(\tau) = \Phi(\tau, \infty) \). This formula will be made use of in the following section for the derivation of asymptotic expressions for \( \Phi(\tau_0; \tau_0) \) as \( \tau_0 \gg 1 \).

2. Medium of Large Optical Thickness: The Function \( \Phi(\tau_0; \tau_0) \)

If the optical thickness of the medium is quite substantial, then, as already noted, the expressions derived above will undergo drastic simplifications. This case is of considerable interest in applications. Consider it in closer detail on that account. The radiation field of a semi-infinite medium with the same optical characteristics [i.e., the same \( \alpha(s) \) and \( \lambda \)] will be assumed known. More accurately, we shall assume that the function \( H(z) = X(z, \infty) \) defined by the equation
\[
H(z) = 1 + \frac{\lambda}{2} \int_0^\infty H(z') G(z') \frac{dz'}{z'}, \tag{32}
\]
is known, as well as the asymptotic behavior of the function \( \Phi(\tau) = \Phi(\tau, \infty) \) as \( \tau \to \infty \). For some particular
forms of the absorption coefficient $\alpha(x)$, $H(x)$ and $\Phi(\tau)$ have been already studied [1, 7]. We shall now use the results reported in those papers.

We restrict ourselves to the case of a pure Doppler absorption coefficient $\alpha(x) = e^{-x^2}$. Then, at large $\tau$ and at $\lambda < 1$

$$\Phi(\tau) \approx \frac{\lambda}{4 \sqrt{\pi} \tau (1 - \lambda)^{7/4}} \sqrt{\ln \tau} \cdot \tag{33}$$

In the pure scattering case, the following asymptotic expression holds for $\Phi(\tau)$ at $\tau \gg 1$:

$$\Phi(\tau) \approx 2 \pi^{-3/4} \tau^{-1/4} (\ln \tau)^{1/4}. \tag{34}$$

The function $H(z)$, as $z$ increases from 0 to $\infty$, from $H(0) = 1$ to $H(\infty) = (1 - \lambda)^{7/2}$. Furthermore, it is known that

$$H_0 = \int_0^\infty H(\zeta') G(\zeta') d\zeta' = \frac{2}{\lambda} (1 - \sqrt{1 - \lambda}). \tag{35}$$

The results of the numerical solution of Eq. (32) for the case of the Doppler absorption coefficient were reported earlier [1].

The probability of emergence of a quantum $P(\tau, z; \infty)$ is also expressed simply in terms of the functions $H(z)$ and $\Phi(\tau)$. At $\tau \gg 1$ and $z \ll \tau$:

$$P(\tau, z; \infty) \equiv P(\tau, z) \approx \frac{\lambda}{4 \pi z} z H(z) \Phi(\tau), \tag{36}$$

where $\Phi(\tau)$ is given by Eq. (33) or (34). The formulas just presented will also come in handy later on.

We now proceed directly to solve the problem of emission of a plane-parallel slab of large optical thickness. First we shall study the asymptotic behavior of $\Phi(\tau_0; \tau_0)$ as $\tau_0 \to \infty$. We see from Eq. (31) that the inner integral in the right-hand member of the equation possesses a logarithmic singularity at $t = 0$. At the same time, $\Phi(\tau_0 + t)$ varies comparatively slowly at large $\tau_0$ values. Accordingly, at $\tau_0 \gg 1$ the integral included in the right-hand member of Eq. (31) may be represented approximately as

$$\frac{\lambda}{2} \int_0^\infty \Phi(\tau_0 + t) dt \int_0^\infty e^{\frac{t}{\tau_0}} X(\zeta'; \tau_0) G(\zeta') d\zeta' \quad \approx \frac{\lambda}{2} \Phi(\tau_0) \int_0^\infty e^{\frac{t}{\tau_0}} dt \int_0^\infty X(\zeta'; \tau_0) G(\zeta') d\zeta' \quad \approx \frac{\lambda}{2} \Phi(\tau_0) X_0(\tau_0). \tag{37}$$

Accurate to principal terms, then, we have

$$\Phi(\tau_0; \tau_0) \approx \frac{1}{2} \Phi(\tau_0) Y_0(\tau_0) \tag{38}$$

in the pure scattering case, and

$$\Phi(\tau_0; \tau_0) \approx \sqrt{1 - \lambda} \Phi(\tau_0) \tag{39}$$

at $\lambda < 1$. Here we make use of the fact that $X_0 = 2 - Y_0$, $X_0 = 1 + \sqrt{1 - \lambda}$ at smaller values of $\lambda$ and $\tau_0 \gg 1$.

Of course, the derivation resorted to here is not rigorous. However, it may be proved that the formulas obtained accurately yield the first term of the asymptotic expansion of $\Phi(\tau_0; \tau_0)$ at $\tau_0 \gg 1$. We shall not spend time on the proof.

We infer from Eq. (28) that, at $\lambda = 1$,

$$\Phi(\tau_0; \tau_0) = -\frac{1}{Y_0} \frac{dY_0}{d\tau_0}. \tag{40}$$

Consequently, from Eq. (38), we find

$$Y_0(\tau_0) = \frac{1}{2} \int_{\tau_0}^\infty \Phi(\tau) Y_0(\tau) d\tau. \tag{41}$$

In the case of the Doppler-profile absorption coefficient, we obtain from the above, with the aid of Eq. (34):

$$Y_0(\tau_0) \approx \frac{1}{2} \pi^{3/4} \tau_0^{-1/4} (\ln \tau_0)^{-1/4}. \tag{42}$$

Substituting this value of $Y_0(\tau_0)$ into Eq. (38), we find, finally, that in the case of pure scattering at $\tau_0 \gg 1$,

$$\Phi(\tau_0; \tau_0) \approx \frac{1}{2\tau_0}. \tag{43}$$

At $\lambda > 1$, we obtain, from Eqs. (39) and (33),

$$\Phi(\tau_0; \tau_0) \approx \frac{\lambda}{4 \sqrt{\pi} (1 - \lambda)^{7/2}} \sqrt{\ln \tau_0}. \tag{44}$$

With $\Phi(\tau_0; \tau_0)$ known at $\tau_0 \gg 1$, it is not difficult to study the behavior of $X_0(\tau_0)$ and $Y_0(\tau_0)$ at large values of $\tau_0$. The pure scattering case was already discussed above [cf. Eq. (42)], so that it now remains to consider only the case $\lambda < 1$.

The first equation in (28) yields

$$X_0(\tau_0) = \frac{2}{\lambda} (1 - \sqrt{1 - \lambda}) \int_{\tau_0}^\infty Y_0(\tau) \Phi(\tau; \tau) d\tau. \tag{45}$$
Substitution of Eq. (45) into the second equation in (28), followed by discarding of higher-order terms, yields

$$Y_0(\tau_0) \approx \frac{2}{\lambda} \sqrt{4 - \lambda} \int_{\tau_0}^{\infty} \Phi(\tau; \tau) \, d\tau. \tag{46}$$

Finally, substitution of (46) into (45) leads to the following definitive expression for $X_0(\tau_0)$ at $\tau_0 \gg 1$:

$$X_0(\tau_0) \approx \frac{2}{\lambda} \left( 1 - \frac{\lambda}{4} \sqrt{4 - \lambda} \right) \int_{\tau_0}^{\infty} \Phi(\tau; \tau) \, d\tau. \tag{47}$$

In the particular case of the Doppler-profile absorption coefficient, Eqs. (46) and (47) assume the form

$$Y_0(\tau_0) \approx \frac{1}{2} \sqrt{\frac{1}{\pi} \frac{1}{\lambda \tau_0} \frac{1}{V \ln \tau_0}}; \tag{48}$$

$$X_0(\tau_0) \approx \frac{2}{\lambda} \left( 1 - \frac{\lambda}{4} \sqrt{4 - \lambda} \right) \int_{\tau_0}^{\infty} \Phi(\tau; \tau) \, d\tau. \tag{49}$$

3. Medium of Large Optical Thickness; Functions $X(z; \tau_0)$, $Y(z; \tau_0)$

In this section, asymptotic formulas are derived for $X(z; \tau_0)$ and $Y(z; \tau_0)$ at large $\tau_0$ values.

Assuming $\tau = \tau_0$ in Eq. (29), and recalling the notation introduced in (13), we have

$$Y(z; \tau_0) = \frac{4\pi}{\lambda} P(\tau_0, z)$$

$$- 2\pi \int_0^\infty P(\tau_0 + t, z) dt \int_0^\infty e^{-i z} X(z'; \tau_0) G(z') \frac{dz'}{z'} \tag{50}$$

This relationship may be used to arrive with ease at an asymptotic expression for $Y(z; \tau_0)$ at $\tau_0 \gg 1$, and at values of $z$ satisfying the inequality $\tau_0 \gg z$. Using Eq. (38), we obtain for this case

$$Y(z; \tau_0) \approx z H(z) \left( \Phi(\tau_0) \right)$$

$$- \frac{\lambda}{2} \int_0^\infty \Phi(\tau_0 + t) dt \int_0^\infty e^{-i z} X(z'; \tau_0) G(z') \frac{dz'}{z'} \tag{51}$$

According to Eq. (31), the quantity enclosed in parentheses is none other than $\Phi(\tau_0; \tau_0)$. Finally, on that account,

$$Y(z; \tau_0) \approx z H(z) \Phi(\tau_0; \tau_0). \tag{52}$$

For $X(z; \tau_0)$, we find, from (27)

$$X(z; \tau_0) = H(z) - \int_{\tau_0}^{\infty} Y(z; \tau) \Phi(\tau; \tau) \, d\tau. \tag{53}$$

Now substituting $Y(z; \tau_0)$ from Eq. (52) here, we obtain

$$X(z; \tau_0) \approx H(z) - z H(z) \int_{\tau_0}^{\infty} \Phi(\tau; \tau) \, d\tau. \tag{54}$$

Equations (52) and (54) yield asymptotic expressions for functions $X$ and $Y$ at $\tau_0 \gg 1$ and $z \ll \tau_0$. In particular, in the case of the Doppler absorption coefficient

$$X(z; \tau_0) \approx H(z) - z H(z) \frac{\lambda^2}{48 \pi (1 - \lambda)^2 \tau_0^2 \ln \tau_0}, \tag{55}$$

$$Y(z; \tau_0) \approx z H(z) \frac{\lambda}{4 \sqrt{\pi (1 - \lambda) \tau_0^2 \ln \tau_0}} \tag{56}$$

at $\lambda < 1$, and

$$X(z; \tau_0) \approx H(z) - \frac{z H(z)}{2 \tau_0}, \tag{57}$$

$$Y(z; \tau_0) \approx \frac{z H(z)}{2 \tau_0} \tag{58}$$

in the case of pure scattering.

It is interesting to note that the method we are using here may be applied (with a few modifications) to the problem of coherent scattering of light in a plane-parallel layer of large optical thickness. In particular, this approach may be followed to derive with ease asymptotic formulas for the Ambartsumyan functions $\phi(\eta; \tau_0)$ and $\psi(\eta; \tau_0)$. These formulas were derived earlier by V. V. Sobolev [4] via an alternative procedure.

When the opposite inequality $z \gg \tau_0$ is satisfied, the matter is a completely different one. It is readily seen that, in this case, $X(z; \tau_0)$ and $Y(z; \tau_0)$ may be represented in the form of power series in reciprocal powers of $z$. This is implied, for example, by Eq. (27). Therefore, at $z \gg \tau_0 \gg 1$, we have, approximately

$$X(z; \tau_0) \approx X(\infty; \tau_0) - \frac{1}{z} x(\tau_0), \tag{59}$$

$$Y(z; \tau_0) \approx X(\infty; \tau_0) - \frac{1}{z} y(\tau_0). \tag{57}$$

Consider the case of a Doppler absorption coefficient. We infer from Eqs. (24) and (42) that, at $\lambda = 1$,

$$X(\infty; \tau_0) \approx 2\pi^{-\eta_0} \left( \ln \tau_0 \right)^{\eta_0}. \tag{58}$$

At $\lambda < 1$, we obtain from (23)

$$X(\infty; \tau_0) \approx \frac{z H(z)}{2 \tau_0}.$$
\[
\frac{1}{\sqrt{1 - \lambda}} \approx \frac{\lambda}{4 \sqrt{\pi} \lambda^{3/2}} \ln \tau_0, \quad (59)
\]

taking Eqs. (47) and (48) into due account. The determination of \( x(\tau_0) \) and \( y(\tau_0) \) does not involve any great labor either. The first equation in (18) and the second equation in (19) yield
\[
\begin{align*}
\int_0^{\tau_0} y(\tau) \Phi(\tau; \tau) d\tau, \\
y(\tau_0) &= \tau_0 X(\infty; \tau_0) - \int_0^{\tau_0} \tau X(\infty; \tau) \Phi(\tau; \tau) d\tau + \int_0^{\tau_0} x(\tau) \Phi(\tau; \tau) d\tau.
\end{align*}
\]  
(60)

From the above, we see readily that, at \( \tau_0 \gg 1 \) and \( \lambda < 1 \),
\[
\begin{align*}
x(\tau_0) &\approx \frac{\lambda}{2 \sqrt{\pi} \lambda^{3/2}} \ln \tau_0, \\
y(\tau_0) &= \frac{\tau_0}{1 - \lambda}.
\end{align*}
\]  
(61)

In the pure scattering case,
\[
\begin{align*}
x(\tau_0) &\approx \frac{1}{2} \pi^{-3/2} \tau_0^{3/2} (\ln \tau_0)^{1/4}, \\
y(\tau_0) &\approx \frac{3}{2} \pi^{-3/2} \tau_0^{3/2} (\ln \tau_0)^{1/4}
\end{align*}
\]  
(62)

Convincing evidence of the validity of formulas (61) and (62) may be secured by substituting them directly into Eqs. (60).

The difference in the behavior of \( X \) and \( Y \) depending on the sign of the inequality \( \tau_0 \approx z \) is readily grasped from physical considerations. For quanta with \( x \) and \( \eta \) such that \( \eta / \alpha(x) \approx z \gg \tau_0 \), the medium is almost transparent, and the quanta may leave the medium virtually without hindrance. When the opposite inequality \( z \ll \tau_0 \) is fulfilled, a quantum cannot traverse any considerable distance inside the medium without experiencing scattering.

4. Average Number of Scattering Events

The estimate of the average number of scattering events experienced by a quantum is of indubitable interest. It is perfectly obvious from physical considerations that the average number of scattering events will be greatly reduced below the level for the case of coherent scattering when the redistribution of quanta relative to frequency is taken into account in an elementary scattering event. This reduction in the number of scattering events is elicited by the possibility of a quantum escaping into the wings of the line, when the absorption coefficient is small. Clearly, the average number of scattering events depends on the geometry of the problem and on the distribution pattern of radiation sources throughout the medium. We shall present an estimate of the average number of scattering events for the simplest case of a plane-parallel slab of thickness \( \tau_0 \) with uniformly distributed radiation sources throughout the slab.

Assume \( g(\tau) = 1 \). Then the total energy emitted in unit time in all directions by sources located in a column of \( 1 \text{ cm}^2 \) cross section is equal to
\[
4\pi \int_0^{\tau_0} g(\tau) d\tau = 4\pi \tau_0.
\]

The total energy reemitted by this volume in unit time is equal to
\[
4\pi \int_0^{\tau_0} B(\tau; \tau_0) d\tau, \quad \text{where } B(\tau; \tau_0) \text{ is the source function for this problem.}
\]

The average number of scattering events experienced by a quantum, \( N_{av} \), is in this case evidently equal to
\[
N_{av} = \frac{1}{\tau_0} \int_0^{\tau_0} B(\tau; \tau_0) d\tau.
\]  
(63)

In order to find the integral of the source function included here, we shall proceed as follows. The intensity of the emergent radiation \( I(0, \tau; \tau_0) \), where \( z = \eta / \alpha(x) \), is related to \( B(\tau; \tau_0) \) by the familiar formula
\[
I(0, \tau; \tau_0) = A \int_0^{\tau_0} e^{\lambda z} B(\tau; \tau_0) d\tau.
\]  
(64)

On the other hand, at \( g(\tau) = 1 \), the intensity of the emergent radiation is
\[
I(0, \tau; \tau_0) = \frac{4\pi}{\lambda} A \int_0^{\tau_0} P(\tau; z; \tau_0) d\tau.
\]  
(65)

With the aid of Eq. (12) for \( P(\tau; z, \tau_0) \), and taking the notation in (13) into account, we find that
\[
\int_0^{\tau_0} P(\tau; z; \tau_0) d\tau
\]
\[
= \frac{\lambda}{4\pi} X(\infty; \tau_0) \{ X(z; \tau_0) - Y(z; \tau_0) \}.
\]  
(66)

Substituting this value of the integral into Eq. (65) and setting the right-hand members of (64) and (65) equal, we find
\[
\int_0^{\tau_0} e^{\lambda z} B(\tau; \tau_0) d\tau
\]
\[
= X(\infty; \tau_0) z \{ X(z; \tau_0) - Y(z; \tau_0) \}.
\]  
(67)
Now setting $z = \infty$ here, we obtain
\[
\int_0^{\tau_0} B(\tau; \tau_0) \, d\tau = X(\infty; \tau_0) [y(\tau_0) - x(\tau_0)],
\]  
(68)

where $x(\tau_0)$ and $y(\tau_0)$ are determined by Eqs. (57).

Accordingly, we finally obtain for the average number of scattering events
\[
N_{av} = \frac{X(\infty; \tau_0)}{\tau_0} [y(\tau_0) - x(\tau_0)].
\]  
(69)

At $\tau_0 \gg 1$ and $\lambda < 1$, and taking (61) and (59) into account, we obtain hence the trivial result
\[
N_{av} \approx \frac{1}{1 - \lambda}.
\]  
(70)

This signifies that, at $\lambda < 1$ and at large values of $\tau_0$, the major factor leading to quanta escaping from the scattering process is the true absorption of the quanta (also, cf. reference [7]).

At $\lambda = 1$, this mechanism of loss of quanta is not operative, and the sole factor determining the average number of scattering events is the escape of quanta from the medium. In the case of the Doppler-type absorption coefficient and $\tau_0 \gg 1$, we obtain for the average number of scattering events, taking (62) and (58) into due account,
\[
N_{av} \approx 2\pi^{-\lambda} \tau_0 \sqrt{\ln \tau_0}.
\]  
(71)

It follows from this formula that $N_{av}$ increases as $\tau_0$ at a slower rate than in the case of coherent scattering. In the latter case, $N_{av}$ increases in proportion to $\tau_0^2$. The slow rate of increase in $N_{av}$ with $\tau_0$ results in the average number of scattering events turning out to be relatively small at large $\tau_0$, e.g., at $\tau_0 = 10^4$, our Eq. (71) indicates $N_{av} \approx 10^4$.

The estimate arrived at here for the average number of scattering events is of interest in relation to the problem of diffusion of $I_\alpha$-radiation in nebulae. The application of the results so obtained to various other problems of astrophysical interest will be presented in one of the subsequent articles in this series.

LITERATURE CITED

2. V. A. Ambartsumyan, Nauchny Trudy, 1, Erevan (1960).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. Some or all of this periodical literature may well be available in English translation. A complete list of the cover-to-cover English translations appears at the back of this issue.