DIFFUSION OF RESONANCE RADIATION IN STELLAR ATMOSPHERES AND NEBULAE I. Semiinfinite Medium

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The problem of diffusion of resonance radiation in a semiinfinite medium is discussed. Completely incoherent scattering is assumed. The function $H(z)$, which in this case is analogous to the $\varphi$-function of Ambartsumyan, is obtained in an explicit form and is computed numerically. Milne's problem is solved in the case of completely incoherent scattering. The intensity of radiation leaving the medium is expressed in terms of the function $H(z)$ for an exponential distribution of sources of radiation in the medium. The resulting line profiles are obtained and are found to be similar to the $L_\alpha$ and $K_\alpha$-$K_\beta$ profiles in the solar spectrum.

1. Introduction

The diffusion of resonance radiation is of major importance in astrophysics. The problem is encountered in the interpretation of resonance absorption-line profiles in stellar spectra, and in the study of gas nebulae (diffusion of $L_\alpha$ radiation), the solar chromosphere, and the transition layer between the corona and the chromosphere. The problem is also encountered in the analysis of the results of a number of physical experiments.

An enormous number of papers concerned with the study of resonance-radiation transport is now available. The various processes which occur in single and multiple scattering of resonance radiation have been investigated in detail.

Henyey [1] has investigated the processes which occur in the elementary (single) scattering event. He considered the change in the frequency of a scattered photon which is due to the Doppler shift associated with the thermal motion of the scattering atoms. He obtained an expression for the probability that when a photon of given frequency is scattered by an elementary volume in a given direction, it will be re-emitted into a given frequency interval. This probability, integrated over all scattering angles, has also been determined by Umno [2] and by V. V. Sobolev [3]. Jefferies and White [4] have used the resulting expression to compute this probability numerically, and concluded that in the central parts of lines (roughly, within three Doppler widths) it may be assumed that, to a good approximation, the probability of re-emission of a photon into a given frequency interval is independent of the frequency of the absorbed photon (this is the so-called complete frequency redistribution, or completely incoherent scattering). Thomas [5] arrived at a similar conclusion in an earlier paper.

The above reason for the change in the frequency of a photon on scattering is not the only one. A stationary atom will frequently scatter photons incoherently (cf. for example [6]), and if this does indeed occur then the above conclusion about the completely incoherent nature of the scattering process will hold even better. Therefore, it is practically always possible to assume that complete photon frequency redistribution occurs in the central parts of lines. We shall use this approximation throughout this analysis.

A large number of papers have also been devoted to the study of multiple scattering of resonance radiation. A brief review of those papers which were concerned with the interpretation of resonance-line profiles in stellar spectra has recently been given by Böhm [7]. A review of papers on the theory of $L_\alpha$-radiation transport in nebulae and stellar envelopes will be found in a paper by V. V. Sobolev [8].

The specific features of resonance-radiation transport consist of the following. Owing to the change in the frequency on scattering, photons having frequencies which are nearly equal to the frequency at the center of the line may be re-emitted in the wings. Since the absorption coefficient in the wing of a line is low, it follows that such photons will travel through a large distance without further scattering in the medium. Hence, a given volume will receive direct radiation from other volumes to a much greater extent than in the case of
coherent scattering. This means that the problem cannot be approximately reduced to the solution of the diffusion equation.

However, modern methods of the theory of multiple scattering of light may be applied to a much wider range of problems than the diffusion approximation. The first application of these methods to incoherent scattering of light is due to V. V. Sobolev [9] (cf. also [10]). The subject was later taken up by Ueno [11], Busbridge [12] and others. It appears to be the most acceptable method at the present time. It may be considered that the systematic application of rigorous methods to resonance-radiation transport will lead to a number of new results. It seems natural to start with a detailed analysis of simple and often rather idealized problems. The solution of such problems will show the characteristic features of more general situations. The solutions will facilitate the solution of more complicated problems which are of major practical interest. Moreover, the rigorous solution of a number of simple problems may be used as a check on the applicability and accuracy of various approximate methods developed for the solution of a wider class of problems.

The present paper is the first of a series of papers which will give an account of a systematic application of modern methods of the theory of multiple scattering of radiation to the study of resonance-radiation transport; it is closely related to previous papers by the present author [13, 14]. The simplest case of a semi-infinite medium is discussed. The case of a medium with a finite optical thickness will be considered in a following paper. It is intended to complete a tabulation of the various functions which are encountered in the theory of resonance-radiation transport. The various results will then be used in the solution of astrophysical problems such as diffusion of Lyman radiation in nebulae and in the interstellar medium, the emission of resonance lines in the solar spectrum and the interpretation of their profiles, and so on.

1. Formulation of the Problem. Basic Equations.

Consider a plane layer whose optical thickness at the frequency corresponding to a particular line is infinitely large, and suppose that there is no absorption in the continuous spectrum. The layer contains sources of radiation whose strength is a function of a single coordinate, i.e., the distance from the boundary of the medium. The sources emit photons at frequencies corresponding to the given line. The photons then diffuse through the medium. Scattering is isotropic and completely incoherent. It is required to compute the radiation field.

The problem may be reduced to the solution of the following transport equation:

\[
- \frac{\lambda A}{2} \alpha(x) \int_{-\infty}^{\infty} \alpha(x') dx' \int_{-1}^{1} I(\tau, \eta', x') d\eta' = A \alpha(x) g(\tau),
\]

where \( \eta \) is the cosine of the angle between the direction of motion of the photon and the normal to the layer; \( \chi \) is a dimensionless frequency representing the distance from the center of the line and expressed in Doppler widths \( \chi = (v-v_0)/\Delta v_0 \), where \( v_0 \) is the frequency at the center of the line and \( \Delta v_0 \) is the Doppler half-width; \( \tau \) is the optical depth at the center of the line, i.e., at \( x = 0 \); \( \alpha(x) \) is the ratio of the absorption coefficient at frequency \( x \) to the absorption coefficient at the center of the line, so that \( \alpha(0) = 1; A \) is a normalizing constant defined by

\[
A \int_{-\infty}^{\infty} \alpha(x') dx' = 1
\]

(2)

(3)

(4)

(5)

(6)

The quantity \( \lambda \alpha(x) dx \) is the probability that the frequency of a photon emitted after scattering will be between \( x \) and \( x + dx \); \( \lambda \leq 1 \) is the ratio of the scattering coefficients (probability of survival of a quantum on scattering); \( I(\tau, \eta, x) \) is the intensity of the radiation, and \( 4\pi g(\tau) \) is the strength of the sources of radiation. It is assumed that the energy emitted by the source and the coefficient of absorption are functions of frequency only.

Equation (1) is to be solved subject to the boundary condition

\[ I(0, \eta, x) = 0 \quad \text{when} \quad \eta < 0, \]

which indicates the absence of incident radiation at the boundary of the medium.

Equation (1) may be derived in the usual way. It is also a consequence of a more general equation for the transport of radiation in the case of incoherent scattering which takes into account absorption in the continuous spectrum as well as in the line (cf. [10]).

It is clear from Eq. (1) that the intensity of the radiation is in fact a function not of the quantities \( \eta \) and \( \chi \) separately but of the ratio

\[
z = \frac{\eta}{\alpha(x)}.
\]

Denoting the intensity as a function of \( \tau \) and \( z \) by \( I \) as before, we obtain the following expressions, instead of Eqs. (1) and (3), respectively:

\[
z \frac{dI(\tau, z)}{d\tau} = I(\tau, z)
\]

\[
- \frac{\lambda}{2} \int_{-\infty}^{\infty} I(\tau, z') G(z') dz' - Ag(\tau)
\]

and

\[ I(0, z) = 0 \quad \text{when} \quad z < 0, \]
where

\[ G(z) = 2A \int_{\frac{z}{x(x)}}^{\infty} x^2(y) \, dy, \quad (7) \]

and

\[ x(z) = 0 \quad \text{when} \quad z \leq 1; \]
\[ x(z) = \frac{1}{z} \quad \text{when} \quad z \geq 1. \]

In the case of the Doppler absorption coefficient \( \alpha = \exp(-x^2) \), the function \( G(z) \) is given by

\[ G(z) = \begin{cases} 
\frac{1}{V^\frac{1}{2}} & \text{when} \quad z \leq 1, \\
\frac{1}{V^\frac{1}{2}} \left( 1 - \frac{2}{V\pi} \int_0^{\frac{3\ln z}{2}} e^{-\alpha t} \, dt \right) & \text{when} \quad z \geq 1.
\end{cases} \quad (8) \]

We note that for large \( z \) we have the asymptotic expansion

\[ G(z) \approx \frac{1}{2V\pi z^2 V\ln z} \times \left( 1 - \frac{1}{4\ln z} + \frac{3}{16(\ln z)^2} - \cdots \right). \quad (9) \]

The integral equation for the source function

\[ B(\tau) = \frac{\lambda}{2A} \int_{-\infty}^{\infty} I(\tau-z') G(z') \, dz' + g(\tau) \quad (10) \]

can easily be obtained from the transport equation (5) subject to the boundary condition (6) and is of the form

\[ B(\tau) = \frac{\lambda}{2} \int_0^{\infty} K(\tau-t') B(t') \, dt' + g(\tau), \quad (11) \]

where

\[ K(t) = \int_0^{\infty} e^{-\frac{t}{x^2}} G(z') \, dz'. \quad (12) \]

Equation (11) is the fundamental integral equation for the problem under consideration and will be considered in detail below.

A further point must be emphasized at the outset. The equation for the source function in the case of diffusion of radiation without change in frequency is

\[ B(\tau) = \frac{\lambda}{2} \int_0^{\infty} Ei(\tau-t') B(t') \, dt' + g(\tau), \quad (13) \]

where

\[ Ei(t) = \int_0^t e^{-\frac{t}{x^2}} \, dx. \quad (14) \]

The similarity between (11) and (13) is very important. It enables us to employ the methods used earlier in the solution of (13) to solve (11). However, it must be remembered that the physical processes described by these two equations are rather different. Thus, in the case of incoherent scattering, the intensity at large distances from the source should fall off comparatively slowly. Physically, this is due to the fact that in each scattering event the photons may be re-emitted in distant wings where the absorption coefficient is small. Mathematically, this property of the intensity is a consequence of the fact that the kernel of Eq. (11) falls off slowly as \( \tau \to \infty \). It may be shown that, for example, in the case of the Doppler absorption coefficient, the following asymptotic expansion will hold for large \( \tau \):

\[ K(\tau) \approx \frac{1}{2V\pi\tau^2 V\ln \tau} \times \left[ 1 + \frac{1}{4}(\Gamma'(2) - \frac{1}{2}\Gamma(2)) \frac{1}{\ln \tau} + \cdots \right] \]

\[ = \frac{1}{2V\pi\tau^2 V\ln \tau} \left( 1 - \frac{0.0386}{\ln \tau} + \cdots \right), \quad (15) \]

where \( \Gamma(\kappa) \) is the gamma function. This relatively slow decrease in \( K(\tau) \) is responsible for the fact that it is impossible to obtain an approximate reduction of the problem to the solution of a differential equation, and limits the applicability to Eq. (11) of approximate methods developed for the determination of the source function satisfying Eq. (13).

2. On the Resolvent of the Fundamental Integral Equation

We shall now summarize the results obtained earlier [14] as a result of the study of Eq. (11), which will be useful in our analysis below. These results were deduced with the aid of the method developed by V. V. Sobolev [15, 16, 10].

Solution of Eq. (11) for an arbitrary \( g(\tau) \) may be reduced to the determination of the resolvent of this equation \( \Gamma(\tau, \tau') \). The latter may be expressed in the terms of the function \( \Phi(\tau) = \Gamma(0, \tau) \) for \( \tau^* > \tau \) as follows:

\[ \Gamma(\tau, \tau') = \Phi(\tau'-\tau) + \int_0^{\tau} \Phi(y) \Phi(y+\tau'-\tau) \, dy. \quad (16) \]

The function \( \Phi(\tau) \) is a solution of the equation

\[ \Phi(\tau) = \frac{\lambda}{2} \int_0^{\infty} K(\tau-t') \Phi(t') \, dt' + \frac{\lambda}{2} K(\tau). \quad (17) \]

It may also be shown that

\[ \overline{\Phi}\left(\frac{1}{2}\right) = H(z) - 1, \quad (18) \]

where \( \overline{\Phi}(t) \) is the result of the application of the Laplace transform to \( \Phi(t) \), i.e.,
\[ \Phi(s) = \int_{0}^{\infty} e^{-s\tau} \Phi(\tau) d\tau, \quad (19) \]

and the function \( H(z) \) satisfies the equation
\[ H(z) = 1 + \frac{\lambda}{2} z H(z) \int_{0}^{\infty} \frac{H(z')}{z + z'} G(z') dz'. \quad (20) \]

The relation given by (18) was used to find the asymptotic formulas for \( \Phi(\tau) \) for \( \tau \gg 1 \). It was found that for the Doppler absorption coefficient and large \( \tau \), this function is given by
\[ \Phi(\tau) \approx \frac{\lambda}{4\pi(1 - 2\eta^2 \tau^2)} \frac{V}{\ln \tau} \quad (21) \]
in the case \( \lambda < 1 \), and by
\[ \Phi(\tau) \approx 2\pi^{-\frac{1}{4}} \tau^{-\frac{1}{4}} (\ln \tau)^{-\frac{1}{4}} \quad (22) \]
in the case \( \lambda = 1 \).

The formulas for \( \Phi(\tau) \) at \( \tau \gg 1 \) do not, of course, yield the complete solution of the problem, although they do reveal the general character of the radiation field at large depths and may be used to find the probability of escape of a photon from low-lying layers in the medium.

Let us denote the probability that a photon absorbed at a depth \( \tau \) will leave the medium at an angle \( \cos^{-1} \eta \) in the solid angle \( d\omega \) and the frequency interval \((x, x+dx)\) by \( P(\tau, \eta, x) d\omega dx = A(\alpha(x) P(\tau, \eta, x)). \) The function \( P(\tau, z) \) satisfies the equation
\[ \frac{\partial P(\tau, z)}{\partial \tau} = -\frac{1}{z} P(\tau, z) + \frac{\lambda}{4\pi} H(z) \Phi(\tau), \quad (23) \]
where
\[ (0, z) = \frac{\lambda}{4\pi} H(z). \quad (24) \]

A knowledge of \( P(\tau, z) \) will enable us to determine the intensity \( I(0, \eta, x) \) of the radiation leaving the medium for an arbitrary distribution of sources \( g(\tau) \). This intensity is given by
\[ I(0, \eta, x) = \frac{4\pi A}{\lambda} \int_{0}^{\infty} g(\tau) P(\tau, \eta, x) \frac{dx}{\eta}. \quad (25) \]

Knowing \( \Phi(\tau) \) for \( \tau \gg 1 \), we can use Eq. (23) to obtain asymptotic formulas for \( P(\tau, z) \) at large \( \tau \). It is found that
\[ P(\tau, z) \approx \frac{\lambda}{4\pi} H(z) \Phi(\tau) \quad \text{when} \quad \tau \gg z \quad (26) \]
and
\[ P(\tau, z) \approx \frac{\lambda}{4\pi} H(z) \left(1 + \int_{0}^{\tau} \Phi(\tau') d\tau'\right) \quad \text{when} \quad z \gg \tau. \quad (27) \]

The difference in the form of the asymptotic formulas for \( P(\tau, z) \) at \( \tau > z \) has a simple physical interpretation. Photons emitted at depths \( \tau \gg 1 \) at frequencies \( x \) and angles \( \cos^{-1} \eta \) such that \( \alpha(x) \tau/\eta = \tau/z \gg 1 \) are practically unable to leave the medium without being scattered. On the other hand, photons emitted with \( \eta > 0 \) in distant wings of a line, or more precisely, photons with \( \alpha(x) \tau/\eta = \tau/z \ll 1 \), will in general leave the medium directly.

3. The Function \( H(z) \)

In the present context the function \( H(z) \) plays the same fundamental role which in the case of coherent isotropic scattering of light in a semitransparent medium is taken by Ambartsumyan's function \( \phi(\eta) \). The use of the function \( H(z) \) leads to a simple expression for the intensity of radiation leaving the medium in the case of certain simple source distributions, including the exponential and constant distributions considered below. The properties of \( H(z) \) determine also the resolvent of the fundamental integral equation for the problem given by Eq. (11). This function is also encountered in the solution of the problem of emission of radiation by a plane layer having a large, though finite, optical thickness. The function \( H(z) \) was investigated in the previous paper [14] for a number of special forms of the absorption coefficient. It was found that for any \( \alpha(x) \)
\[ \frac{\lambda}{2} \int_{0}^{\infty} H(z') G(z') dz' = 1 - \sqrt{1 - \lambda} \quad (28) \]
and
\[ H(\infty) = \frac{1}{\sqrt{1 - \lambda}}. \quad (29) \]

Equation (20) was used to investigate the asymptotic behavior of \( H(z) \) for \( z \to \infty \). It was found that for the Doppler absorption coefficient and \( z > 1 \)
\[ H(z) \approx \frac{1}{\sqrt{1 - \lambda}} - \frac{\lambda}{4\pi} \frac{V}{\ln z} \quad (30) \]
in the case of \( \lambda < 1 \), and
\[ H(z) \approx 2\pi^{-\frac{1}{4}} z^{-\frac{1}{4}} (\ln z)^{-\frac{1}{4}} \quad (31) \]
in the case of pure scattering.

It is possible to take the analysis of the equation for \( H(z) \) much further than was done in [14] and obtain its exact solution in an explicit form. The transport equation (5) can be solved by the method of Chandrasekhar by approximately replacing the integral on the right-hand side of (5) by an integral sum. On repeating exactly all the steps given in the case of diffusion of radiating without change in frequency (cf. [17], Chapters III and V) it is found that in the case of multiple scattering with complete frequency redistribution
\[ \ln H_n(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \ln T_n(w) \frac{dw}{w^2 - z^2}, \quad (32) \]
where

$$T_n(w) = 1 - n w^2 \sum_{j=1}^{n} \frac{A_j G(z_j)}{w^2 - z_j^2}. \quad (33)$$

Here $H_\nu(z)$ is the $H$-function in the $n$-th approximation of Chandrasekhar, and $A_j$ and $z_j$ are the weights and the division points in the quadrature formula

$$\sum_{j=1}^{n} f(z_j) dz = \sum_{j=1}^{n} A_j f(z_j).$$

The choice of the specific quadrature formula is immaterial, and Eq. (32) will hold for any such formula.

It may be assumed that by letting $n$ tend to infinity in Eq. (32), we will obtain the exact solution for the equation $H(z)$. We shall not consider the rigorous proof of the existence of this limit, or more precisely, the limitations which must be imposed on $\alpha(x)$ in order that this limiting procedure should be a proper one. We shall simply assume that for those coefficients of absorbance with which we will be concerned, including the Doppler absorption coefficient, the function $H_\nu(z)$ will tend to the solution of Eq. (30) as $n \to \infty$. We shall see later that this is in fact so. On taking the limit corresponding to $n \to \infty$ in Eqs. (32) and (33), we find that

$$\ln H(z) = \frac{1}{2 \pi i} \int_{-\infty}^{+\infty} \ln T(w) \frac{dw}{w^2 - z^2}, \quad (34)$$

where

$$T(w) = 1 - w^2 \int_{0}^{\infty} \frac{G(x) dx}{w^2 - x^2}. \quad (35)$$

Substituting $w = iv$ into Eq. (34), we obtain the following final expression:

$$\ln H(z) = -\frac{1}{\pi} \int_{0}^{\infty} \ln S(v) \frac{dv}{v^2 + z^2}, \quad (36)$$

where

$$S(v) = 1 - \lambda v^2 \int_{0}^{\infty} \frac{G(x) dx}{x^2 + v^2}. \quad (37)$$

These formulas are in fact the exact solution of Eq. (20) in an explicit form.

We shall not investigate this solution in any great detail and will only indicate how it can be used to obtain the above formulas for $H(z)$ at $z \gg 1$ (in the case of the Doppler absorption coefficient). Using (9) it is easy to show that when $\nu \gg 1$,

$$S(v) \approx 1 - \lambda + \frac{\lambda}{4} \frac{V \pi}{\nu} \frac{1}{\sqrt{\ln \nu}}. \quad (38)$$

It may be shown that when $z \gg 1$, the magnitude of the integral given by Eq. (36) is largely determined by those values of the integrand which correspond to $\nu \gg 1$, where the expression given by Eq. (38) will hold. Equations (30) and (31) can easily be deduced from this result. They were found earlier [14] by a direct analysis of Eq. (20).

From the point of view of applications, the Doppler profile of the absorption coefficient is of the maximum interest. The function $H(z)$ was therefore computed for this case for a number of values of $\lambda$. It was found by numerical integration of Eq. (20) by the method of successive approximations. The calculations were largely carried out on the electronic computer "Ural". The computer calculations were carried out together with Yu. F. Kazarinov.

Figure (1) shows graphs of the function $H(z)$ for different $\lambda$, and is based on the above computer calculations. The values of $H(z)$ in the case of pure scattering are given in the table. Detailed tables of the function $H(z)$ will be published in a subsequent paper.

4. Milne's Problem for Incoherent Scattering

It is well known that in the case of diffusion of radiation without change in frequency, it is possible to pose the following problem: What is the radiation field in a semi-infinite medium in the absence of sources and for a given flux of radiation leaving the medium? Pure scattering is assumed. This problem is known as Milne's problem, and in the case of isotropic scattering may be reduced to the solution of the homogeneous equation

$$B(\tau) = \frac{\lambda}{2} \int_{0}^{\infty} Ei(|\tau - \tau'|) B(\tau') d\tau' \quad (39)$$

with $\lambda = 1$. 
The Function $H(z)$ in the Case $\lambda = 1$
(Doppler Absorption Coefficient)

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Milne's problem can also be formulated in another way: What is the radiation field in a semi-infinite medium in the case where the sources lie at an infinitely large depth within the medium? In this form the problem may be set up for any $\lambda$ and corresponds to the solution of Eq. (39) for an arbitrary $1 \leq \lambda$. It is evident that in order to obtain a finite flux at the surface for $\lambda < 1$, the strength of the sources at infinity must be regarded as infinitely large. An exact solution of the problem has been found both for $\lambda = 1$ [18, 19] and $\lambda < 1$ [16].

The problem may also be formulated when there is a complete frequency redistribution on scattering. It is found that the solution of this problem can be expressed in terms of $\Phi(\tau)$ and $H(z)$ in an exceptionally simple fashion.

It is clear that the solution of Milne's problem for incoherent scattering may be reduced to the determination of the nonzero solution of the equation for the source function

$$B(\tau) = \frac{\lambda}{2} \int_{0}^{\infty} K(|\tau - \tau'|)B(\tau')d\tau',$$  \hspace{1cm} (40)

where $K(\tau)$ is given by Eq. (12). We shall proceed as in the case of coherent scattering [16]. Differentiating (40) with respect to $\tau$ we find that

$$B'(\tau) = \frac{\lambda}{2} \int_{0}^{\infty} B'(\tau')K(|\tau - \tau'|)d\tau' + \frac{\lambda}{2} K(\tau).$$  \hspace{1cm} (41)

Since the solution of Eq. (40) is determined to within a constant factor, it may be assumed that $B(0) = 1$.

Comparing (41) with Eq. (17) which defines $\Phi(\tau)$, we conclude that

$$B'(\tau) = \Phi(\tau) + kB(\tau),$$  \hspace{1cm} (42)

where $k$ is a constant. It may be shown using (42) that $k$ must satisfy the relation

$$\int_{0}^{\infty} \frac{H(\tau')}{1 - k\tau'}G(z')dz' = 1.$$  \hspace{1cm} (43)

Since for pure scattering $\int_{0}^{\infty} H(z')G(z')dz' = 2$ [cf. Eq. (28)], it follows from (43) that in this case $k = 0$. It may be shown that Eq. (43) has no roots when $\lambda < 1$.

Hence, for any $\lambda$,

$$B(\tau) = 1 + \int_{0}^{\tau} \Phi(\tau')d\tau'.$$  \hspace{1cm} (44)

This follows directly from (42) since $B(0) = 1$. Equation (44) is a solution of Milne's problem for incoherent scattering.

Knowing $B(\tau)$, it is easy to find the intensity of the radiation leaving the medium. In the present case we have

$$I(0, \eta, x) = A \int_{0}^{\infty} B(\tau')e^{-\frac{a(x)}{\eta} \tau'} a(x) \frac{d\tau'}{\eta},$$

$$= A \left(1 + \int_{0}^{\infty} e^{-\frac{a(x)}{\eta} \tau'} \Phi(\tau')d\tau'\right) = AH\left(\frac{\eta}{a(x)}\right).$$  \hspace{1cm} (45)

Thus the function $H(\eta/a(x))$ gives the relative frequency and angular distribution of radiation leaving the medium when the sources of radiation are located at an infinite depth.

Fig. 2. Line profiles in the case of diffusion radiation through an infinitely thick layer of gas (Doppler absorption coefficient). Radiation leaves along the normal to the layer. Figures on the right are the values of $\lambda$.  

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We note that this result may also be obtained in another way. Let us assume that the sources are distributed in accordance with the law \(1/\theta(\tau')\) \(\delta(\tau - \tau')\), where \(\delta(x)\) is the Dirac delta function, i.e., the sources are in the form of a thin layer at a depth \(\tau\). Hence when \(\tau \gg 1\), we find from (25) and (26) that

\[
I(0, \eta, x) \approx AH \left( \frac{\eta}{\alpha(x)} \right). \tag{46}
\]

If now \(\tau\) tends to infinity, the approximate relation becomes exact, since in the limit \(\tau \to \infty\) the formula given by (26) is exact.

The line profiles arising as a result of diffusion of radiation through an infinitely thick gas layer is shown in Fig. 2, which is based on Eq. (45). The curves are plotted for the Doppler absorption coefficient and \(\eta = 1\) (radiation leaves the medium along the normal to the boundary). As can be seen, this yields an ordinary absorption line. Radiation entering the medium at large depths, largely in the form of photons with frequencies approaching the frequency of the center of the line, exhibits a radical change in its spectral composition. It is transformed by the medium into a continuous spectrum.

The problem considered above is very idealized. However, its solution is interesting in many respects. In particular, it shows the type of transformation which a beam of radiation will undergo on diffusing through a thick layer of gas.

One would expect that when sources of radiation lie within the medium, the spectral line profiles will be very broad, with a minimum at the center. The depth of this minimum should depend on the particular source distribution function. It increases with increasing depth of sources in the medium. In the following section we shall show that these qualitative considerations are in fact correct.

5. Spectral Line Profiles

The above results may be used to interpret the profiles of emission lines in the solar spectrum, including the recently obtained \(L_\alpha\) profile. The associated problems will be discussed in detail in a future paper. However, it seems useful to report here some of the results which may be of practical interest.

In view of the applications mentioned above, we shall now derive an expression for the intensity of radiation leaving the medium in the case where the sources of radiation are distributed so that

\[
g(\tau) = Ce^{-m\tau}. \tag{47}
\]

Equation (25) is then of the form

\[
I(0, \eta, x) = \frac{4\pi A}{\lambda} C \int_0^\infty e^{-m\tau} P \left( \frac{\eta}{\alpha(x)} \right) \alpha(x) \frac{dx}{\eta}. \tag{48}
\]

The integral in this expression may easily be found from Eq. (23). To do this, Eq. (23) should be multiplied by

![Fig. 3. Line profiles in the case of an exponential distribution of sources in the medium. The Doppler absorption coefficient is assumed. The curves correspond to \(\lambda = 1\), \(\eta = 1\) and \(m = \infty\), 2.00, 0.50, 0.30, 0.20 and 0.15; the width of the line increases with decreasing \(m\).](image)
exp(-mτ) and then integrated with respect to τ between 0 and ∞. The result is

$$I(0, \eta, z) = ACH \left( \frac{1}{m} \right) \frac{H \left( \frac{\eta}{\alpha(x)} \right)}{1 + m \frac{\eta}{\alpha(x)}}.$$  \hspace{1cm} (49)

In particular, when the source distribution is uniform (m = 0) and λ < 1, we find that since H(∞) = (1-λ)^{-1/2},

$$I(0, \eta, z) = AC \frac{H \left( \frac{\eta}{\alpha(x)} \right)}{\sqrt{1 - \lambda}}.$$  \hspace{1cm} (50)

Equation (49) may be used to determine the characteristic features of line profiles. For sufficiently small m, the intensity at the center of a line should increase with decreasing |x|, owing to the increase in H[η/α(x)]. Conversely, in the distant wings of a line [when α(x) ≈ mη], the intensity of the radiation leaving the medium decreases in proportion to H[η/α(x)]. Since for λ < 1, $H(\eta/\alpha(x)) \approx (1-\lambda)^{-1/2}$ when |x| → ∞, it follows that the intensity in the wing decreases in proportion to $\alpha(x)$. When λ = 1 the situation is different. In the case of the Doppler absorption coefficient, the function H(z) is roughly proportional to $z^{1/2} \ln(z)$ when z → ∞, i.e., if one considers the frequency dependence only, it is proportional to $\sqrt{|x|} e^{x/2}$. Hence, the intensity in the wing decreases in proportion to $\sqrt{|x|} e^{x/2}$, i.e., much more slowly than $\alpha(x) = \exp(-x^2)$.

Equation (49) has been used to compare a large number of line profiles in the case of the Doppler absorption coefficient. The values of the function H(z) given in Section 3 were used in these calculations. Figure 3 shows the line profiles, calculated for a number of values of m and λ = 1. It was assumed that the radiation leaves the medium along the normal to the boundary (η = 1). These graphs show that the width of the lines increases with decreasing m. The depth of the minimum at the center and the distance between the maxima are also found to increase with decreasing m. It should be noted that the computed profiles reproduce the characteristic features of the solar Lα profiles well.

Equation (49) can also be used to investigate the change in the profiles with varying η, i.e., in the transition from the center of the disc to the limb in the case of the sun. The change in the profiles with varying η can be seen in Fig. 4. The graphs shown in the latter figure were plotted for λ = 1 and m = 0.30.

Observations [20] appear to indicate that the central minimum in the profile of the solar Lα line is deeper at the limb than at the center of the disc, and the distance between the maxima increases between the center of the disc and the limb. These regularities are also observed for the emission cores of the H and K lines [21]. It is clear from Fig. 4 that the theory and the observations are in a qualitative agreement. As was pointed out above, a quantitative comparison between the theory and the observational data will be given in a future paper.

Fig. 4. The change in the line profiles with η (λ = 1, m = 0.30).
LITERATURE CITED
13. V. V. Ivanov, Vestn. LGU, No. 19, 117 (1960).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. Some or all of this periodical literature may well be available in English translation. A complete list of the cover-to-cover English translations appears at the back of this issue.