LETTERS TO THE EDITOR

A GENERAL VARIATIONAL PRINCIPLE GOVERNING THE RADIAL
AND THE NON-RADIAL OSCILLATIONS OF GASEOUS MASSES

The non-radial oscillations of gaseous masses have not received the attention they
might deserve; and the reason for this, one might suppose, is, in part, the complexity of
the equations which follow from a spherical harmonic analysis of the normal modes. (For
an account of the extant information see Ledoux and Walraven 1958; also, Chandrase-
khari and Lebovitz 1962, 1963.) However, in this case, as in others, a variational for-
mulation may make possible a systematic attack on the problem along rigorous lines. We shall
show that such a formulation is possible.

Consider the result of a slight perturbation of a spherically symmetric equilibrium
configuration governed by the equations

$$\frac{d \rho}{dr} = \rho \frac{d \mathcal{B}}{dr} \quad \text{and} \quad \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \mathcal{B}}{dr} \right) = -4\pi G \rho, \quad (1)$$

where the various symbols have their standard meanings. Let the departures from equi-
librium caused by the perturbation be described in terms of a Lagrangian displacement
of the form

$$\xi(x) e^{\text{i} \omega t}, \quad (2)$$

where $\sigma$ denotes the characteristic frequency that is to be determined. The linearized
equation of motion governing the perturbation is, then,

$$\sigma^2 \rho \xi = \frac{\partial}{\partial x_i} \delta \rho - \frac{\delta \rho}{\rho} \frac{\partial \rho}{\partial x_i} - \rho \frac{\partial}{\partial x_i} \delta \mathcal{B}, \quad (3)$$

where $\delta \rho_i$, $\delta \rho$, and $\delta \mathcal{B}$ are the Eulerian changes in the respective quantities caused by the
displacement. On the further assumption that the changes in the pressure and the density
take place according to the laws appropriate to a gas subject to adiabatic changes, we
have

$$\delta \rho = -\xi_k \frac{\partial \rho}{\partial x_k} - \gamma \rho \text{ div } \xi \quad \text{ (4)}$$

and

$$\delta \rho = -\xi_k \frac{\partial \rho}{\partial x_k} - \rho \text{ div } \xi = -\text{ div } (\rho \vec{\xi}), \quad (5)$$

where $\gamma$ denotes the ratio of the specific heats.

Let $\sigma^{(\lambda)}$ denote a characteristic frequency; and let the proper solutions belonging to
it be distinguished by the same superscript. Consider equation (3) belonging to $\sigma^{(\lambda)}$ and
after multiplication by $\xi^{(\mu)}$ (belonging to a different characteristic frequency $\sigma^{(\mu)}$)
integrate over the volume occupied by the fluid. We obtain

$$\left[ \sigma^{(\lambda)} \right]^2 \int_V \rho \xi^{(\mu)} \cdot \xi^{(\mu)} d \mathbf{x} = \int_V \xi^{(\mu)} \frac{\partial}{\partial x_i} \delta \rho^{(\lambda)} d \mathbf{x} - \int_V \frac{\delta \rho^{(\lambda)}}{\rho} \xi^{(\mu)} \frac{\partial \rho^{(\lambda)}}{\partial x_i} d \mathbf{x} - \int_V \rho \xi^{(\mu)} \frac{\partial}{\partial x_i} \delta \mathcal{B}^{(\lambda)} d \mathbf{x}. \quad (6)$$
Let us integrate by parts the first and the last integrals on the right-hand side and remembering that \( \delta \rho \) and \( \rho \) are required to vanish on the boundary of \( V \), we have

\[
[s] \int_V \rho \xi^{(\lambda)} \cdot \xi^{(\mu)} \, dx = \int_V \left[ \xi_{\lambda}^{(\lambda)} \frac{\partial \rho}{\partial x_k} + \gamma \rho \text{ div } \xi^{(\lambda)} \right] \text{ div } \xi^{(\mu)} \, dx
\]

\[
+ \int_V \left[ \xi_{\lambda}^{(\lambda)} \frac{\partial \rho}{\partial x_k} + \text{ div } \xi^{(\lambda)} \right] \xi_{\mu}(\mu) \frac{\partial \rho}{\partial x_i} \, dx - \int_V \Delta \rho^{(\mu)} \Delta \xi^{(\lambda)} \, dx,
\]

where we have substituted for \( \delta \rho^{(\lambda)} \), \( \partial \rho^{(\lambda)} \), and \( \text{ div } [\rho \xi^{(\mu)}] \) in accordance with equations (4) and (5). Rearranging the various terms on the right-hand side of equation (7) and expressing \( \delta \xi^{(\lambda)} \) in terms of \( \delta \rho^{(\lambda)} \) by means of Poisson's integral, we finally have

\[
[s] \int_V \rho \xi^{(\lambda)} \cdot \xi^{(\mu)} \, dx = \gamma \int_V \rho \text{ div } \xi^{(\lambda)} \text{ div } \xi^{(\mu)} \, dx
\]

\[
+ \int_V \frac{\partial \rho}{\partial x_k} \left[ \xi_{\lambda}^{(\lambda)} \text{ div } \xi^{(\mu)} + \xi_{\lambda}^{(\lambda)} \text{ div } \xi^{(\lambda)} \right] \, dx
\]

\[
+ \int_V \frac{\xi^{(\lambda)} \cdot x [\xi^{(\mu)} \cdot x]}{r^2 \rho} \frac{d \rho}{dr} \frac{d \rho}{dr} \, dx
\]

\[
- G \int_V \int_V \frac{\delta \rho^{(\mu)}(x) \delta \rho^{(\lambda)}(x')}{|x - x'|} \, dx \, dx'.
\]

The right-hand side of equation (8) is clearly symmetric in \( \lambda \) and \( \mu \); accordingly

\[
\int_V \rho \xi^{(\lambda)} \cdot \xi^{(\mu)} \, dx = 0 \quad (\lambda \neq \mu).
\]

The underlying characteristic value problem is, therefore, self-adjoint; and the equation obtained by setting \( \lambda = \mu \) in equation (8) and suppressing the distinguishing superscripts, namely,

\[
\sigma^2 \int_V \rho \left| \xi \right|^2 \, dx = \int_V \left[ \gamma \rho (\text{ div } \xi)^2 + \frac{2}{r} \frac{d \rho}{dr} (x \cdot \xi) \text{ div } \xi \right] \, dx
\]

\[
+ \int_V \frac{(x \cdot \xi)^2}{r^2 \rho} \frac{d \rho}{dr} \frac{d \rho}{dr} \, dx - G \int_V \int_V \frac{\text{ div } (\rho \xi) \cdot x [\text{ div } (\rho \xi)]^2}{|x - x'|} \, dx \, dx',
\]

provides a variational base for determining the characteristic frequencies.

It can be readily verified that equation (10), when suitably specialized (by setting \( \xi = xf \), where \( f \) is a function of \( r \) only), reduces to the well-known equation (cf. Ledoux and Walraven 1958, p. 468) giving the frequencies of radial oscillations. Also, on the basis of equation (10), the role of the virial equations in determining the characteristic frequencies of non-radial oscillations (in the manner of Chandrasekhar and Lebovitz 1962, 1963) can be clarified, as well as the occurrence of two classes of modes—the \( p \)- and the \( g \)-modes of Cowling (1942)—with different attributes. These matters, as well as detailed applications of equation (10) to a systematic treatment of non-radial oscillations, are considered in a forthcoming paper.

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References


