CHROMOSPHERIC LINE PROFILES

V. V. Ivanov

Astronomical Observatory, Leningrad State University
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Line profiles are computed for a set of heights in the chromosphere. The transformation of the chromospheric emission lines to Fraunhofer lines of the solar limb is obtained. It is assumed that the mechanism of line formation is scattering with complete redistribution in frequency. The results are compared with observational data.

Several papers have been devoted to a theoretical calculation of the profiles of chromospheric lines, and the problem of transforming the chromospheric emission spectrum to the Fraunhofer spectrum at the limb of the solar disk [1-3]. In all these papers, it was assumed that the scattering in the line occurs without change of frequency. In the present paper chromospheric line profiles are computed for several heights above the limb, using a certain model for the chromosphere, but now with a different assumption about the scattering mechanism: it is here assumed that the scattering leads to a complete redistribution in frequency. A comparison with profiles computed for the same chromospheric model, assuming coherent scattering, shows that the line profiles do not differ very greatly from each other in the two cases. The chromospheric model adopted is too crude, however, to permit a detailed comparison with the observational data.

The calculation of the profile of a spectral line near the solar limb reduces to computation of the integral

\[ I_x(h) = \int_{-\infty}^{+\infty} \epsilon_x(h') \exp \left\{ -\int_{h'}^{\infty} \left[ \alpha_x(h) + \alpha(h') \right] ds' \right\} ds. \]  

(1)

Here \( I_x(h) \) is the intensity of the radiation at the dimensionless frequency \( x \) as observed at the distance \( h \) from the limb; \( \epsilon_x(h') \) is the emission coefficient; and \( \alpha_x(h') \) and \( \alpha(h') \) are the absorption coefficients for the line and the continuum respectively.

The meaning of the quantities \( h, h', \) and \( s \) will be clear from Fig. 1 (O marks the center of the sun, R its radius, and AB the direction to the observer); \( x \) is the distance from the center of the line, expressed in Doppler widths.

Because of the rapid fall in density with height, only the region \( h' < R \) contributes significantly to this integral; therefore

\[ h' = \frac{\nu^2}{2R} + h. \]  

(2)

It is clear that \( \alpha_x \) and \( \alpha \) are given by the chromospheric model adopted, while \( \epsilon_x \) is found by solving the corresponding problem for diffuse radiation.
We shall take a model for the chromosphere in which the density varies exponentially with height; that is,

$$\alpha_x (h') = \alpha_x (0) e^{-\beta h'};$$

the ratio

$$\frac{\alpha_x}{\alpha} = \eta_x$$

is regarded as independent of $h$. Let $t(h)$ denote the optical thickness in the continuous spectrum along the line of sight $AB$. We have

$$t (h) = \alpha (0) \sqrt{\frac{2\pi R}{\beta}} e^{-\beta h}. $$

The quantity $h$ is so calibrated that $t(0) = 1$.

This gives

$$\alpha (0) \sqrt{\frac{2\pi R}{\beta}} = 1. $$

We note that the optical depth of the point $O'$, as measured along the radius, is given by

$$\tau (h) = \frac{t (h)}{\sqrt{2\pi R \beta}}. $$

We shall consider lines formed by scattering, assuming that each such line is formed with complete redistribution in frequency. We also take into account any true absorption that may be present. Since $1/R$ is small compared to $\beta$, we may take $\epsilon_x (h')$ to be the emission coefficient appropriate for a plane, semi-infinite medium.

It is not difficult to show that this is tantamount to neglecting quantities of order $2/R$ compared with $\beta$.

The required emission coefficient is therefore given by the sum of the last two terms on the right-hand side of the equation

$$\cos \Theta \frac{dI_x (h', \theta)}{dh'} = -(\alpha_x + \epsilon_x) I_x (h', \theta) + (t - \epsilon_x) \frac{\int_{-\infty}^{+\infty} I_x (h') \alpha_x dx'}{\int_{-\infty}^{+\infty} \alpha_x dx'} +$$

$$+ (\alpha_x + \epsilon_x) B, $$

where

$$\tilde{I}_x (h') = \int I_x (h', \theta) \frac{d\theta}{4\pi}. $$

The usual notation is used here. The introduction of $\epsilon$ allows for the true absorption.

In the following discussion, we shall regard the Planck function $B$ as constant: $B = B_0$. The boundary condition has the form

$$I_x (0, \theta) = 0, \ \theta > \frac{\pi}{2}. $$

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To solve Eq. (4), we make two approximations. First, when developing $I_x(h, \theta)$ in a series of Legendre polynomials, we retain only the zeroth and first terms (Eddington approximation). Secondly, we shall assume that at the central frequency, $x = 0$, diffusion occurs without redistribution in frequency, and

$$
\int_{-\infty}^{+\infty} \int_0^{+\infty} \alpha_x d\alpha' dx' = \int_0^{+\infty} \alpha_x dx' = \tilde{T}_0(h').
$$

This approximation was introduced by Spitzer [4].

Solving Eq. (4) under the boundary condition (5), and using the approximations mentioned, we obtain an expression for $\varepsilon_\chi(h')$ in the form

$$
\varepsilon_\chi(h') = (1 - \varepsilon) A \alpha_x e^{-\lambda \tau} + B_0 (\alpha_x + \alpha), \quad (6)
$$

where

$$
\tau = \frac{\int_{h'}^\infty d\alpha h^*}{h} = \frac{\alpha(0)}{\beta} e^{-\beta h'},
$$

$$
\lambda^2 = 3 (1 + \eta_0) (1 + \varepsilon \eta_0),
$$

$$
A = \frac{B_0}{1 + \frac{2\lambda}{3(\eta_0 + 1)}}. \quad (7)
$$

We remark that the emission coefficient for the same problem, but with coherent scattering assumed, can be obtained from (6) by replacing $\eta_0$ by $\eta_\chi$ in (7) and (8).

We now introduce into (1) the explicit formulations of $\alpha_\chi(h'), \alpha(h')$ and $\varepsilon_\chi(h')$, first writing $\varepsilon_\chi(h')$ in the form

$$
\varepsilon_\chi(h') = B_0 (\alpha_x + \alpha) + (1 - \varepsilon) A \alpha_x + (1 - \varepsilon) A \alpha_x (e^{-\lambda \tau} - 1).
$$

From (2) and (3) we obtain, after some transformations,

$$
I_x(h) = B_0 \left\{ 1 - \exp \left\{ -\left( \eta_x + 1 \right) e^{-\beta h} \right\} \left\{ 1 + (1 - \varepsilon) \frac{A \eta_x}{B_0} \eta_x + 1 \right\} - \right. 
$$

$$
- (1 - \varepsilon) A \frac{2\lambda}{V 2\pi R_B} \frac{\eta_x + 1}{\eta_x + 1} e^{-\beta h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-z \times z' \times z'} \left. \right\} dz
$$

$$
\times \exp \left( -\frac{\lambda}{V 2\pi R_B} e^{-\beta h} e^{-z \times z'} \frac{\eta_x + 1}{V \pi} e^{-\beta h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-z' \times z'} dz' \right) dz. \quad (9)
$$

*It is easy to see that if $\alpha_\chi = \alpha_0 e^{-\chi^2}$, this last equality is none other than the Gauss formula for numerical quadrature,

$$
\int_{-\infty}^{+\infty} e^{-x^2} f(x) dx = \sum_{k=-\infty}^{n} A_k f(x_k)
$$

with $n = 0$. Here the $x_k$ are the roots of the $(2n+1)$th-order Hermite polynomials, and the $A_k$ are weight factors.
In many cases the following expression for $I_x(h)$, identical with (9), is more convenient for computation:

$$I_x(h) = B_0 \left( 1 - \exp \left[ -(\eta_x + 1) e^{-\beta h} \right] \right) \left\{ 1 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \frac{\eta_x}{\eta_x + 1} \right\}$$

$$- (1 - \varepsilon) A \frac{4h}{\sqrt{2\pi R^B}} \frac{\eta_x}{\eta_x + 1} \exp \left( -\beta h - \frac{\eta_x + 1}{2} e^{-\beta h} \right) \int_0^\infty z \times$$

$$\times \exp \left( -z^2 - \frac{\lambda}{\sqrt{2\pi R^B}} e^{-\beta h} e^{-z^2} \right) \text{sh} \left( \frac{\eta_x + 1}{2} e^{-\beta h} \text{erf} z \right) dz,$$

where

$$\text{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

If the optical thickness along the line of sight at the frequency $x$ is

$$t_x(h) = (\eta_x + 1) e^{-\beta h} \ll 1,$$

then we have from (10), to within terms of order $t_x^2(h)$,

$$I_x(h) = B_0 \left\{ 1 + \left[ 1 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \right] \eta_x e^{-\beta h} - \frac{B_0}{2} \left[ 1 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \right] \eta_x^2 + \right.$$  

$$+ \left[ 2 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \left( 1 + \frac{\lambda}{\sqrt{2\pi R^B}} \right) \right] \eta_x + 1 \right\} e^{-\beta h}.$$

In particular, it follows that if

$$t_0(h) = (\eta_0 + 1) e^{-\beta h} \ll 1$$

(an optically thin layer), the profile of an emission line is given by the formula

$$1 + \left[ 1 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \right] \eta_x.$$

This result can be derived from (1) and (6) directly by taking the exponential factor there as unity. This is justified, because

$$\int_{-\infty}^{+\infty} (\alpha_x + \alpha) ds' \ll \int_{-\infty}^{+\infty} (\alpha_0 + \alpha) ds' = t_0(h) \ll 1$$

and

$$\lambda \tau(h) \ll \lambda \tau(h') \leq \sqrt{3}(1 + \eta_0) \frac{\alpha(0)}{\beta} e^{-\beta h} = \sqrt{\frac{3}{2\pi R^B}} t_0(h) \ll t_0(h) \ll 1.$$

Finally, by passing to the limit in (10) as $R \to \infty$ and $h \to \infty$, we obtain

$$I_x = B_0 \left[ 1 + \left( 1 - \varepsilon \right) \frac{A}{B_0} \frac{\eta_x}{\eta_x + 1} \right].$$

(11)

This is the intensity of the radiation from the solar limb, derived without allowance for curvature of the layers.
Formula (11) can also be obtained readily from (6) in view of the known relation

$$\lim_{\theta \to \pi} I_x(0, \theta) = B_x(0),$$

where

$$B_x(\tau) = \frac{e_x}{\alpha_x + \alpha}.$$ 

We note that the term with the integral in (10) does allow for curvature. It is found to be comparable to, and in many cases even exceeds, the first term.

As already mentioned, upon replacing $\eta_0$ in all the formulas derived by $\eta_x$, we obtain a solution to the problem of the profile of a chromospheric line formed by coherent scattering. This problem was treated in detail by Woolley. He found an expression for the intensity of the emergent radiation corresponding to (10), if $\eta_0$ is replaced by $\eta_x$, and the integral term neglected. This term, which allows for the curvature of the layers, arises because of the factor $e^{-\lambda \tau}$ in the first term of formula (6). Woolley had taken it equal to unity on the ground that, for observations near the limb, $\tau \ll 1$ for the layers concerned. Actually, however, the quantity $\lambda \tau = \sqrt{(1 + \eta_x)(1 + \epsilon \eta_x)} \tau$ is not always small.

As one passes to a consideration of noncoherent scattering, the role of the curvature of the layers must increase as compared with the case where the scattering induces no redistribution in frequency. This is perfectly clear physically. If the scattering is coherent, a quantum of frequency $\omega_0$ reaching the observer arises mainly in a thin layer at an optical depth of the order of 1. For noncoherent scattering, quanta of frequency $\omega_0$ arise, to a marked extent, in shallower layers, where they appear because of transformation of quanta of frequencies $|\chi| < |\omega_0|$. Mathematically, the greater role that the second term in (10) plays in the noncoherent scattering is related to the fact that the quantity $\lambda \tau$ in (6) does not depend on frequency in this case.

![Graph](image)

*Fig. 2.*

Line profiles were computed from formula (10) for different heights above the limb. The computations were carried out for the following values of the parameters: $\beta = 10^{-8}$ $\text{cm}^{-1}$; $\eta_0 = (1/3) 10^4$; $\epsilon = 0$. As an
example, the absorption coefficient in the line was taken as Doppler, so that $\eta_x = \eta \theta e^{-x^2}$; the calculation nevertheless is no more complicated for an arbitrary choice of $\alpha(x)$.

The results are presented in Figs. 2 and 3. Line profiles were also computed for the coherent-scattering case, using the same values of the parameters. The expression obtained from (10) upon replacing $\eta \theta$ by $\eta_x$ was employed for this purpose. The results are presented in Fig. 4. We see that the profiles do not differ very strongly for coherent and noncoherent scattering. They well reproduce the characteristic features of the $H\alpha$, $H$, and $K$ profiles at various heights in the chromosphere, as derived from observations by V. A., and T. V. Krat [5] and Mrs. Smith [6]. In view of the crude model for the chromosphere adopted here, this agreement must be regarded as satisfactory.

There is hardly any point in making a detailed comparison of the theoretical and observed profiles. But one result ought specially to be noted. Figure 2 shows that the line profiles have a characteristic saddle-shaped appearance when the optical thickness along the line of sight lies between about 20 and 60 at the center of the line. The observations indicate [5] that the profile of chromospheric $H\alpha$ already has this shape at $h = 3000$ km. This allows an estimate to be made of the optical thickness $\tau_\theta(0)$ of the chromosphere along a radius, at the center of $H\alpha$:

$$\tau_\theta(0) = \frac{t_\theta(0)}{V \frac{2\pi R^3}{3}} = \frac{t_\theta(h) e^{\beta h}}{V \frac{2\pi R^3}{3}}.$$

If we take $t_\theta(3 \times 10^5) = 60$ to 60, and $\beta h = 3$, we obtain $\tau_\theta(0) = 6$ to 20. We therefore see that the chromosphere is opaque at the center of $H\alpha$. This value of $\tau_\theta(0)$ agrees with the estimate given by McMath et al. [7]. They found that the optical thickness of the chromosphere at the center of $H\alpha$ is considerably greater than 1, but can hardly exceed 50.

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We should mention that of the assumptions we have made (exponential density variation with height, \( \eta_0 = \text{const}, B = \text{const} \)), the roughest is the assumption that \( \eta_0 \) is independent of \( h \). But if we were to refrain from making this restriction, the problem would become much more complicated without, it would seem, materially affecting the results. This has been shown by Woolley [1] for the case of coherent scattering.

In conclusion, I wish to thank V. V. Sobolev for suggesting this topic, and for valuable guidance in carrying out the work.

**LITERATURE CITED**