THE BALMER: PASCHEN RATIO IN THE CHROMOSPHERE AND THE
EQUILIBRIUM POPULATIONS OF HYDROGEN
ANGULAR-MOMENTUM STATES

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ABSTRACT

The work reported in this paper is a test of the hypothesis that the angular-momentum states of hydrogen are not populated according to their statistical weights. The anomalous ratio of the hydrogen Balmer to Paschen lines observed at the 1952 eclipse is investigated in relation to both observations and theoretical calculations. The Paschen decrement is used to test for a reduced population of states of higher angular momentum; such a depopulation of states was previously suggested as accounting for the anomalous ratio. The decrement does not confirm the suggestion. A statistical equilibrium analysis of the angular-momentum states is used to determine departures in the populations from their LTE values. The results indicate that departures are too small to give the observed Balmer:Paschen ratio. An additional calculation shows that it is reasonable to average together the substates of different angular momentum in hydrogen chromospheric calculations.

I. INTRODUCTION

The 1952 eclipse observations have shown an anomalous ratio of intensities between the Balmer and Paschen lines of hydrogen which originate from common upper levels (Athay and Zirin 1957). If the substates of different angular momentum for a given value of the principal quantum number were populated in proportion to their statistical weights and if self-absorption could be neglected, the ratio of the intensities should be in the ratio of the corresponding \( A \nu \)'s, where \( A \) is an average transition probability and \( \nu \) is the frequency. The average \( A \) is defined as

\[
A_{nn'} = \frac{1}{n^2} \sum_{l, l'} (2l + 1) A_{nl', n'l'},
\]

where the prime refers to the lower state of the transition. For \( n \) large \( A \nu \) is proportional to \( 1/n^2 \), so that the Balmer to Paschen ratio should be \( \sim 3.4 \). But, in fact, the observed ratio is about 8 for \( n > 14 \). Self-consistency checks using the continuum and Ca \( \pi \) lines on the eclipse plates indicate that this anomalous behavior is not due to photometric error.

To explain this discrepancy, Athay and Zirin postulated that the substates from \( f \) and beyond must have a very low population relative to the \( s \), \( p \), and \( d \) states. Such a condition might exist if the upper levels were populated for the most part from the \( n = 1 \) and 2 levels and if the collisions between the substates of the same \( n \) were too slow to redistribute the electrons completely according to the statistical weights. If only the \( s \), \( p \), and \( d \) terms were populated, the Balmer to Paschen ratio would be about 6, which approaches more closely to the observed value.

The objective of the work reported in this paper is twofold: (1) to try to find observational evidence which would confirm the suggested explanation for the anomalous relationship between the Balmer and Paschen lines and (2) to investigate the problem theoretically by means of a statistical equilibrium analysis to see whether it is possible to reduce the population of \( f \) terms and higher relative to the \( s \), \( p \), and \( d \) terms. Each of these points will be discussed in turn.
II. OBSERVATIONS

The basis for the discussion of the observational data is the Paschen decrement. This decrement is used to test for an underpopulation of terms of higher angular momentum. As has been mentioned previously, if the principal levels of hydrogen were populated from the levels $n = 1, 2$ mainly by induced radiative transitions, then the electrons must come into the $s, p,$ and $d$ substates and be, to an unknown extent, redistributed in the level $n$ by collisions. On the other hand, for sufficiently large $n$, electrons would come into the level mainly from immediately adjacent levels by induced transitions and from the continuum by recombination. These higher levels would have the electrons distributed in the substates more nearly according to the statistical weights than the terms of lower $n$ because transitions to all the substates would be allowed from the adjacent levels and the continuum. If the change-over in the method of populating the levels takes place at a high enough value of $n$ and if the interlevel collisions are not fast enough to distribute the electrons in the level in proportion to $(2l + 1)$, then the resulting lower population of the terms beyond $p$ might decrease the Paschen emission to a value consistent with the observed Balmer to Paschen ratio; i.e., “cutting off” the terms beyond $p$ would not appreciably affect the Balmer emission, but the Paschen emission would be decreased. However, when the value of $n$ is reached where the change in mode of populating the levels occurs, the Paschen line intensities should show a relative increase. The question next arises as to where this change-over in the mode of populating the levels takes place. If the change takes place anywhere in the range of $n = 11$ to $n = 28$, it should be detectable in the Paschen decrement.

Figure 1 shows two decrements calculated for $T_e > 10^4$. The only effect of a lower $T_e$ would be to steepen somewhat the decrement for smaller values of $n$. The upper decrement takes into account all substates as contributing to the radiation, whereas the lower decrement has only the $s, p,$ and $d$ terms populated and giving rise to emission. In both cases the population of the levels are assumed to have their LTE values. According to what was mentioned previously, the emission might follow the lower decrement for low values of $n$ and the upper curve for large values of $n$. If the effects of departures from LTE were included, it would result in merely increasing the slope of the decrements for low values of $n$, but they would asymptotically approach the plotted LTE decrements for large values of $n$. One should note that the difference in the decrements that are shown is only a constant shift without any change in shape.

Also plotted in Figure 1 are points representing the average of several measured decrements from the 1952 eclipse (Thomas and Athay 1961) and a decrement from a prominence observed at Climax. The observations are normalized to fit the upper decrement for high $n$. The eclipse decrement clearly shows no transition to the lower calculated decrement. However, the prominence decrement does show an apparent transition. It is difficult to reconcile the fact that two greatly different processes might be giving rise to the hydrogen emission in the prominence and the chromosphere; therefore, it is felt that too much weight should not be given the prominence decrement because of the limitation of only one observation. Further attempts are being made at Climax to observe both the Paschen decrement and the Balmer to Paschen ratio in prominences, but it is a difficult task because of the weakness of the lines. The shape of the decrement is independent of $n_e$ and only very slightly dependent on $T_e$; and, since the spectra of quiescent prominences are essentially the same as the chromospheric flash spectrum near 1500 km (Zirin 1960), prominences should provide a good means for studying this problem if enough observations could be obtained to yield reasonable statistics.

The conclusion to be drawn from the observations is that no transition in the mode of populating the upper states is observed from the eclipse decrement in the range of $n = 11$ to $n = 30$. Therefore, the eclipse decrement suggests that if the explanation as offered by Athay and Zirin for the origin of the anomalous Balmer to Paschen ratio is valid, the
transition in the mode of populating the levels takes place for \( n > 30 \). In the next section we investigate this possibility by calculating the equilibrium populations for hydrogen.

### III. CALCULATION OF POPULATIONS

The object of the following calculations is to see whether one can theoretically predict a depopulation of the higher substates relative to \( s \), \( p \), and \( d \). The manner in which this analysis is carried out is to calculate the departures in the populations of levels from that of LTE by means of the equations of statistical equilibrium. A hydrogen atom of five principal quantum levels plus continuum with all the allowed values of \( l \) is taken as a model. This gives a total of fifteen bound levels. An additional set of calculations is also

![Paschen decrements, observed and calculated](image-url)
made for a hydrogen atom of five bound levels plus a continuum, but with the substates treated as if they were populated according to their statistical weights, thus considering only a total of five averaged levels plus a continuum. These calculations consequently serve two purposes: (1) as part of the main problem, to see how the substates of a given \( n \) are populated, and (2), as an interesting side line, to test the assumption which is usually made, that the states of different angular momentum may be averaged together for hydrogen calculation involving collisional interactions. Point 2 is accomplished by comparing the calculations for the case in which the different states of angular momenta are considered in detail with the case in which they are lumped together.

\( \text{a) Transition Rates} \)

All rates to be used in the equations of statistical equilibrium are calculated under the assumption of \( n_e = 10^{11} \) and \( T_e = 2 \times 10^4 \). This value of \( T_e \) was chosen because the transition rates for this temperature had been previously computed. The general conclusions to be drawn from the subsequent calculations would not be changed by changing the value of \( T_e \) within the limits expected for the solar chromosphere and quiescent prominences. The assumed radiation field for all optically thin subordinate transitions is a Planck distribution with a radiation temperature of 6000° K and a dilution factor of 0.5. For the resonance lines, detailed balancing in the radiative transitions is assumed. In the case of the Balmer lines and the Lyman continuum, the two extremes—detailed balance and optical thinness—are treated. The general procedure was to calculate a rate in one direction and then equate it to the inverse process in thermodynamic equilibrium to obtain the rate in the opposite direction. By balancing rates to find the inverse rates, one is assured that the equations of statistical equilibrium will be exactly satisfied in LTE.

The collisional rates\(^1\) which are calculated for this problem are (1) between all levels which correspond to optically allowed transitions plus the transition between the 1s–2s levels; (2) between adjacent substates of the same principal quantum number, i.e., from \( n, l \) to \( n, l - 1 \) and \( n, l + 1 \); and (3) between all \( n, l \) levels and the continuum. Measured cross-sections were used whenever available, and when measurements were not available, a simplified form of the Born cross-section was used. Experimental cross-sections which are available in the literature are 1s–2p (Fite 1959), 1s–2s (Lichten 1959), 1s–3p (Mott and Massey 1949), and 1s–continuum (Fite 1958). For collisions between principal levels, those from the ground states turn out to be the most important for this problem. Collision rates between higher levels of different principal quantum numbers, although faster than those from the ground state, are not significant compared with other rates between these levels.

The form of the Born cross-section used in the cases where measurements were not available is (Massey 1956)

\[
Q_{ij} = \frac{2E_i}{E} |z|^2 \ln \frac{4E}{E_0},
\]

where \( E_i \) is the energy of the level from which the excitation takes place, \( E_0 = E_i - E_j \), and \( |z|^2 \) is the square of the \( z \) component of the radial matrix element. A low-energy cutoff factor \( (E - E_0)/E \), was multiplied into the cross-section to give a more reasonable behavior near threshold. The logarithmic term in the Born cross-section was assumed not to vary significantly over the range of integration so that it could be taken outside the integral for the rate calculation and could be evaluated at the threshold energy.

\(^1\)Recently a set of collision cross-sections has been published for collisions between levels 3 and 4 (McCoyd, Milford, and Wahl 1960; Fisher, Milford, and Pomilla 1960). The cross-sections used in this paper are consistently lower than those given in the references mentioned above. However, they are not lower by a large enough factor to make a significant change in the results, since the transitions between the upper levels would still be dominated by radiative transitions.
The collisional transition rates between the states of the same value of \( n \) but with different values of \( l \) were calculated from equation (1) also. Special mention can be made of a few important points in considering this type of collision. First, the matrix element for the transition between two substates \( n, l \) to \( n, l - 1 \) is given, in atomic units, by

\[
|r| = \frac{3}{2} n \sqrt{(n^2 - l^2)}
\]  

(Condon and Shortley 1957). It might be pointed out that this matrix element increases rapidly as \( n \) increases, indicating that for high values of \( n \) these collisions are very rapid.

The energy levels that were used to calculate threshold energies were taken from the centers of gravity of the \( j \)-level splitting, as given in the Atomic Energy Level Tables (Moore 1949). A Born type cross-section should be very good for this type of collision because the splitting is so small. These collisions are the ones that are important in redistributing the electrons among the substates of a given \( n \). It is principally the inclusion of these collisional transitions that allows one to test the question of the departure of the population of the substates from a proportionality to their statistical weights.

The one collisional transition which corresponds to an optically forbidden transition, \( 1s - 2s \), was included because, as has been stated before, collisions out of the ground state turn out to be the most important rates in determining the population of the ground state. Also it turns out that the \( 1s - 2s \) collision rate is approximately the same as the \( 1s - 2p \) rate.

The ionization cross-sections which were used were again of the Born type with the one exception of the measured cross-section, \( 1s - \text{continuum} \), which was mentioned previously. The formula for the Born cross-section for ionization, as given in Mott and Massey (1949) in c.g.s. units, is

\[
Q_{n, i} = \frac{\pi e^4}{E} \frac{c_{n, i}}{|E_{n1}|} \log \frac{4E}{C_{n1}},
\]  

where \( c_{n, i} \) is proportional to the square of the radial matrix element to the continuum and the values are tabulated in Mott and Massey for the first four principal quantum numbers; the values for the fifth level were obtained by extrapolation; \( C_{n1} \) is a tenth of the ionization energy of the shell. Using this type of cross-section for ionization from the sublevels, i.e., one which depends on \( n \) and \( l \), brings out the important fact that it is more difficult to ionize states of higher angular momentum. The calculations of the rates were treated in the same manner as the excitation rate, i.e., the logarithmic term is assumed constant and taken outside the integral.

All the radiative rates for hydrogen are either well known (spontaneous) or easily calculated (induced and recombinations) for the assumed thermodynamic conditions.

In Table 1 are listed rates to and from \( 4p \) as an example. Values are not listed for the radiative transitions \( 4p - 1s \) or \( 1s - 4p \), since they are assumed to balance in detail. In general, the radiative rates dominated over the collisional rates for level \( n = 4 \). The rates for the collisions between substates vary from a minimum of \( 7.4 \times 10^3 \) for the \( 2p - 2s \) to a maximum of \( 4.6 \times 10^6 \) for \( 5s - 5p \) (units: transitions per sec per atom in initial state of transition). The rates for the five-level model are constructed from the rates calculated for the fifteen-level model by properly averaging the rates between two levels giving rise to the transition.

### b) Statistical Equilibrium Equations and Results

The equations of statistical equilibrium are solved to determine the departures of the population of the levels from that of thermodynamic equilibrium. The equations are written in terms of absolute rates in the following form:

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\[ \sum_{i=1}^{m} n_i P_{ij} = n_j P_{jj} \quad (j = 1, 2, 3, \ldots, m), \quad (4) \]

where \( P_{ij} \) is the total transition rate from \( i \) to \( j \); \( P_{jj} \) is the sum of all ways out of the level; \( n_i \) is the population of level \( i \); \( m \) is the number of levels including the continuum. The solution of the equations may be written as

\[ \frac{n_j}{n_\kappa} = [p]^{-1}p_{\kappa j}, \quad (5) \]

where \([p]^{-1}\) is the inverse matrix of the coefficients of the system of linear equations, \( \kappa \) denotes the continuum, and \( n_j/n_\kappa \) and \( p_{\kappa j} \) are both column vectors. Use of the modified Boltzmann-Saha equation in the form

\[ \frac{n_i}{n_\kappa} = \left( \frac{h^2}{2 \pi m k} \right)^{3/2} \frac{n_e \omega_i}{T_e^{3/2}} \frac{b_i \exp \left( \frac{x_i}{kT_e} \right)}{2}. \quad (6) \]

TABLE 1

<table>
<thead>
<tr>
<th>Sample Rates ( (P_{ij})'s ) for 4p Term*</th>
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<table>
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<tr>
<th>From 4p to i</th>
<th>To 4p from i</th>
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<tr>
<td></td>
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</tr>
<tr>
<td>1s</td>
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<td>2s</td>
<td>9 ( \times 10^6 )</td>
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<tr>
<td>2p</td>
<td>0</td>
</tr>
<tr>
<td>3s</td>
<td>3 ( \times 10^6 )</td>
</tr>
<tr>
<td>3p</td>
<td>0</td>
</tr>
<tr>
<td>3d</td>
<td>3 ( \times 10^6 )</td>
</tr>
<tr>
<td>4s</td>
<td>0</td>
</tr>
<tr>
<td>4p</td>
<td>(Total=1 ( \times 10^7 ))</td>
</tr>
<tr>
<td>4d</td>
<td>0</td>
</tr>
<tr>
<td>4f</td>
<td>0</td>
</tr>
<tr>
<td>5s</td>
<td>0</td>
</tr>
<tr>
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<td>5g</td>
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* All rates in units of transitions per sec per atom in initial level of the transition.

coupled with the solutions \( n_i/n_\kappa \), then allows one to obtain the \( b_i \)'s which measure the departure of the population of the level \( n_i \) from the LTE population for a given \( n_\kappa \).

A true solution to this problem would involve the simultaneous solution of the equations of statistical equilibrium coupled with an equation of transfer for each transition. In order to simplify the problem, as mentioned earlier, two conditions were selected which are assumed to bracket the true solution that would be obtained by including the equation of radiative transfer: case I—all the Lyman lines are in detailed balance and case II—the Lyman lines plus the Lyman continuum and the Balmer lines are in detailed balance. The opacity of the chromosphere probably lies between these two extremes.

The calculated \( P_{ij} \)'s, described above, were programmed in a Bendix G-15 digital computer which solved the simultaneous equations for the two hydrogen atom models under
the two cases of opacities. The results of the calculations in terms of the $b_i$ parameters are listed in Table 2.

If one considers the model where all the substates are taken into account, it can be seen from the table that the population of the different substates of a level—$n = 4$, for example—may differ by as much as a factor of 2 in case I but differ very little for case II. One should note here that if the substates were populated according to their statistical weights, the $b_i$'s for a particular value of $n$ would all be equal but not necessarily equal to 1, as in the case of thermodynamic equilibrium. In regard to the proposed explanation for

<table>
<thead>
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<th>Term</th>
<th>Case I</th>
<th>Case II</th>
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<tr>
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<tr>
<td>1s</td>
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</tr>
<tr>
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<td>5g</td>
<td>$5 \times 10^2$</td>
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the anomalous Balmer to Paschen ratio, these results indicate that for the levels of higher $n$ it would not be possible to decrease the population of the $f$ state and beyond relative to the $s$, $p$, and $d$ states for the following reasons:

1. Although some indication of differences in the population of substates has been obtained for opacity case I, only the first five principal quantum levels have been considered. In order to approach the observed ratio, the emission from terms beyond $p$ would have to be considerably reduced for levels with $n \gg 5$. This does not appear possible because the collisions within a principal quantum level will increase rapidly as the higher values of $n$ are considered, because of the dependence upon $n$ of the matrix element. Also, induced radiative transitions from immediately adjacent levels become important even for small values of $n$. Hence the substates would be more nearly populated according to their statistical weights.

2. The actual case of opacity most likely lies between the two assumed cases and again
most likely closer to case II. The differences between cases I and II show, as one expects, that the more lines and continua that are assumed to be in detailed balance, the closer the atom will be to thermodynamic equilibrium. If one considers the hydrogen atom in the chromosphere to be near to opacity case II, it again appears impossible to suppress the radiation beyond the d terms because the substates of n = 4 and n = 5 are already nearly populated according to their statistical weights.

If one now considers the calculations where the substates are averaged together, it can be seen that the b's for any of the levels differ very little from the average of the b's for the substates for that same level. It is true that the five-level model consistently gives b's that are slightly smaller than the average of the b's for the corresponding value of n in the fifteen-level model, but the difference is not significant, considering the accuracy of the knowledge of the rates. The results therefore indicate that it is a reasonable assumption to average the substates of different angular momentum when calculating the population of hydrogen levels under the physical conditions which have been assumed to be typical for the chromosphere and quiescent prominences.

To summarize, the results of this paper have yielded two conclusions: (1) the explanation offered by Athay and Zirin for the anomalous Balmer to Paschen ratio could not be confirmed either observationally (except possibly for the one prominence observed) or theoretically, and (2) the substates of different angular momentum may be averaged together in hydrogen chromospheric and prominence calculations. The fact still remains that the anomalous ratio was observed at the 1952 eclipse. The investigations of this paper indicate that it would be of great interest to obtain further observations of the Balmer:Paschen ratio.

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