Stellar Orbits in the Galaxy (I. Two-dimensional Case)

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(Received April 12, 1960)

abstract

Two-dimentional stellar orbits in the galactic plane were studied by assuming rotational symmetry of the gravitational potential. In the usual series-expansion of $\xi^2$ which is derived from the equations of motion referred to the local standard of rest, its convergency becomes slower as $|\xi|$ ($=\dot{\varpi}_0 =$ galactocentric distance of the local standard of rest) increases to $\dot{\varpi}_0$. This shortening was avoided in our treatment, so that even if we use our first approximation formulæ, it may be possible to get the orbital parameters with a fair degree of accuracy.

The main results in this study are summarized in the last paragraph, then it is mentioned only that for convenience's sake of the statistical studies is given here the velocity-diagram in which both the peri- and the apo-galacticon distances and also the period of the radial motion can be read from a pair of the velocity-components.

Introduction

The stellar orbits in the galactic system have been studied by several investigators, as summarized well in Lindblad's article[1]. Concerning recent results only, Torgård has shown the various features of the two-dimensional orbits in her extensive numerical investigation[2], while Contopoulos has generalized Lindblad's formulæ of the epicyclic orbit[3]. As readily suspected from the results of Torgård, the orbits of the near-by stars are not limited in such a small interval of the galactocentric distance as a few kiloparsecs, except for stars with low velocities less than about 20 kilometer per second. Contopoulos' second approximation formulæ are valid well with respects to the low velocity stars, but they become less accurate as the stellar velocity increases, because a large velocity corresponds to a wider interval of the galactocentric distance. The reason why the usual formulæ for stellar orbit extending in a wider interval turn out to be inaccurate is that the convergency of the usual series-expansion of $\xi^2$ obtained from the equations of motion becomes slower as $|\xi|$ ($=\dot{\varpi}_0 =$ galactocentric distance of the local standard of rest) approaches to $\dot{\varpi}_0$. Our study was set about in order to get a diagram in which the orbital parameters can be read from any pair of the velocity-components, but it forced to look for the other adequate expressions for stellar orbits.

§1. Equations of Motion.

Let ($\dot{\varpi}, \theta, z$) be the cylindrical fixed coordinates of a star $S$ referred to the galactic center $C$. The $z$-axis is coincided with the rotational axis of the galactic
system, consequently the plane \((\tilde{\omega}, \theta)\) is laid in the galactic plane. By taking the local standard of rest \(O (\tilde{\omega} = \tilde{\omega}_0)\) as the origin, we define a new rotating coordinate-system \((\xi, \eta, \zeta)\) with a constant angular velocity \(\omega_0\) around the \(z\)-axis as shown in Fig. 1. The \(\xi\)-axis is directed as usual, while the \(\eta\)-axis is along the circular locus of \(O\), so that \(\eta\) in measured by an arc length \(OQ\) cut off by two radius-vectors \(CO\) and \(CP\). Hence, it follows that

\[
\begin{align*}
\dot{\xi} &= \dot{\omega} - \tilde{\omega}_0, \quad \dot{\eta} = \tilde{\omega}_0 (\theta - \theta_0); \\
\dot{\zeta} &= \dot{\omega}, \quad \dot{\eta} = \tilde{\omega}_0 (\theta - \theta_0) = \tilde{\omega}_0 (\omega - \omega_0); \\
\dot{\Omega} &= \omega, \quad \dot{\theta}_0 = \omega_0.
\end{align*}
\]

We assume here that the gravitational potential in the galactic plane \(V(\omega) = V(\xi)\) is rotationally symmetric with respect to the galactic center. Then, the equations of motion in the two-dimensional case are written as

\[
\begin{align*}
\dot{\xi} &= (\omega_0 + \dot{\xi})(\omega_0 + \frac{\dot{\eta}}{\omega_0})^2 + \frac{\partial V(\xi)}{\partial \xi} = 0, \\
\frac{d}{dt} \left\{ \frac{1}{\omega_0}(\omega_0 + \dot{\xi}^2)(\omega_0 + \frac{\dot{\eta}}{\omega_0}) \right\} &= 0.
\end{align*}
\]

From the second equation, we get

\[
(\omega_0 + \dot{\xi}^2)(\dot{\eta} + \dot{\theta}_0) = C\omega_0^2, \quad \dot{\theta}_0 = \omega_0\omega_0.
\]

\(C\) is an integration constant which corresponds to the circular linear velocity at \(\xi = 0\), so far as \(\xi = 0\) be possible in the course of stellar motion, that is

\[
C = \Theta_0 + \dot{\eta}_0.
\]

The equations of motion are, therefore, rewritten as

\[
\begin{align*}
\dot{\xi} &= -\frac{C^2 \omega_0^2}{(\omega_0 + \dot{\xi})^2} + \frac{\partial V(\xi)}{\partial \xi} = 0, \\
\dot{\eta} + \dot{\theta}_0 &= \frac{C}{(1 + \dot{\xi}/\omega_0)^2}.
\end{align*}
\]

Fig. 1. Coordinate-systems.
C: Galactic center, P: Star, O: Local standard of rest

The gravitational potential at any point \( \xi \), \( \gamma (|\xi| < \tilde{\omega}_0) \) in the galactic plane may be expanded into the following Taylor series,

\[
V(\xi) = V_0 + \left( \frac{\partial V}{\partial \omega} \right)_0 \xi + \frac{1}{2} \left( \frac{\partial^2 V}{\partial \omega^2} \right)_0 \xi^2 + \frac{1}{6} \left( \frac{\partial^3 V}{\partial \omega^3} \right)_0 \xi^3 + \frac{1}{24} \left( \frac{\partial^4 V}{\partial \omega^4} \right)_0 \xi^4 + \frac{1}{120} \left( \frac{\partial^5 V}{\partial \omega^5} \right)_0 \xi^5 + \cdots,
\]

where the suffix 0 signifies the value at the local standard of rest (\( \xi = 0 \)). The expression for the force-function is therefore

\[
\frac{\partial V}{\partial \xi} = \tilde{\omega} \omega^2 \equiv \frac{\partial^2}{\partial \omega^2} \equiv \tilde{\omega}_0 \omega_0^2 + J_0 \xi + H_0 \xi^2 + K_0 \xi^3 + L_0 \xi^4 + \cdots,
\]

where

\[
\left( \frac{\partial V}{\partial \omega} \right)_0 = \tilde{\omega}_0 \omega_0^2 \equiv \frac{\partial^2}{\partial \omega^2} \equiv \omega_0 (\omega_0 - 4A_0),
\]

\[
H_0 = \frac{1}{2} \left( \frac{\partial^3 V}{\partial \omega^3} \right)_0, \quad K_0 = \frac{1}{6} \left( \frac{\partial^4 V}{\partial \omega^4} \right)_0, \quad L_0 = \frac{1}{24} \left( \frac{\partial^5 V}{\partial \omega^5} \right)_0 \cdots.
\]

Provided that a relation between \( \omega \equiv \tilde{\omega}_0 + \xi \) and \( \Theta \equiv \tilde{\omega}_0 \) has been known empirically over a wide range of \( \xi \), the first several coefficients in (5) may be determined numerically by the method of the least squares. Some observational relations have hitherto been given from the 21 cm radiation of the neutral hydrogen⁴ as well as from the stellar motions of the B-stars³, the cepheids⁵ and so on. But, the data from the neutral hydrogen are restricted in the inner zone (\( \xi < 0 \)) remaining the central part of the galactic system (\( \tilde{\omega} < 3 \) kpc) as ambiguous, while the ones from the stellar motions cover hardly a range of \( \xi \) more than a few kiloparsecs from the sun. Accordingly, it is impossible to collect the observed data over a sufficient range of \( \xi \) though not a few stars now found in our vicinity may be movable further outside.

Hence, we provisionally adopted the data of \( \Theta \) from the neutral hydrogen due to Kwee et al.⁴ for the region of \( 3 \) kpc \( \leq \tilde{\omega} \leq \tilde{\omega}_0 = 8.2 \) kpc and the numerical values of \( \omega \) based on Schmidt's model⁵ or those given Lindblad's paper⁶ for the region of \( 8.2 \) kpc \( \leq \tilde{\omega}_0 \leq \tilde{\omega} \leq 14 \) kpc. As the basic constants to be used throughout in this paper, it was decided to assume

\[
\tilde{\omega}_0 = 8.2 \text{ kpc}, \quad \omega_0 = 26.4 \text{ (km/s)/kpc} \quad (\Theta_0 = 216.5 \text{ km/s}),
\]

by letting the Oort's constants be adjustable, though \( A = 19.5 \text{ (km/s)/kpc} \) has besides the above constants, been used for ascertaining the above value of \( \tilde{\omega}_0 \). By reading the values of \( \Theta \) for every round kpc of \( \tilde{\omega} \) on a smoothed curve of the \( \omega-\Theta \) relation drawn from the adopted data, the values of \( \frac{\partial V}{\partial \omega} \) were calculated and then they were solved by the method of least squares. Two sets of the result were determined respectively by changing an available range of \( \xi \). The one for a range of \( 3 \) kpc \( \leq \xi \leq 14 \) kpc was found to be

\[
J_0 = -1064.9 \text{ (km/s)/kpc}, \quad H_0 = +60.589 \text{ (km/s)/kpc},
\]

\[
K_0 = +1.6632 \text{ (km/s)/kpc}, \quad L_0 = -0.03976 \text{ (km/s)/kpc}.
\]

with the mean error of \( 142 \text{ (km/s)/kpc} \), while the other for an alternative range of \( 2 \) kpc \( \leq \tilde{\omega} \leq 14 \) kpc was
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\[ J_0 = -976.25 \frac{\text{km/s}^2}{\text{kpc}^2}, \quad H_0 = +70.302 \frac{\text{km/s}^2}{\text{kpc}^1}, \]
\[ K_0 = -3.8620 \frac{\text{km/s}^2}{\text{kpc}^1}, \quad L_0 = -0.11750 \frac{\text{km/s}^2}{\text{kpc}^1}, \]  
with the mean error of \( \pm 370 \frac{\text{km/s}^2}{\text{kpc}} \).

On the other hand, basing on Schmidt’s values of \( \partial V / \partial \phi \) (= \( K_\sigma \)) over a whole range of \( |\xi| \leq 6 \text{kpc} \), interpolated values were read graphically at every round kpc of \( \xi \) and the method of the least squares was applied to these data too. The resulted value of \( J_0 \), however, was found to be unreasonable, when the first four constants were solved simultaneously. Then, by fixing \( J_0 = -1073 \frac{\text{km/s}^2}{\text{kpc}^2} \) just as in (8), the other constants were determined trially by the method of the least squares as given below.

\[ J_0 = -1064.9 \frac{\text{km/s}^2}{\text{kpc}^2} \text{ (fixed)}, \quad H_0 = +43.265 \frac{\text{km/s}^2}{\text{kpc}^1}, \]
\[ K_0 = -0.48214 \frac{\text{km/s}^2}{\text{kpc}^1}, \quad L_0 = +0.78506 \frac{\text{km/s}^2}{\text{kpc}^1}; \]  
\[ J_0 = -1064.9 \frac{\text{km/s}^2}{\text{kpc}^2} \text{ (fixed)}, \quad H_0 = +66.445 \frac{\text{km/s}^2}{\text{kpc}^1}, \]
\[ K_0 = -0.48214 \frac{\text{km/s}^2}{\text{kpc}^1}, \quad L_0 = 0 \text{ (fixed)}. \]  

As the above results show, a least squares solution depends more or less on the range of \( \phi \) (or \( \xi \)) as well as a number of unknowns taken into account, because of abrupt increase of the force-function towards the galactic center. Accordingly, there is a room to discuss further, but leaving it afterwards (8), we will adopt exclusively the numerical values in (8) for a time as the constants of the empirical force-function.

It is noted that \( J_0 \) is given by \( \omega_0 (\omega_0 - 4A_0) \), hence if \( A_0 = 19.5 \frac{\text{km/s}}{\text{kpc}} \) is assumed, it follows \(-1362.2 \frac{\text{km/s}^2}{\text{kpc}^2} \) against the adopted least squares solution (8) which corresponds to \( A_0 = 16.7 \frac{\text{km/s}}{\text{kpc}} \). In connection with this, it is added that \( A_0 = 17.5 \frac{\text{km/s}}{\text{kpc}} \) seems to be the most probable according to the recent investigations\(^{3,16}\). The force-function defined by (8) and those due to Schmidt and Torgård are illustrated in Fig. 2.

§3. Approximations of the Equations of Motion.

Putting (5) in the first equation of (4) we get

\[ \ddot{\xi} - \frac{C^2 \ddot{\omega}_0^2}{\omega_0 + \xi^2} + \ddot{\omega}_0 \omega_0 \dot{\xi} + J_0 \dot{\xi} + H_0 \xi^2 + K_0 \xi^3 + L_0 \xi^4 + \ldots = 0. \]

After integration, it becomes to be

\[ \frac{d\xi}{dt} = \pm \frac{1}{(1 + \xi/\omega_0)^{\frac{1}{2}}} \sqrt{\xi_0^2 + a \xi + b \xi^2 + c \xi^3 + d \xi^4 + \ldots}, \]

where

\[ a = \frac{2}{\omega_0} (\ddot{\xi}_0^2 + C^2) - 2 \ddot{\omega}_0 \omega_0 \dot{\xi}_0, \quad b = \frac{1}{\omega_0^2} (\ddot{\xi}_0^2 + C^2) - 4 \dot{\omega}_0 \omega_0 - J_0, \]
\[ c = - \frac{2}{\omega_0} \omega_0^2 \ddot{\xi}_0 - 2 \ddot{\omega}_0 \omega_0 - \frac{2}{3} H_0, \quad d = - \frac{J_0}{\omega_0^2} - \frac{4}{3} \dot{H}_0 - \frac{K_0}{2}, \ldots \]

As the double signs discriminate merely the direction of motion, going and coming, we disregard the sign hereafter.

The factor \( \frac{1}{(1 + \xi/\omega_0)} \) has not been expanded into an infinite series deliberately but reserved as it stands, for with such expansion the convergency becomes
Fig. 2  Force-function in the galactic plane.

Circles: Schmidt’s model ($K_\infty$).
Squares: Torgård’s model (G1).

Upper Part: Designations are corresponded with those in Table II.

Lower Part:

Solid line:
\[
\frac{\partial V}{\partial \omega} = 5715.1 - 1064.9\xi + 60.589\xi^2 + 1.6632\xi^3 - 0.039676\xi^4
\]

Dashed line:
\[
\frac{\partial V}{\partial \omega} = 5715.1 - 8732.2\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right) - 5715.1\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^2
- 2698.0\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^3 (e=0, d=e=\ldots=0).
\]

Dot-and-dashed line:
\[
\frac{\partial V}{\partial \omega} = 5715.1 - 8732.2\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^2 + 2677.5\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^2
- 3377.2\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^3 (e=10.101, d=e=\ldots=0).
\]

Dotted line:
\[
\frac{\partial V}{\partial \omega} = 5715.1 - 8732.2\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^2 + 1358.0\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^2
- 6016.2\left(\frac{\xi}{\omega_0}\right)^2 \left(1 + \frac{\xi}{\omega_0}\right)^3 (e=49.349, d=e=\ldots=0).
\]
slower as \( |\xi| (< \tilde{a}_0) \) increases. Whereas, the series in the square root of (9) is convergent rapidly as suspected from (10). In fact, with respects to the two extreme cases of the force-function such as (i) \( \frac{\partial V}{\partial \tilde{a}} = k_1 \tilde{a} \) (a homogeneous model) and (ii) \( \frac{\partial V}{\partial \tilde{a}} = k_2 / \tilde{a}^2 \) (a point model), these constants of (10) become as follows.

\[
\begin{align*}
\text{(i)} & \quad \frac{\partial V}{\partial \tilde{a}} = k_1 \tilde{a} : \quad A_0 = 0, \quad J_0 = \omega_0^2, \quad H_0 = K_0 = L_0 = \ldots = 0; \\
& \quad b = -\frac{1}{\omega_0^2} (\xi_0^2 + C^2) - 5\omega_0^2, \quad c = -\frac{4\omega_0^2}{\omega_0^2}, \quad d = -\frac{\omega_0^2}{\omega_0^2} = \frac{c}{4\omega_0^2}, \\
& \quad e = f = \ldots = 0; \\
\text{(ii)} & \quad \frac{\partial V}{\partial \tilde{a}} = \frac{k_2}{\tilde{a}^2} : \quad A_0 = \frac{3\omega_0}{4}, \quad J_0 = -2\omega_0^2, \quad H_0 = \frac{3\omega_0^2}{\omega_0^2}, \\
& \quad K_0 = -\frac{4\omega_0^2}{\omega_0^2}, \quad L_0 = \frac{5\omega_0^2}{\omega_0^2}, \quad \ldots ; \\
& \quad b = -\frac{1}{\omega_0^2} (\xi_0^2 + C^2) - 2\omega_0^2, \quad c = d = e = \ldots = 0.
\end{align*}
\]

Meanwhile, the corresponded empirical values computed from (8)-(8″′) turn out to be respectively as follows,*

\[
\text{a} = (\xi_0^2 + C^2) \times 0.24390 - 11432, \quad \text{b} = (\xi_0^2 + C^2) \times 0.014872 - 1723.4, \\
\text{c} = +49.349, \quad \text{d} = +5.1588, \quad \text{e} = -0.78724 \quad \text{[corresponded to (8)].} \quad (12)
\]

\[
\text{a} = (\xi_0^2 + C^2) \times 0.24390 - 11432, \quad \text{b} = (\xi_0^2 + C^2) \times 0.014872 - 1811.6, \\
\text{c} = +21.252, \quad \text{d} = +5.0187, \quad \text{e} = -0.02500 \quad \text{[corresponded to (8′)].} \quad (12′)
\]

\[
\text{a} = (\xi_0^2 + C^2) \times 0.24390 - 11432, \quad \text{b} = (\xi_0^2 + C^2) \times 0.014872 - 1723.4, \\
\text{c} = +60.898, \quad \text{d} = +9.0935, \quad \text{e} = -0.68418 \quad \text{[corresponded to (8″)].} \quad (12″)
\]

\[
\text{a} = (\xi_0^2 + C^2) \times 0.24390 - 11432, \quad \text{b} = (\xi_0^2 + C^2) \times 0.014872 - 1723.4, \\
\text{c} = +45.445, \quad \text{d} = +5.2744, \quad \text{e} = -0.71758 \quad \text{[corresponded to (8″′)].} \quad (12″′)
\]

The numerical values are throughout this paper expressed by the units of (km/s) and kpc for the velocity and the distance respectively, unless stated otherwise.

From a practical point of view, therefore, following successive steps of approximation may be taken into consideration.

\[
\frac{d\xi}{dt} = \frac{E}{1 + \frac{\xi}{\tilde{a}_0}} \quad \text{or} \quad dt = \frac{(1 + \frac{\xi}{\tilde{a}_0}) d\xi}{E}
\]

\[
\begin{align*}
1\text{st approximation:} & \quad E \equiv \sqrt{\xi_0^2 + a\xi + b\xi^2}, & (13) \\
2\text{nd approximation:} & \quad E′ \equiv \sqrt{\xi_0^2 + a\xi + b\xi^2 + c\xi^3}, & (13′) \\
3\text{rd approximation:} & \quad E'' \equiv \sqrt{\xi_0^2 + a\xi + b\xi^2 + c\xi^3 + d\xi^4}. & (13′′)
\end{align*}
\]

Such any order of approximation corresponds to make the force-function (5) specify into a respective form according to its order as will be seen later.

* Judging from (11), it likes to be \( c \leq 0, d \leq 0 \) in general, but this does not necessarily hold in such a model due to Schmidt.
§4. **Orbits in the first Approximation.**

The force-function specified to the first approximation is derived from $c=d=\ldots=0$ by referring to (5) and (10), namely

$$
\frac{\partial V}{\partial \omega} = \tilde{\omega}_0\omega^2 \left\{ 1 - \sum_{n=1}^{\infty} (-1)^n(n+1)(n-1) \left( \frac{\xi}{\tilde{\omega}_0} \right)^n \right\} - J\tilde{\omega}_0 \sum_{n=1}^{\infty} (-1)^n n(n+1) \left( \frac{\xi}{\tilde{\omega}_0} \right)^n.
$$

For $|\xi| < \tilde{\omega}_0$, this infinite series converges absolutely and can be reduced into the following finite expression.

$$
\frac{\partial V}{\partial \omega} = \tilde{\omega}_0\omega^2 \left\{ 1 + 3 \left( \frac{\xi}{\tilde{\omega}_0} \right) \right\} \left( 1 + \left( \frac{\xi}{\tilde{\omega}_0} \right) \right)^3 + J\tilde{\omega}_0 \left( \frac{\xi}{\tilde{\omega}_0} \right) \left( 1 + \left( \frac{\xi}{\tilde{\omega}_0} \right) \right)^2.
$$

(14)

If the numerical values of (8) are used, both the coefficients become as $\tilde{\omega}_0\omega^2 = 5715.1 \text{ (km/s)}^2/\text{kpc}$ and $J\tilde{\omega}_0 = -8732.2 \text{ (km/s)}^2/\text{kpc}$. The force-function (14) with these values is illustrated in Fig. 2 as the curve $c=0$.

In reality, as the value of $c$ may be an order given in (8)-(8′′), it is in general much smaller than the preceding constants $a$ and $b$. Consequently omission of the terms after $b$ seems not to be so serious, unless the stellar orbit extends in a wider range of $\xi$.

In the first approximation, $E^2=\dot{\xi}^2+a\xi+b\xi^2 > 0$ should be satisfied for any orbit. Let us examine this condition for a moment. From (10)

$$a^2 - 4b\dot{\xi}_0^2 = \frac{4}{\omega_0^2} \left\{ \dot{\xi}_0^2(C^2+2\Theta_0^2+\dot{\omega}_0^2J_0) + (C^2-\Theta_0^2)^2 \right\} > 0$$

holds always except at $\dot{\xi}_0=0$ where $a^2=4b\dot{\xi}_0^2=0$.** Hence, if $b=0$, the sign of $E^2$ is reverse against that of $b$ whenever $\xi$ lies between two real roots of $E^2=0$, while both signs are the same so far as $\xi$ is outside this interval.

For a special case $b=0$, corresponded to

$$\dot{\xi}_0^2+(\dot{\xi}_0+\Theta_0)^2=4\Theta_0^2+\dot{\omega}_0^2J_0=(340.4)^2,$$

the condition is to be $\xi > -\frac{\dot{\xi}_0^2}{a} = -\left( \frac{\dot{\xi}_0}{130} \right)^2$, because of

$$E^2=\dot{\xi}_0^2+a\xi=\dot{\xi}_0^2+\frac{2}{\omega_0} \left\{ \dot{\xi}_0^2+(\dot{\xi}_0+\Theta_0)^2-\Theta_0^2 \right\} \xi$$

$$=\dot{\xi}_0^2 + \frac{2}{8.2} \left\{ (340.4)^2-(216.3)^2 \right\} \xi = \dot{\xi}_0^2+(130)^2 \xi.$$

Thus, if we take $\alpha$ inwards $(\xi<0)$ and $\beta$ outwards $(\xi>0)$ and put

$$E^2=b(\xi+a)(\xi-\beta), \quad -b(\beta-\alpha)=a, \quad -b\alpha\beta=\dot{\xi}_0^2,$$

(15)

then the possible orbits may be classified as follows.

**Case (i) $b>0$, $\xi \leq -\alpha$: unbounded orbit,**

no peri-center, apo-center $(\xi=-\alpha)$.

$b>0$, $\xi \geq \beta$: unbounded orbit, peri-center $(\xi=\beta)$, no apo-center.

**Case (ii) $b=0$, $\xi \leq -\frac{\dot{\xi}_0^2}{a}$:** unbounded orbit,

peri-center $(\xi=-\frac{\dot{\xi}_0^2}{a})$, no apo-center.

**Case (iii) $b<0$, $-\alpha \leq \xi \leq \beta$: periodic orbit,**

peri-center $(\xi=-\alpha)$, apo-center $(\xi=\beta)$.

---

* $2\Theta_0^2+\dot{\omega}_0^2J_0=\dot{\omega}_0^2(3\omega_0^2-4A_0) \geq 0$ is satisfied over a region not far from the sun.

** Those orbits lying wholly outside or inside the circle $(\omega=\tilde{\omega}_0)$ are out of consideration here (see §5).
In Fig. 3, the domains of \( b > 0, b < 0; a > 0, a < 0 \) are indicated in the \( \xi_0 - \eta_0 \) diagram.

The circle \( b = 0 \) or \( \xi_0^2 + (\eta_0 + \Theta_0)^2 = (340.4)^2 \) corresponds to the so-called escape velocity at \( \dot{\omega} = \dot{\omega}_0 \), and if \( \xi_0 = 0 \), it follows \( \gamma_0 = +126 \text{ km/s} \), which is much larger than the observed value of 63 km/s due to Oort\(^9\). Further, the condition \( b = 0 \) in the case of the point model \( \left( \frac{\partial V}{\partial \omega} = k^2 / \omega^2 \right) \) corresponds to \( \xi_0^2 + (\eta_0 + \Theta_0)^2 = (306)^2 \), while that for the homogeneous model \( \left( \frac{\partial V}{\partial \omega} = k_1 \omega \right) \) does to \( \xi_0^2 + (\eta_0 + \Theta_0)^2 = (485)^2 \), both of which are much larger than \( (280)^2 \) corresponded to Oort's escape velocity. Such a degree of discrepancy may be inevitable in the first approximation, hence the problem will be considered again in the later paragraph from a view point of the second approximation. It may be said, however, that the stellar orbit having \( b > 0 \) but with \( \xi_0^2 + (\eta_0 + \Theta_0)^2 > (280)^2 \) can be represented, at least over a region

![Fig. 3. Curve of the velocities of escape.](image)

Dots are velocity points of the eleven stars listed in Table 1.
Outer solid line circle: \( b = 0 \ (c = 0) \)
Inner solid line circle: \( a = 0 \)
Outer dashed line curve: \( c = 10.101 \)
Inner dashed line curve: \( c = 49.349 \)
\( \Theta = 216.5 \text{ km/s} \) (Rotational velocity of local standard of rest)
not far from the origin, with the periodic orbit in the first approximation.

§4.1. Stellar Orbit in the Case (i)—Spiral Orbit.

In this case, $b > 0$ or $\xi^2 + (\eta_0 + \theta_0)^2 > 0$ and $a > 0$ always as seen in Fig. 3, accordingly there exist two different real roots $-\alpha$ and $\beta$, where

$$
\alpha = \frac{a + \sqrt{a^2 - 4b\xi_0^2}}{2b}, \quad \beta = \frac{a - \sqrt{a^2 - 4b\xi_0^2}}{2b} \equiv -\beta' < 0 \quad (\alpha > \beta' = -\beta).
$$

(17)

Accordingly, the stellar motions are restricted within either $\xi \leq -\alpha$ ($<0$) or $\xi \geq \beta = -\beta'$ ($<0$).

By integrating (13), we find

$$
t - t_1 = \int_{\xi_1}^{\xi} \frac{(\bar{\omega}_0 + \xi)d\xi}{\bar{\omega}_0 b(\xi + \alpha)(\xi - \beta)}
$$

$$
= \left\{ \begin{array}{ll}
\frac{1}{\sqrt{b} \bar{\omega}_0} \sqrt{(\xi + \alpha)(\xi - \beta)} + 2\bar{\omega}_0 + \beta - \alpha \log \left( \frac{1 + \sqrt{\xi + \alpha}}{1 - \sqrt{\xi - \beta}} \right)_{\xi_1}^{\xi} & (\xi \leq -\alpha), \\
\frac{1}{\sqrt{b} \bar{\omega}_0} \sqrt{(\xi + \alpha)(\xi - \beta)} + 2\bar{\omega}_0 + \beta - \alpha \log \left( \frac{1 + \sqrt{\xi - \beta}}{1 - \sqrt{\xi + \alpha}} \right)_{\xi_1}^{\xi} & (\xi \geq \beta \ (<0)).
\end{array} \right.
$$

(18)

The terms in brackets of (18) and (18') vanish when $\xi = -\alpha$ and $\xi = \beta = -\beta'$ respectively. In a case, $\xi \geq \beta$ ($= -\beta' < 0$), if a star starts from $\xi_1 (> -\beta')$ with a velocity-component $\frac{d\xi}{dt} < 0$, it moves up to the peri-galacticon by increasing $\frac{d\xi}{dt}$ to zero, and then it comes back by increasing $\frac{d\xi}{dt} (> 0)$ from zero and goes far away through the starting circle $\bar{\omega}_1 = \bar{\omega}_0 + \xi_1 > \bar{\omega}_0$. In another case, $\xi < \alpha$, the stellar motion appears to be similar except that the direction of the $\xi$-axis is reversed, nevertheless, it must be remarked that $\xi$ has no usual meaning there because the star in question does not reach up to $\xi = 0$.

As for the $\gamma$-component, we have from (13) and (14)

$$
\gamma - \gamma_1 + \theta_0 (t - t_1) = \bar{\omega}_0 (\theta - \theta_1) = \int_{t_1}^{t} \left\{ \frac{Cd\xi}{\left(1 + \xi/\bar{\omega}_0\right) \sqrt{\xi_0^2 + a\xi + b\xi^2}} \right\} = \int_{\xi_1 (1 + \xi/\bar{\omega}_0) \sqrt{\xi_0^2 + a\xi + b\xi^2}}^{\xi} \frac{Cd\xi}{\sqrt{b(\xi - \alpha)}(\xi + \alpha)}
$$

$$
\cdot \left( \begin{array}{l}
C\bar{\omega}_0 \\
\sqrt{b(\xi - \alpha)}(\xi + \alpha)
\end{array} \right) \left\{ \begin{array}{ll}
\log \sqrt{\frac{\bar{\omega}_0 - \alpha}{\bar{\omega}_0 + \beta}} + \sqrt{\frac{\xi + \alpha}{\xi - \beta}} & (\xi \leq -\alpha), \\
\log \sqrt{\frac{\bar{\omega}_0 - \alpha}{\bar{\omega}_0 + \beta}} + \sqrt{\frac{\xi - \beta}{\xi + \alpha}} & (\xi \geq \beta \ (<0)).
\end{array} \right.
$$

(19)

Hence, both orbits are, viewed from the galactocentric fixed coordinates, of some complex hyperbolic spirals. The behavior of stellar motion at $\xi = -\bar{\omega}_0$ (the galactic center) in the case of $\xi \leq -\alpha$ or that at $\xi = +\infty$ in the other case of $\xi \geq \beta$ may be readily deduced from (18) and (19) or (18') and (19') respectively.
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§4.2. Stellar Orbit in the Case (ii)—Parabolic Orbit.

In this case $b=0$ or $\xi^2 + (\eta + \Theta) = 4\Theta^2 + \omega_0 J = (430)^2$ holds. Noticing

$$a = \frac{1}{\omega_0} \left\{ \omega_0^2 + (\eta + \Theta)^2 - \Theta^2 \right\} = (130)^2 > 0,$$

the integral of (13) is to be

$$t - t_1 = \frac{1}{\omega_0} \left[ \begin{array}{c}
\omega_0 + \xi d\xi = \frac{2 \sqrt{\xi^2 + a^2}}{3 a^2 \omega_0} \left\{ 3 a \omega_0 - 2 \xi^2 + 2 a^2 \xi \right\} \end{array} \right]_{\xi_1}^{\xi} \left( - \frac{\xi^2}{a} = - \left( \frac{\xi_0}{130} \right)^2 \leq \xi \right). \quad (20)$$

While, the integral concerning the $\eta$-component is as before

$$\eta = \eta_1 + \Theta (t - t_1) = \frac{C dt}{1 + \xi/\omega_0} = \frac{C dt}{1 + \xi/\omega_0 \sqrt{\xi^2 + a^2}},$$

or

$$\eta = \eta_1 + \Theta (t - t_1) = \frac{2 C \omega_0}{a \omega_0 - \xi^2} \left\{ \arctan \frac{\sqrt{\xi^2 + a^2}}{a \omega_0 - \xi^2} \right\} \xi_1. \quad (21)$$

It is needless to say that at $\xi = \xi_1$, it becomes to be $t - t_1 = 0$ and $\theta - \Theta_1 = 0$. From (20) and (21) we get

$$\omega_0 + \xi = \frac{2 (\omega_0 - \xi^2 / a)}{1 + \cos \frac{\sqrt{a \omega_0 - \xi^2} (\theta - \Theta_1)}{C}}. \quad (22)$$

Thus, in this case the relation between radius vector and position angle referred to the fixed space is far simpler than before. It is nothing but an equation of the parabola whose focus coincides with the galactic center and latus rectum is equal to $4 \left( \frac{\omega_0 - \xi^2}{a} \right)$.

§4.3. Stellar Orbit in the Case (iii)—Periodic Orbit.

In this case $b<0$ or $\xi^2 + (\eta + \Theta)^2 < (430)^2$ and $a^2 - 4b\xi^2 > 0$, accordingly there are a positive root $\beta$ and a negative one $-\alpha$, except only such a special case $\alpha = 0$, or $\alpha = \beta = 0$.

$$\beta = -\sqrt{a^2 - 4b\xi^2} = 0, \quad \alpha = -\sqrt{a^2 - 4b\xi^2} = 0. \quad (23)$$

The integral of (13) will be

$$t - t_1 = \int_{\xi}^{\xi_1} \frac{(\xi + \omega_0) d\xi}{\sqrt{b \omega_0 - b (\xi + \alpha)(\beta - \xi)}} = \int_{\xi}^{\xi_1} \left\{ (2 \omega_0 + \beta - \alpha) \arctan \frac{\xi + \alpha}{\beta - \xi} \right\} \xi_1. \quad (24)$$

Clearly, the corresponded stellar motion is periodic in the $\xi$-component between the peri-galacticon $\xi = -\alpha$ and the apo-galacticon $\xi = \beta$.

Letting the period of the $\xi$-motion be $T_\xi$, then it follows

$$T_\xi = \frac{\pi}{\sqrt{-b \omega_0}} \left( 2 \omega_0 + \beta - \alpha \right) = \frac{\pi}{\sqrt{-b \omega_0}} \left( \omega_{\text{max}} + \omega_{\text{min}} \right),$$

$$= 3.746 \times 10^8 \times \frac{16.4 + (\beta - \alpha)}{\sqrt{-b}} \text{ yr} \left( \alpha, \beta : \text{ in kpc} \right). \quad (25)$$
Especially, if $\alpha = \beta = 0$ or if the orbit is a circle with a radius of $\bar{\omega}_0$, $T_\xi$ is reduced to

$$
\left( \frac{2\pi}{\sqrt{-b}} \right)_{\alpha = \beta = 0} \rightarrow \frac{2\pi}{\sqrt{4\omega_0(\omega_0 - A)}} = \frac{2\pi}{\kappa_0}.
$$

Meanwhile, the second equation of (4) concerning the $\gamma$-component is

$$
\gamma = \gamma_1 + \Gamma_0(t - t_1) = \int_{t_1}^{t} \frac{Cdt}{(1 + \xi/\bar{\omega}_0)^2} = \int_{t_1}^{\xi} \frac{Cdt}{(1 + \xi/\bar{\omega}_0)^2} + \bar{\omega}_0(\theta - \theta_1)
$$

$$
= \frac{2C\bar{\omega}_0}{\sqrt{-b(\bar{\omega}_0 - \alpha)(\bar{\omega}_0 + \beta)}} \left[ \arctan \sqrt{\frac{\bar{\omega}_0 + \beta}{\bar{\omega}_0 - \alpha}} \sqrt{\frac{\alpha + \xi - \bar{\omega}_0}{\beta - \xi}} \right].
$$

Putting $\xi_1 = -\alpha$, $\xi = \beta$ in the above equation, is obtained an advance of the position angle around the galactic center during a half period of the $\xi$-motion;

$$
\theta_\beta - \theta_{-\alpha} = \pi \frac{\pi C}{\sqrt{-b(\bar{\omega}_0 - \alpha)(\bar{\omega}_0 + \beta)}}.
$$

A time interval between $\theta$ and $\theta + \pi$, that is, a half-period of the galactic rotation of an individual star cannot be independent of $\theta$, because of the non-regular motion in the $\gamma$-component. But, a mean revolitional period $T_\theta$ defined by the following is considered for any orbit.

$$
T_\theta = \frac{\pi}{\theta_\beta - \theta_{-\alpha}} = \frac{\sqrt{-b(\bar{\omega}_0 - \alpha)(\bar{\omega}_0 + \beta)}}{C} \times \left( \text{geometric mean of } \bar{\omega}_{\text{max}} \text{ and } \bar{\omega}_{\text{min}} \right) \equiv k,
$$

or

$$
T_\theta = \frac{\pi(2\bar{\omega}_0 + \beta - \alpha)}{C} \sqrt{\frac{(\bar{\omega}_0 - \alpha)(\bar{\omega}_0 + \beta)}{\bar{\omega}_0^2}}.
$$

Now, if we put

$$
\sqrt{\alpha + \xi} = M \sin \frac{\varphi}{2}, \quad \sqrt{\beta - \xi} = M \cos \frac{\varphi}{2}, \quad M^2 = \alpha + \beta,
$$

the formulae (24) and (26) can be transformed into the more simplified expressions. In fact, the formula (24) is replaced by

$$
\varphi - \frac{\alpha + \beta}{2\bar{\omega}_0 + \beta - \alpha} \sin \varphi = \frac{2\sqrt{-b\bar{\omega}_0}}{2\bar{\omega}_0 + \beta - \alpha} (t - t_1),
$$

where $t_1$ is an epoch corresponding to $\varphi = 0$ or $\xi = -\alpha$. With this equation, $\varphi$ is determined for a given $t$. The formula for $\xi$ in terms of $\varphi$ is readily obtained from (30).

$$
\sqrt{\frac{\alpha + \xi}{\beta - \xi}} = \tan \varphi, \quad \therefore \xi = \frac{\beta - \alpha}{2} - \frac{\beta + \alpha}{2} \cos \varphi,
$$

or

$$
\bar{\omega}_0 + \xi = \frac{2\bar{\omega}_0 + \beta - \alpha}{2} \left[ 1 + \frac{\beta + \alpha}{2\bar{\omega}_0 + \beta - \alpha} \cos \varphi \right].
$$

As for the formula for $\theta$, we get from (29)

$$
\tan \left[ \frac{\sqrt{-b(\bar{\omega}_0 - \alpha)(\bar{\omega}_0 + \beta)}}{2C}(\theta - \theta_1) \right] = \sqrt{\frac{\bar{\omega}_0 + \beta}{\bar{\omega}_0 - \alpha}} \tan \frac{\varphi}{2}.
$$

Both the equations (31) and (32) are quite the same formulae as encountered in the two-bodies problem, though these formulae concern only the $\xi$-motion and accordingly $\varphi$ is nothing but a parameter in place of $t$. Nevertheless, we will use
conventionally the words such as the eccentricity, the mean radius and the mean motion by defining

\[ n \text{ (mean motion)} = \frac{2\sqrt{-b(\tilde{\omega}_0-\alpha)}}{2\tilde{\omega}_0+\beta-\alpha}, \]

\[ e \text{ (eccentricity)} = \frac{\beta+\alpha}{2\tilde{\omega}_0+\beta-\alpha}, \]

\[ a \text{ (mean radius)} = \frac{2\tilde{\omega}_0+\beta-\alpha}{2}. \]

Then, the formulae of the stellar orbit in the case (iii) are summarized simply as

\[ \varphi-e\sin\varphi=n(t-t_1), \]

\[ \tilde{\omega}_0+\tilde{\xi}=a(1-e\cos\varphi), \]

\[ \tan\left\{\frac{\sqrt{-b(\tilde{\omega}_0-\alpha)(\tilde{\omega}_0+\beta)}}{2C}\right\} = \sqrt{\frac{\tilde{\omega}_0+\beta}{\tilde{\omega}_0-\alpha}} \tan\frac{\varphi}{2}. \]

Eliminating \( \varphi \) between (32) and (33), is obtained the expression of a periodic orbit referred to the fixed galactocentric coordinates.*

\[ \tilde{\omega}_0+\tilde{\xi} = \frac{a(1-e^2)}{1+e\cos\{k(\theta-\theta_1)\}}, \]

where

\[ k = \frac{T_\theta}{T_\xi} = \frac{a\sqrt{-b(1-e^2)}}{C} = \sqrt{1+\frac{3-4A_0/\tilde{\omega}_0}{(1+\tilde{\eta}_0/\tilde{\theta}_0)^2}}. \]

Then, provided that the angular coordinate \( (\theta-\theta_1) \) is replaced by \( k(\theta-\theta_1) \), any periodic orbit can be represented by an ellipse. Viewing from the fixed space, however, the periodic orbit is either closed or not according as \( k \) or \( 1/k \) is an integer or not, and especially for \( k=1 \) or \( 2 \) the orbit becomes to be an ellipse whose focus or center coincides respectively with the galactic center.

The parameter \( k \) which characterizes the figure of periodic orbit as illustrated schematically in Fig. 4** depends on \( A_0/\tilde{\omega}_0 \) and \( \tilde{\eta}_0/\tilde{\theta}_0 \) but not on \( \tilde{\xi}_0 \). Based on Schmid's model \( (A_0=19.5 \text{ km/s/kpc at } \tilde{\omega}_0=8.2 \text{ kpc}) \), the force-field of the Galaxy seems to be characterized in a following way by assuming \( A=19.5 \text{ km/s/kpc} \) tentatively.

* Similar transformation is possible for the case (i) too. That is, by putting

\[ M = \alpha + \beta \geq 0, \quad \tanh\frac{\varphi}{2} = \frac{\xi+\alpha}{\sqrt{\xi-\beta}} \quad \text{(for } \xi \leq -\alpha), \quad \tanh\frac{\varphi}{2} = \frac{\xi-\beta}{\sqrt{\xi+\alpha}} \quad \text{(for } \beta \leq \xi), \]

the expression becomes.

\[ \varphi+e\sin\varphi=n(t-t_1); \quad n = \frac{2\sqrt{-b\tilde{\omega}_0}}{2\tilde{\omega}_0+\beta-\alpha}, \quad e = \frac{\beta+\alpha}{2\tilde{\omega}_0+\beta-\alpha}, \]

and

\[ \varphi' = \frac{\sqrt{b(\tilde{\omega}_0-\alpha)(\tilde{\omega}_0+\beta)}}{C}(\theta-\theta_1); \quad \tanh\frac{\varphi'}{2} = \sqrt{\frac{\tilde{\omega}_0+\beta}{\tilde{\omega}_0-\alpha}} \tan\frac{\varphi}{2} \quad \text{(for } \xi \leq -\alpha), \]

\[ \tanh\frac{\varphi'}{2} = \sqrt{\frac{\tilde{\omega}_0-\alpha}{\tilde{\omega}_0+\beta}} \tan\frac{\varphi}{2} \quad \text{(for } \beta \leq \xi). \]

As for the orbital expressions, it follows

\[ \tilde{\omega}_0+\tilde{\xi} = \frac{a(1-e^2)}{1+e\cosh\varphi'} \quad \text{(for } \xi \leq -\alpha), \quad \tilde{\omega}_0+\tilde{\xi} = \frac{a(1-e^2)}{1-e\cosh\varphi'} \quad \text{(for } \beta \leq \xi). \]

** For a given periodic orbit extending from \( \tilde{\omega}_1 = \tilde{\omega}_0 - \alpha \) to \( \tilde{\omega}_2 = \tilde{\omega}_0 + \beta \), it is possible to adopt any \( \tilde{\omega}_0 \) in an interval of \( \tilde{\omega}_1 \leq \tilde{\omega}_0 \leq \tilde{\omega}_2 \), but the change of \( \tilde{\omega}_0 \)-value and accordingly that of \( (A_0/\tilde{\omega}_0)\)-value reflects upon the \( \tilde{\eta}_0 \)-value, so that in principle no ambiguity appears.
Fig. 4. Periodic orbits with different values of $k$.
C: Galactic center; P: Peri-galactic; A: Apo-galactic.
An arrow indicates a rotational direction of CA (for example, in the upper left figure, if $k$ is slightly larger than 4, CA rotates counterclockwise).

$3\omega_0 > 4A: \bar{\omega} < \text{ca} 8 \text{kpc}$  (inner zone): $k > 1$,
$3\omega_0 \approx 4A: \text{ca} 8 \text{kpc} < \bar{\omega} < \text{ca} 10 \text{kpc}$  (middle zone): $k \approx 1$,
$3\omega_0 < 4A: \bar{\omega} > \text{ca} 10 \text{kpc}$  (outer zone): $k < 1$.

Only in a special case $3\omega_0 = 4A_0$, $k = 1$ holds irrespectively of $\dot{\varphi}_0$ value, otherwise $k$ depends on $\dot{\varphi}_0$. Concerning the parameter $k$, there remain some interesting discussions, such as the possibility of the periodic orbits with $k = 3$ or $4$ near the galactic nucleus in connection with the radial motion of the neutral hydrogen, the dispersion orbits advocated by Lindblad\(^9\) and so on. But, the matters will be taken up in a separate paper.

It is noted that the expressions (24) and (26) are reduced to the usual formulae due to Lindblad,\(^{11}\) if we neglect the smaller terms. In reality, if we take $t = t_1$ when $\xi = -\alpha$, they turn out to be*

$$
\begin{align*}
\xi &= \frac{\beta - \alpha}{2} - \frac{\beta + \alpha}{2} \cos \kappa_0 (t - t_1), \\
\kappa_0^2 &= 4\omega_0 (\omega_0 - A_0), \\
\gamma &= \gamma_1 - \left( \frac{\beta - \alpha}{2} \right)^2 2A_0 (t - t_1) + \sqrt{\frac{\omega_0}{\omega_0 - A_0}} \left( \frac{\beta + \alpha}{2} \right) \sin \kappa_0 (t - t_1). 
\end{align*}
$$
\tag{38}

For comparison, with respects to three examples of the periodic orbits such as (i) $\alpha = 3 \text{kpc}$, $\beta = 0 \text{kpc}$, (ii) $\alpha = 0 \text{kpc}$, $\beta = 3 \text{kpc}$ and (iii) $\alpha = 3 \text{kpc}$, $\beta = 3 \text{kpc}$, their epicyclic orbits referred to the rotating local origin were computed with our formulae (24) and (26) as well as the ones (36) corresponded to Lindblad's. The results are graphically shown in Fig. 5,* where discrepancies in the $\gamma$-direction should be noticed as the extremum values of $\xi$ have been identified.

* In Lindblad's formulae, $A(\bar{\omega} = \bar{\omega}_0)$ in the second term of $\gamma$ has been replaced by $A$ or a mean value of $A$ between $\bar{\omega} = \bar{\omega}_0$ and $\bar{\omega} = \frac{2\bar{\omega}_0 + 5 - \alpha}{2}$. The broken curve in Fig. 4, however, is drawn with $A(\bar{\omega} = \bar{\omega}_0) = 16.7 \text{ (km/s)/kpc}$.

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Fig. 5. Epicyclic orbits.
Solid lines: Orbits due to formulae (24) and (26) ....... [S]
Dashed lines: Orbits due to Lindblad's formulae (38) ....... [L]
Units of figures in the diagram: \( \dot{\xi}_0, \dot{\eta}_0 \) (km/s); \( T_\xi, T_\eta \) (10^8 yr).
§5. Velocity-Diagram for Getting Orbital Parameters in the Case (iii).

Among the orbital parameters such as $\alpha$, $\beta$, $T_\xi$, $T_\eta$, $n$, $a$, $e$ and $k$ defined by (23), (25), (29), (34) and (37), the first two regarded as foundamental can be
derived from the velocity components at $\dot{\omega}=\dot{\omega}_0$ $(\xi=0)$ or $\dot{\eta}_0$ and $\dot{\eta}_0$, as readily seen from (15), (10) and (3). Hence, it is possible to draw two sets of the curves
showing the values $\alpha$ and $\beta$ in the velocity-diagram of $\dot{\xi}_0$ and $\dot{\eta}_0$.

In practice, formation of this diagram was carried out in the following way. For each pair of properly chosen values of $\alpha$ and $\beta$ $(\alpha+\beta>0)$ in $0 \text{ kpc} \leq \alpha, \beta \leq 5$
$kpc$, $(\xi^2_0+C^2)$'s were determined at first by

$$
\frac{1}{2\dot{\omega}_0+\dot{\beta}-\alpha} \{2\dot{\omega}_0\dot{\theta}_0+a(4\dot{\theta}_0^2+\dot{\omega}_0^2J_0)(\beta-\alpha)\}
-\frac{768,688+115,881.4(\beta-\alpha)}{16.4+16(\beta-\alpha)}

(39)
$$

which is reduced from

$$
\beta-\alpha=-\frac{-\frac{2}{\dot{\omega}_0}(\xi^2_0+C^2)+2\dot{\omega}_0\dot{\omega}_0}{-\frac{1}{\dot{\omega}_0}(\xi^2_0+C^2)+4\dot{\omega}_0^2+J_0}.

(40)
$$

Then, after evaluating $b$ from

$$
b=\frac{1}{\dot{\omega}_0^2}(\xi^2_0+C^2)-4\dot{\omega}_0^2-J_0=(\xi^2_0+C^2)\times 0.014,872-1,723.4,
$$

we got a pair of the corresponded values of $\dot{\xi}_0=\pm\sqrt{-b\alpha\beta}$ and $\dot{\eta}=C-\dot{\theta}_0=\sqrt{(\xi^2_0+C^2)-\xi^2_0}-216.5$. It is remarked that a curve of a constant $\beta+\alpha$ is nearly
an ellipse with its center at the coordinates-origin $(\dot{x}_0=\dot{y}=0)$, while that of a
constant $\beta-\alpha$ is also nearly an ellipse with its center at $\dot{x}_0=0$, $\dot{y}_0=-\frac{\omega_0-A_0}{2\omega_0-A_0}\dot{\theta}_0$.

The velocity-diagram thus obtained is shown in Fig. 6 in which the values of $\alpha$ and $\beta$ may be read in a few tenth kiloparsecs. The curve giving $T_\xi$ are also
indicated there.

On use of this diagram, however, a fact must be taken into account that the
values of $\dot{\xi}_0$ and $\dot{\eta}_0$ have been defined at the local standard of rest, consequently
the diagram cannot be available except for the near-by stars. Hence, for distant
stars with non-negligible coordinates $(\xi, \eta)$, their residual velocity-components
$(\dot{\xi}, \dot{\eta})$ at $(\dot{x}, \dot{y})$ should be transformed into the ones which will be taken at the
local standard of rest $(\xi=\eta=0)$. From a practical view point, moreover, as the
observed kinematical quantities are given usually by the rectangular coordinates,
they must have been converted into $\dot{x}$ and $\dot{y}$, $\dot{x}_0$, $\dot{y}_0$ beforehand.

Let $(x, y)$ and $(u, v)$ be respectively the rectangular coordinates and the
velocity-components of a star, lying at a distance $r$ from the local standard of
rest in a direction of the galactic coordinated $(l, b)$. Then,

$$
\begin{align*}
x &= -r \cos b \cos (l-L_e), & u &= \dot{z}, \\
y &= r \cos b \sin (l-L_e), & v &= \dot{y}, \\
\end{align*}

(41)
$$

where $l_e$ is the longitude of the galactic center ($l_e=327^\circ$). If these values are
given, the corresponded values of $(\xi, \eta)$ and $(\dot{\xi}, \dot{\eta})$ can be calculated by the following relations with approximations up to an order of $\left(\frac{x}{\dot{\omega}_0}\right)^2$. 

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Now, as for the transformation of \((\xi, \eta)\) into \((\xi', \eta')\) we get from (2), (3) and (7),

\[ C = (\eta_0 + \Theta_0) \left( 1 + \frac{\xi}{\Theta_0} \right)^2, \quad \eta' = C - \Theta_0, \]

\[ \xi' = \xi + (\eta + \Theta_0)^2 + \frac{2\Theta_0^2}{\Theta_0} \xi + J_0 \xi^2 + \frac{2}{3} H_0 \xi^3 + \frac{K_0}{2} \xi^4 + \ldots. \]
or alternatively from (9)

\[ \dot{\xi}^2 = \dot{\xi}^2 \left( 1 + \frac{\xi}{\omega_0} \right)^2 - a \xi - b \xi^2 - c \xi^3 - \ldots \]

(44')

With these processes \( \dot{\gamma}_0 \) and \( \dot{\xi}_0 \) will be reduced. If \( \dot{\xi}_0 \geq 0 \), then the diagram in Fig. 6 is applicable. But, if not, in other words, if the stellar orbit in question lies wholly inside or outside the circle \( \alpha = \omega_0 \), Fig. 6 remains as useless. In such cases, at least two more similar diagrams are needed which must be prepared independently from the beginning by choosing suitable values of \( \omega_0 \), the one being \( \omega_0 = \omega_1 < 8.2 \text{ kpc} \) and the other being \( \omega_0 = \omega_2 > 8.2 \text{ kpc} \).

Instead of constructing these \((\xi_0, \gamma_0)\)-diagrams, however, the alternative diagrams of \((C = \Theta_0 + \gamma_0, \sqrt{-1} \dot{\xi}_0)\) available for reading the values of \( \alpha \) and \( \beta \) were made for both cases of \( \alpha > - \beta > 0 \) (inner orbits) and \( \beta > - \alpha > 0 \) (outer orbits) by the same process of calculation as in the case of Fig. 6. They are shown in Figs. 7 and 8 corresponded to the inner and the outer orbits respectively.


Now, as the examples for practical applications of our formulae, let us show the numerical results concerning the orbital parameters of the eleven near-by stars, whose kinematical data as well as the galactic orbits have been given in Torgård's paper²). Her results on the galactic orbits of these stars (her Table 22) have been calculated with aids of BESK by the method of the numerical integration, basing on the plane model of her Galaxy 1.

In Torgård's Table 21, the velocity-components have been given in terms of \( \Pi, \Theta \), but we regard them as \( \xi_0, \gamma_0 \) respectively by neglecting all the small corrections due to (41)-(44). As all of these stars can be classified into the case (iii) (see Fig. 3), their numerical values of \( \alpha, \beta, T_\xi \) and \( T_\gamma \) were calculated by our respective formulae in §4.3. Our results and the corresponding ones due to Torgård are given in the columns of \((\alpha_s', \beta_s', T_{\xi s'}, T_{s s'})\) and \((\alpha_T, \beta_T, T_{\xi T})\) of Table 1, respectively.

Torgård's force-function is not quite the same as our one specified for the first approximation (14), as seen in Fig. 2. But, reflecting that we have not expected our first approximation formulae to hold beyond \( |\xi| = 5 \text{ kpc} \), both sets of the parameter-values may be said to be accordant fairly well, except the \( \beta \)-value of HD 10700. As for this star, \( \beta_s' \) becomes as 3.6 kpc against \( \beta_T > 9.8 \text{ kpc} \). It is remarked, however, that this star has a large \( c^2 \)-value \( (C^2 \omega_0^2 \text{ in our notation}) \) close to the maximum value of Torgård's model and its \( r^2 \)-curve \( (\xi^2 \text{ in our notation}) \) lies between the two lowest curves in her Fig. 19, consequently its apogalacticon distance is sensitively influenced by a small change of the \( r^2 \)-curve.

It is mentioned further that differences between \( T_{\xi s'} \) and \( T_{s s'} \) appear not to be so small as predicted before (§4.3). But this is due to a fact that the present computations are based on \( J_0 = 1064.9 \) corresponding to \( A_0 = 16.67 \) \( (A_0/\omega_0 = 0.84) \) but not on \( A_0 = 19.5 \) \( (A_0/\omega_0 = 0.985) \) assumed previously.

§7. Orbits in the Second Approximation.

Now, in order to see how far the results in the first approximation deviate from those in the second one, let us consider the latter by referring to (13'). The force-function specified for the second approximation is similarly obtained
from the condition $d=e=\ldots=0$, or

$$\frac{\partial V}{\partial \omega} = \bar{\omega}_0 \omega_0^2 + \frac{J\bar{\omega}_0}{2} \sum_{n=1}^{\infty} (-1)^n(n+1)(n-2)\left(\frac{\xi}{\bar{\omega}_0}\right)^n + \frac{\bar{\omega}_0^2H}{3} \sum_{n=1}^{\infty} (-1)^n(n-1)(n+1)\left(\frac{\xi}{\bar{\omega}_0}\right)^n. $$

For $|\xi|<\bar{\omega}_0$, this is reduced into

$$\frac{\partial V}{\partial \omega} = \bar{\omega}_0 \omega_0^2 + J\bar{\omega}_0 \left\{ \frac{\left(\frac{\xi}{\bar{\omega}_0}\right)}{1+\left(\frac{\xi}{\bar{\omega}_0}\right)} + \frac{\left(\frac{\xi}{\bar{\omega}_0}\right)^2}{1+\left(\frac{\xi}{\bar{\omega}_0}\right)^2} \right\} + \frac{H\bar{\omega}_0^2}{3} \left\{ \frac{\left(\frac{\xi}{\bar{\omega}_0}\right)^2}{1+\left(\frac{\xi}{\bar{\omega}_0}\right)} + \frac{2\left(\frac{\xi}{\bar{\omega}_0}\right)^2}{1+\left(\frac{\xi}{\bar{\omega}_0}\right)^2} \right\}. \tag{45} $$

![Image](image-url)  
**Fig. 7.** Velocity-diagram for getting $\alpha$ and $\beta$. [For the inner orbits]

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Fig. 8. Velocity-diagram for getting $\alpha$ and $\beta$.
[For the outer orbits]
Table 1. Orbital Elements of Eleven Stars.

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<th>$v$</th>
<th>$w$</th>
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<th>$\alpha_s''$</th>
<th>$\beta_T$</th>
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<th>$\beta_s''$</th>
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$\alpha_T, \beta_T, T_{\xi_T}$: Torgård's values; $\alpha_s', \beta_s', T_{\xi_s'}$: Our values in the first approximation ($c=0$).

$\alpha_s'', \beta_s'', T_{\xi_s''}$: Our values in the second approximation ($c=0.349$); ($\alpha_s''$, $\beta_s''$, $T_{\xi_s''}$): Our values in the second approximation ($c=10.101$).
In Fig. 2, this function is illustrated by the curve \( c=49.349 \), for which the numerical values of (8) are applied namely as for the curve \( c=0 \).

From (13') \( E''=0 \) is a cubic equation about \( \xi \), so that it has whether (i) three real roots or (ii) one real root and two imaginary ones. A periodic orbit occurs in the case (i), while an unbounded orbit does in the case (ii), because \( E'' \) must be \( \geq 0 \) for any real value of \( \xi \). But we limit ourselves here only to the periodic orbit.

Denoting the three real roots as \( -\alpha, \beta \) and \( \gamma \) (\( -\alpha \leq \beta \leq \gamma \)), the following formulae are readily obtained with the same notations as before.

\[
E'' = \xi^3 + a\xi + b\xi^2 + c\xi = c(\xi + \alpha)(\xi - \beta)(\xi - \alpha), \quad (-\alpha \leq \beta \leq \gamma)
\]

\[
t - t_1 = \frac{2}{\omega_0\sqrt{c(\alpha + \gamma)}} \left\{ (\omega_0 + \gamma)F(k, \varphi) - (\alpha + \gamma)F(k, \varphi) \right\},
\]

\[
\varphi = \arcsin \sqrt{\frac{\xi + \alpha}{\beta + \alpha}}, \quad k = \sqrt{\frac{\beta + \alpha}{\gamma + \alpha}},
\]

\[
F(k, \varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad E(k, \varphi) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \varphi} d\varphi,
\]

\[
T_\xi = \frac{4}{\omega_0\sqrt{c(\alpha + \gamma)}} \left\{ (\omega_0 + \alpha)F(k, \frac{\pi}{2}) - (\alpha + \gamma)E(k, \frac{\pi}{2}) \right\},
\]

\[
\theta - \theta_1 = \frac{2C}{(\omega_0 - \alpha)\sqrt{c(\alpha + \gamma)}} II(\varphi, n, k), \quad n = \frac{\beta + \alpha}{\omega_0 - \alpha},
\]

\[
II(\varphi, n, k) = \int_0^\varphi \frac{d\varphi}{(1 + n \sin^2 \varphi)\sqrt{1 - k^2 \sin^2 \varphi}}.
\]

By means of these formulae the orbital parameters of the eleven stars were calculated by adopting the numerical values given in (12) or (8), accordingly by taking an additional constant \( c = +49.349 \) \( \frac{\text{km/s}^2}{\text{kpc}^2} \) \( (H = +60.544 \) \( \frac{\text{km/s}^2}{\text{kpc}^2} \) together with the same values of \( a \) and \( b \) as before. The results are shown in the columns of \( \alpha_s''', \beta_s''' \) and \( T_{\xi s}''' \) of Table 1.

Comparing \( (\alpha_s', \beta_s', T_{\xi s'}) \) with \( (\alpha_s''', \beta_s''', T_{\xi s'''}) \) for each star, we find that though both corresponding force-functions differ remarkably in an inner region of \( \xi > -3 \) kpc as seen in Fig. 2, both sets of the peri-galactocentric distance are unexpectedly accordant, namely within a deviation of 13%, while discrepancy between both force-functions in an outer zone, say \( \xi > 3 \) kpc, seems to affect considerably on the apo-galactocentric distance as well as the period of \( \xi \)-component. As the first approximation corresponds to adopting \( c=0 \) \( (H_0=134.61 \) \( \frac{\text{km/s}^2}{\text{kpc}^2} \) ), the above-mentioned fact indicates that, letting \( \omega_0\alpha_0 \) and \( J_0 \) be fixed, the value of \( c \) affects sensibly on the apo-galactocentric distance \( \beta \), especially when \( \xi \) exceeds about +3 kpc.


Here, it will be examined whether the adopted \( c \)-value might be actually consistent with the observed value of the velocity of escape or not. The velocity of escape occurs in such a case where \( E''=0 \) has two real roots or its discriminant \( D \) vanishes to zero.

If we accept the velocity of escape in the direction of the galactic rotation at \( \omega = \omega_0 \) (\( \xi = 0 \)) to be \( \xi_{oc}=0 \), \( \eta_{oc}=63.5 \) km/s \( (C_0=\Theta_0+\eta_{oc}=280 \) km/s), then the condition that \( E''=\xi(\alpha + b\xi + c\xi^2) \) has two real equal roots should be
Numerically, letting our adopted $J_0 = -1064.9$ (km/s)/kpc be kept unchanged, it becomes as

$$c = +10.101 \text{ (km/s)}^2/\text{kpc}^3, \quad (H_0 = 119.46 \text{ (km/s)}^2/\text{kpc}^3)$$

but, if $J_0 = -1360$ (km/s)$^2$/kpc$^2$ is assumed, it is to be

$$c = +2.101 \text{ (km/s)}^2/\text{kpc}^3, \quad (H = +239.27 \text{ (km/s)}^2/\text{kpc}^3).$$

Whereas, in a general case where $\xi_{oe} = 0$ accordingly $E^2 = \xi_{oe}^2 + a\xi + b\xi^2 + c\xi^3$, the condition for the escape velocity is written as

$$D = \left( \frac{27c^2\xi_{oe}^2 - 9abc + 2b^2}{27c^3} \right)^2 - 4\left( \frac{3ca - b^2}{3c^2} \right)$$

$$= -\frac{1}{27c^4} \{ 27c^2\xi_{oe}^4 + 2b(2b^2 - 9ca)\xi_{oe}^2 - a^2(b^2 - 4ca) \} = 0,$$

consequently the $\xi$-component of escape velocity must be

$$0 \leq \xi_{oe}^2 = \frac{b(9ca - 2b^2) + \sqrt{b^3(9ca - 2b^2)^2 - 27c^2a^2(4ca - b^2)}}{27c^2} \leq \xi_{oe}^2 + C^2.$$

For any trial values of velocity of escape at $\xi = 0$ or $\sqrt{\xi_{oe}^2 + C^2}$, the values of $a$ and $b$ are determined, hence with these and an assumed $c$-value the formula (49) enables us to evaluate $\xi_{oe}^2$ and thereafter to ascertain whether the trial value corresponds really to the velocity of escape or not. Then, if the value of $c$ is fixed, from a proper set of trial values about $\sqrt{\xi_{oe}^2 + C^2}$, we can get a closed curve limiting the velocities of escape on the velocity-diagram as illustrated in Fig. 3. There are given two curves corresponded to $c = +10.101$ and $c = 49.349$ respectively. (the circle $b = 0$ corresponds to $c = 0$).

It is seen there that the closed curve limiting velocities of escape is generally not circular but elliptic, and further its form as well as its dimension depend on an adopted value of $c$ or $H_0$. Moreover, such circumstances that the $\beta_s''$-values of HD 10700 and 157089 differ considerably from the respective $\beta_s$' while for HD 140283 no definite values of $\beta_s$' exists, or in other words, its orbit becomes to be unbounded, may be understandable by making reference to the curve. Because the velocity-points of these stars are close to the curve with $c = 49.349$, the former two being inside and the latter outside.

It is not probable, however, that any of the stars given in Table I is going to escape outward, therefore $c = 49.349$ used for the calculations of $(\alpha_s'', \beta_s''$)'s in Table 1 seems not to be appropriate. To see how a pair of the values $\alpha_s''$ and $\beta_s''$ depends upon the value of $c$ and to examine with what value of $c$ the stellar orbit begins to be unbounded outward or inward, the values of $(\alpha_s', \beta_s')$'s for various values of $c$ were computed with respects to the above-mentioned three stars. Fig. 9 represents the results graphically. We find then that the orbits of HD 10700, 157089 and 140283 are unbounded inward in a certain $c$-range beginning from $c = -66$, though such mention may not interpreted rigorously because of $\alpha \leq \bar{\alpha}_0 = 8.2$ kpc.

Judging from the matters above-mentioned, it may be concluded that the probable value of $c$ does not differ appreciably from that given by (48), if
as $\bar{\omega}_0=8.2$ kpc, $\omega_0=26.4$ (km/s)/kpc and $J_0=-1069.4$ (km/s)/kpc may be accepted.

* On the ground of the third approximation, it is possible to derive the value of $d$ consistent with an escape velocity of $(\xi_{oc}=0$, $\eta_{oc}$=say 63.5 km/s). Considering possible combinations of real roots from $E/z=\xi(a+b\xi+c\xi^2+d\xi^3)$, an unbounded orbit occurs if $d$ as well as the discriminant of $a+b\xi+c\xi^2+d\xi^3$ are non-negative. Accordingly

$$d = b(9ac-2b^2) \pm \sqrt{b^2(9ac-2b^2)^2-4a^2c^4(4ac-b^3)}}$$

Then, if $c=b^2/4a$ ($a>0$, $b>0$), there exists one value of $d \geq 0$. As both the double signs are corresponded in order of arrangements, it is likely to be $b \approx 0$ for any escape velocity, provided that $c=b^2/4a$ is satisfied.

So far as the stellar orbits are studied on the basis of any of our approximation formulae, to evaluate its force-constants with a specified force-function under consideration may be more favourable than to do by mean of a Taylor series as done in § 2. On this line, the coefficients such as $\omega_0 \omega^2$, $J_\delta \omega^3$, $(H_0/3) \omega^3$, ... were determined directly from the empirical data with the method of the least squares.

The formulae of the force-function specified for both the first and the second approximation have been given in (14) and (45) respectively. So we add here the one for the third approximation. Under a restriction of $|\xi| < \omega_0$, this will be

$$ \frac{\partial V}{\partial \omega} = \omega_0 \omega^2 + J_\delta \omega^3 - \frac{H_0}{3} \omega^2 \sum_{n=1}^{\infty} (-1)^{n+1}(n-1)(n-3) \left( \frac{\xi}{\omega_0} \right)^n, $$

$$ - \frac{K_0}{4} \omega^2 \sum_{n=1}^{\infty} (-1)^n(n-1)(n+1) \left( \frac{\xi}{\omega_0} \right)^n $$

or

$$ \frac{\partial V}{\partial \omega} = \omega_0 \omega^2 + J_\delta \omega^3 + \frac{H_0}{3} \omega^2 \left\{ 4 \left( \frac{\xi}{\omega_0} \right) - \frac{3(\xi/\omega_0)}{1+((\xi/\omega_0)^2)} \right\} - \frac{(\xi/\omega_0)^2}{1+((\xi/\omega_0)^2)} \right\} + \frac{K_0}{2} \omega \left\{ \left( \frac{\xi}{\omega_0} \right) - \frac{(\xi/\omega_0)}{1+((\xi/\omega_0)^2)} \right\} - \frac{(\xi/\omega_0)^2}{1+((\xi/\omega_0)^2)^2}. $$

These (14), (15) and (50) were used by turns for getting the least squares solutions shown in Table II. We can notice there that the least square solution depends only on the order of approximation but also remarkably on the considered range of $\omega$, as did the solution due to the Taylor series. Moreover, the percentage fluctuations among the tabulated values of $\omega_0 \omega^2$, $J_\delta \omega^3$, $(H_0/3) \omega^3$ and so on, being mainly of the systematic nature, appear to become larger in this sequence.

Nevertheless, among the resulted force-functions, irrespective of either the second or the third, those characterized by such respective sets of constants that are satisfied nearly by the relation (47) or $c=b^2/4a$ (assuming $\xi_0=0$, $\gamma=63.5$ km/s) come out to represent nearly the empirical force data for the outer region of the Galaxy. The fact is seen from Table II and Fig. 2. In Table II, the $c$-values computed directly from (10) are shown in comparisons with those due to (47) and the solutions bringing comparatively large residuals in the outer region are distinguished by the parentheses. While in Fig. 2 (upper part), some of the force-functions are illustrated as examples.

On the other hand, it was remarked before that the force-constants are not necessarily equivalent to $\frac{\partial V}{\partial \omega}$, $\frac{1}{2} \frac{\partial^2 V}{\partial \omega^2}$, $\frac{1}{6} \frac{\partial^3 V}{\partial \omega^3}$, ..., at $\omega=\omega_0$ but something respective mean values. Hence, it is no wonder to get a different numerical set of force-constants by changing the adopted range of $\omega_0$. In connection with this, let us examine the dependencies of the force-derivatives on the galactocentric distance $\omega$ to draw some information concerning the force-constants. From the potential values given in Schmidt's Table 10, the values of $\frac{\partial V}{\partial \omega}$, $\frac{\partial^2 V}{\partial \omega^2}$ and $\frac{\partial^3 V}{\partial \omega^3}$ for every round kpc of the galactocentric distance ($3\text{kpc} \leq \omega \leq 15\text{kpc}$) were computed by the Stirling's formula. And then, the values of $\omega = \sqrt{\left( \frac{\partial V}{\partial \omega} \right) / \omega}$ and...
Table 2. Least squares solutions of the force-function.

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$\omega^2 \bar{\omega}, \ldots, \frac{K_0}{2} \bar{\omega}^3$ are least squares solutions. $J_0, H_0, \ldots, d$ are derived from the solutions by assuming $\bar{\omega}_0=8.2$ kpc.

Order of approximation is self-evident, but it is shown with a respective suffix of designated letter.
\[ A = \frac{\omega}{4} - \frac{3}{4a} \frac{\partial^2 V}{\partial \omega^2} \]

were calculated for every corresponding values of \( \omega \). The differences between \( \omega \)'s computed and those given by Kwee et al. were found to be less than 0.8%, consequently all the computed values are regarded as practically equivalent to the corresponding ones from our data. The computed values of \( \left( \frac{\partial V}{\partial \omega} \right)'s \), \( \left( \frac{\partial^2 V}{\partial \omega^2} \right)'s \), \( A 's \), \( \left( \frac{\partial^3 V}{\partial \omega^3} \right)'s \) were plotted against the galactocentric distance \( \omega \)'s as illustrated in Fig. 10.

In the diagram, the circles at \( \omega_0 = 8.2 \) kpc represent the adopted values of \( J_0 \) and \( 2H_0 \) given in (8), which have been determined in a range of \( 3 \) kpc \( \leq \omega \leq 14 \) kpc by the method of the least squares. At glancing Fig. 10 the following

![Graph](image)

**Fig. 10.** Variations of \( \frac{\partial V}{\partial \omega} \), \( \frac{\partial^2 V}{\partial \omega^2} \), \( \frac{\partial^3 V}{\partial \omega^3} \), \( A \) and \( c \) due to Schmidt's model G1.

- ●: ⊙: correspond to the numerical values of (8).

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may be noticed. (i) Over a region about $4 \, \text{kpc} \leq \omega \leq 10 \, \text{kpc}$, the curve of $\frac{\partial V}{\partial \omega}$ is not monotonously decreasing with $\omega$ but is superposed by a slight undulation and the influence appears the more distinctly on the curves of $\frac{\partial^2 V}{\partial \omega^2}$ and $\frac{\partial^3 V}{\partial \omega^3}$ in order. (ii) As for $\frac{\partial^3 V}{\partial \omega^3}$, the undulation exceeds the systematic change and especially near the sun ($\omega_0=8.2 \, \text{kpc}$) its change is very rapid. (iii) As expected each circle is situated as if it were along by some "mean" curve which makes the undulation even. (iv) But the path of respective "mean" curve is not definitely defined, especially with respect to $\frac{\partial^3 V}{\partial \omega^3}$ and $c(\omega) = \frac{2}{\omega} \left( \frac{\partial V}{\partial \omega} - \frac{2}{\omega \omega_0} \left( \frac{\partial^2 V}{\partial \omega^2} - \frac{1}{3} \frac{\partial^3 V}{\partial \omega^3} \right) \right)$ whose fluctuations are so large. It is clear that the runs of these "mean" curves may depend on the range of $\omega$ under consideration. (v) The value of $(c)_{esc}$ or $(2H_0)_{esc}$ given in (48) being consistent with the observed escape velocity is also situated as if $(c)_{esc}$ or $(2H_0)_{esc}$ were along another some "mean" curve different from that for the adopted $c$- or $2H_0$-value. (vi) The matters mentioned in (iii)~(v) still hold similarly to the least squares solutions in Table II.

By taking into account the consistency with the observed escape velocity, the $c$-value must be identical to the $(c)_{esc}$-value computed from (47). Then, let us consider a set of the force-constants combining the adopted values of $\omega_0$, $\omega_0$ and $J_0$ given in (8) with $c=10.101$ obtained in (48) instead of $c=49.349$ adopted previously. The corresponding force-function or

$$\frac{\partial V}{\partial \omega} = 5715.1 - 8782.2 \left\{ \frac{\xi/\omega_0}{(1+\xi/\omega_0)^2} + \frac{(\xi/\omega_0)^2}{(1+\xi/\omega_0)^3} \right\}$$

$$+ 2677.5 \left\{ \frac{(\xi/\omega_0)^2}{(1+\xi/\omega_0)^2} + \frac{2(\xi/\omega_0)^2}{(1+\xi/\omega_0)^3} \right\}$$

(51)

is illustrated in Fig. 2 as the curve $c=10.101$. It is found there that the force-function with $c=49.349$ represents closely the empirical force-data in the outer region while this force-function does less satisfactorily. This means that our adopted values of $\omega_0$, $\omega_0$ and $J_0$ cannot be consistent with our empirical force-data so far as the escape velocity is concerned. Letting the empirical data be as they stand, nevertheless, a slight modification of these adopted constants will remove or at least make lessen considerably this contradiction. In reality, as indicated before, among the least squares solutions in Table II there exist such ones as that the condition $c=(c)_{esc}$ is nearly satisfied under the intended conditions about the order of approximation and the range of $\omega$. Consequently, for determining the stellar orbit in the second approximation within the range of $3 \, \text{kpc} \leq \omega \leq 15 \, \text{kpc}$, such a least squares solution, say $A_2$ in Table II, seems to be preferable to (51).

It is expected, however, that the empirical data themselves are subjected to non-negligible errors. So, if our adopted values of $\omega_0$, $\omega_0$ and $J_0$ are correct under the intended conditions, then the empirical data should be revised in such a way that they bring $c=10.101$ in place of $c=49.349$. At any rate, we have nothing to say definitely at present concerning the force-data, so that it is reserved here to shape the matter concretely.

Nevertheless for comparisson, by supposing the latter's view the revised
orbital parameters due to the force-function (51) were calculated for all the stars in Table II. These results are shown also in the columns \((\alpha_s')\), \((\beta_s')\) and \((T_{\xi s}')\) of the same table. As \(c=10.101\) is nearer to \(c=0\) than to \(c=49.349\), the revised stellar orbits in the second approximation should be nearer to those in the first approximation. Comparison between \((\alpha_s', \beta_s', T_{\xi s}')\)’s and \((\alpha_s''), (\beta_s''), (T_{\xi s}'')\)’s agree with this anticipation. On the other hand, the differences of the parameter-values between Torgård’s and our revised second approximation may be ascribed to the difference of both the adopted force-functions.

§9. Conclusion.

The main results in this study are summarized as follows.

1) In the two dimensional model of the Galaxy assumed as axially symmetric, the stellar orbit may be represented well even by our first approximation formulea. They give three different kinds of orbit, namely viewed from the fixed galactocentric coordinate-system, (i) an orbit of some complex kind of hyperbolic spiral, (ii) a parabolic orbit and (iii) a periodic orbit which appears elliptic to a suitably chosen rotating coordinate-system. Lindblad’s expression of an epicyclic orbit is reduced from our first approximation formulea as a special case. (§§ 4.1~4.3)

2) With respect to the periodic orbit, were made the diagrams available for readings of both peri- and apo-galacticon distances as well as a period of the radial motion. (§§ 4.1~4.3)

3) The formulea in the second approximation were given, but except for high velocity stars a gain of accuracy may be a little, unless the absolute value of a constant \(c\) is large. That \(|c|\) is large, however, contradicts the observed value of the escape velocity. (§§ 4.7)

4) On the velocity-diagram, the escape velocities are limited by a closed curve whose shape and dimension depend sensibly on the \(c\)-value so far as \(\omega_0\), \(\alpha_0\) and \(J_0=\omega_0(\omega_0-4A_0)\) are kept unchanged. Accepting \(\omega_0=5.2\) kpc, \(\alpha_0=26.4\) (km/s)/kpc, \(J_0=1064.9\) (km/s)/kpc² \((A=16.67\) (km/s)/kpc) and the escape velocity of 63.5 km/s, the \(c\)-value is to be +10 (km/s)/kpc³ \((H=120\) (km/s)/kpc³). With the first approximation corresponded to \(c=0\), the escape velocity is expected to be too large as +126 km/s, so that some unbounded orbit may be represented as periodic in the first approximation. Nevertheless, even for the stars moving really along the unbounded orbits, their paths near the sun will be known fairly well by the first approximation formulea. (§§ 4,7)

5) The different order of approximation corresponds to adopt the different analytic expression for the force-function and the number of the parameters or the force-constants increases by one at each advanced step of approximation. (§§ 4,7,8)

6) The force-function was evaluated in various ways from the same empirical data. The numerical results on the force-constants depend systematically on the conditions imposed initially, such as the order of approximation, the range of \(\omega\) and so on. But, as we must be satisfied with some sort of approximate orbit, it may be the best way to make use of the force-function matched with the intended conditions such as the order of approximation and the considered range of \(\omega\). (§§ 2,8)

7) Provided that \(\omega_0 = 8.2\) kpc, \(\alpha_0 = 26.4\) (km/s)/kpc and \(J_0 = -1064.9\)
be fixed, the force-function expressed by (51) may be probable for the second approximation over a range of $3 \text{kpc} \leq \bar{\omega} \leq 15 \text{kpc}$, and there is no need of the third approximation because of $c = 10.101$ and accordingly of $d = 0$. (§ 7)

8) The force-constants determined by the method of least squares from the empirical data do not coincide with the respective force-derivatives at $\bar{\omega} = \bar{\omega}_0$ but may be regarded as something like respective mean values around $\bar{\omega} = \bar{\omega}_0$. (§§ 2, 8)

9) For the eleven actual stars, of which orbits have been calculated numerically by Torgård, we also determined their orbital parameters with respects to three different cases, namely in the first approximation ($c = 0$) and in the second approximations with $c = 49.349$ and $10.101$. The results are as expected and the slight discrepancies among Torgård's and three sets of ours may be ascribed to the differences of the respective force-functions. (§§ 6, 7, 8)

**Literatures**