SOURCE FUNCTION IN A NON-EQUILIBRIUM ATMOSPHERE. VI. THE
FREQUENCY DEPENDENCE OF THE SOURCE FUNCTION
FOR RESONANCE LINES

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ABSTRACT

The frequency dependence of the line source function is investigated for the case of pure coherent
scattering in the reference frame of the atom. It is shown that the thermal redistribution due to Doppler
effect gives a form of scattering similar to complete redistribution in the line core and coherency in the
wings. Using a modified form for this redistribution and allowing for some residual non-coherency due to
collisions in the frame of the atom, an algebraic solution of the transfer equation is obtained, and emergent
line profiles are computed, for an isothermal atmosphere. It is shown that the line shape in the transition
region from line core to wing is strongly influenced by the proportion of this residual non-coherency. It is
finally suggested that, until the strength of collisional perturbations is better understood from theoretical
or laboratory studies, theoretical work on line spectra should adopt complete redistribution in scattering.

I. INTRODUCTION

In Paper I of this series Thomas (1957) investigated the frequency incoherence in
scattering which is introduced by Doppler redistribution. He showed that, over the line
core, the scattering is essentially purely non-coherent. Subsequent papers have developed
some of the consequences of this fact for the cores of resonance lines. In this paper we
investigate the form of the source function over the whole line profile, not just the line
core, for the case when scattering in the frame of the atom is purely coherent, and we
suggest some consequences on the shape of the line profile.

II. THE SOURCE FUNCTION

Warwick (1955) has written the transfer equation in the convenient form

\[ -\mu \frac{d I_\nu}{dx} = I_\nu (N_L B_{LU} - N_U B_{UL}) \phi_\nu - N_L B_{LU} \int \int R(q, \phi_\nu, I_{\nu'}, N_{\nu'}) d\nu' d\omega' \]

where \( R(q, \nu', \nu, \omega', \omega) \) is the fraction of a scattered radiation absorbed at \( \nu' \) from a direction
\( \omega' \) which is scattered into \( \nu \) and \( \omega \); \( q \) is the fraction of radiation which is scattered; \( \phi_\nu \) is
the normalized profile of the absorption coefficient; and the subscriptions \( L \) and \( U \) refer,
respectively, to the lower and upper states of the transition. The form of \( R \) is controlled
by the frequency dependence of the scattering, and it is with the specification of this
function that this paper is primarily concerned. Pure non-coherent scattering or complete
redistribution corresponds to the case \( R = \phi_\nu / 4\pi \), and, for this, equation (1) takes a
particularly simple form.

We distinguish first between those processes, outlined, e.g., by Spitzer (1944), which
produce non-coherency in scattering in a stationary atom and the inevitable redistribution
resulting from the Doppler shift due to motion of the scattering atom. Those under the
first heading come from the natural breadth of the states or from perturbations by

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collisions or radiation which modify the lifetime of the upper or lower state and so control
the frequency coherence in the atom itself. In general, it has been shown by Woolley and
Stibbs (1953) and by Zanstra (1941) that for a stationary atom these perturbations result
in a redistribution characterized as partly coherent and the remainder purely non-coherent.
While this is only approximate, it is entirely adequate at the present phase of spectral line
interpretation.

Two cases are therefore to be considered, according to whether the scattering in the
frame of the atom is (a) purely coherent or (b) completely redistributed, and we wish to
determine the effect of the atomic velocities on the frequency dependence of the scattering
as seen by an external observer. The general form for $R$ in equation (1) will then be
obtained by adding the two components with appropriate weights.

Case b can be handled quite easily. If complete redistribution applies in the atom's
frame, it will hold in an external frame of reference, provided that one makes the appropria
t change from an atomic to a Doppler-broadened absorption coefficient.

Case a has been considered by Henyey (1940) and Unno (1951), while Thomas (1957)
has applied it to the line core. For a monochromatic beam of frequency $v_1$ and unit in-
tensity in a given direction, Henyey shows that the amount $f(v_1; v_2; a)$ scattered isotropi-
cally per unit solid angle through an angle $a$ in a unit frequency range about $v_2$ is

$$f(v_1; v_2; a) = rac{a_D(v_0)}{4\pi^{3/2}} \sin a \exp \left[ -\frac{1}{2} \left( \frac{v_2 - v_1}{\Delta v_0} \right)^2 \right]$$

where $v_1$ and $v_2$ are the frequencies from the line center, in units of the Doppler width $\delta$;
$a_D(v_0)$ is the Doppler absorption coefficient at the line center; and $a_2 = \delta/b \sec a/2$, with
$4\pi\delta$ the sum of the spontaneous transition probabilities from the upper state to all lower
states. The scattered component of the emission in a specific direction is then given by the
integral, over all incident frequencies and directions, of the specific intensity, $I(v_1, \theta)$,
weighted by $f(v_1; v_2; a)$, where the integration is to be consistent with the emission being
in the given direction.

The integration over direction is readily performed, if $I(v_1; \theta)$ is isotropic, to give

$$\epsilon(v_2) = 2\pi \int_{-\infty}^{+\infty} I(v_1) \int_0^{2\pi} f(v_1; v_2; a) \sin \alpha \, dv_1 \, d\alpha,$$

where

$$I(v_1) = \frac{1}{4\pi} \int_{4\pi} I(v_1, \theta) \, d\omega,$$

so that the component contributed to the source function may be written

$$S_L(v_2) = \frac{\epsilon(v_2)}{\alpha(v_2)} = \int_{-\infty}^{+\infty} I(v_1) g(v_1; v_2) \, dv_1.$$

Values of the probability amplitude, $g(v_1; v_2)$, are given in Figure 1 for a damping width,
$\alpha$, of $10^{-3}$. The specific value of the damping width is not critical to the form of the curves.
Comparison of equations (1) and (4) shows that, for isotropic scattering, $R = g(v_1; v_2) \phi(v_2)/(4\pi \phi(v_1))$.

1 In Henyey's published result the exponent is misprinted as

$$\frac{1}{2} (v_2 - v_1)^2 \cos^2 \frac{\alpha}{2}.$$
Equation (3) also applies to the case where $I(v_i; \theta)$ is a linear function of $\cos \theta$. This is so, since, as is readily shown, the contribution from frequency $v_i$ in an incident beam to the scattered intensity in a certain direction is equal to that from frequency $-v_i$ in the same beam to the intensity scattered in the opposite direction. Thus that component of the intensity which is proportional to $\cos \theta$ must cancel in the scattered beam.

It is evident from Figure 1 that the basic coherency in the scattering has been very seriously modified in the sense that, the farther from the line center, the greater the degree to which the re-emission tends to be centered about the incident frequency. For complete redistribution, $g(v_1, v_2) = \delta(v_1)$, so that the curves in Figure 1 would degenerate into a single one centered about the origin.

The physical explanation of the results in Figure 1 is as follows. In the atom’s frame of reference the absorption coefficient has an extremely sharp peak at the line center, so that most absorptions and re-emissions take place at the line center of those atoms which are moving in such a way as to present this maximum to the photons. The pattern of re-emission is thus essentially symmetric about the line center and is broadened by the atom’s thermal motion. Away from the line center, however, the dominant absorption is in the wings, since, at these frequencies, few atoms are moving fast enough to present the center of the absorption pattern to the photon. In this case most absorptions are by atoms with small velocities, since these are the most abundant. The absorbed and emitted photons have much the same frequencies because the re-emission is coherent in the atom’s frame of reference, which is, as above, effectively at rest for these frequencies.

Because the redistribution function shown in Figure 1 is too complex for application,

Fig. 1.—The thermal redistribution function in a coherently scattering atmosphere versus the frequency before scattering, $v_1$, relative to line center, in units of the Doppler width with the frequency after scattering, $v_2$, as a parameter.

We are indebted to Dr. R. G. Giovanelli for drawing attention to this possibility.
we approximate it by a sum of two components—one coherent and one completely redistributed—by writing

\[ g(v_1; v_2) = a(v_2) \delta(v_1 - v_2) + [1 - a(v_2)] \phi(v_1), \]

where \( \delta \) is the Dirac delta function. To determine the \( a(v_2) \), we note that the fundamental difference in these two limiting characterizations lies in the fact that for complete redistribution the scattered photons have a distribution centered about the central line frequency, whereas the coherently scattered photons have a distribution about the incident frequency. We therefore divide each curve for a given \( v_2 \) in Figure 1 into two components: one symmetric about \( v = 0 \) and one centered about \( v_1 = v_2 \). We then adopt the areas of these two curves as giving, respectively, the relative proportions of complete redistribution and of coherent scattering. Figure 2 shows the variation of \( a(v_2) \) thus obtained.

In contradiction to these results, Woolley and Stibbs (1953) have concluded that the Doppler effect will not modify the coherency. Their argument leads to the following expression for the line emission at frequency \( \nu \) from a unit volume scattering isotropically and coherently in the reference frames of the atoms,

\[ i_\nu = a_D(\nu) I_\nu - a_D(\nu_0) \frac{a\nu V_0}{c} \frac{dI_\nu}{d\nu} \int_{-\infty}^{+\infty} \frac{x e^{-x^2} dx}{a^2 + (\nu - x)^2}, \]

where \( a_D(\nu) \) is the Doppler-broadened absorption coefficient; \( V_0 \) is the mean thermal velocity; \( \nu = \Delta \nu/\Delta \nu_0 \), and where terms in second and higher derivatives are neglected. In fact, there are errors in this formula, but we neglect those here. Woolley and Stibbs state that the second term in equation (6) is negligible, so that

\[ j_\nu = a_D(\nu) I_\nu, \]

which is precisely the condition for coherent scattering. In fact, however, the integral on the right has the value \( (2\pi/a)e^{-\nu^2} \) in the line core, so that equation (6) becomes

\[ i_\nu = a_D(\nu) I_\nu - a_D(\nu - \nu_0) \frac{dI_\nu}{d\nu}, \]
or, to the accuracy of the two terms of the Taylor series used to characterize \( I_r \),

\[
j_r = a_D (v) \, I_{r*},
\]

or the source function is frequency-independent, a form of non-coherent scattering closely akin to complete redistribution.

III. THE INFLUENCE ON LINE PROFILES

a) Residual Non-Coherency

For radiative damping, Woolley and Stibbs (1953) show that the degree of coherency in the atomic scattering in a line formed between levels \( L \) and \( U \) is \( \gamma_r (v) \). For a resonance line, this quantity is unity, neglecting the very slight broadening under solar atmospheric conditions of the lower state due to collisions and absorption of radiation; but the degree of non-coherency will be appreciable for a subordinate line.

Zanstra (1941) concludes that, for a stationary atom in the presence of collisional and radiative damping, a fraction, \( \gamma_c / (\gamma_c + \gamma_n) \), of the absorbed radiation will be emitted non-coherently, while the remainder will be emitted coherently, where \( \gamma_c \) and \( \gamma_n \), respectively, are the half-widths of the upper state for collisional and radiative broadening. This conclusion has been questioned by Edmonds (1955), who argues that Zanstra's results give too much coherency by a factor of about 20 when statistical broadening is significant. He suggests that random fluctuations in the perturbing force will occur at a much more rapid rate than \( 1/\gamma_c \), which is the characteristic time of the perturbations required to obtain Zanstra's result. The value of \( \gamma_c \) depends on the type of interaction.

For discrete encounters with van der Waals interaction and quadratic Stark effect due to ions, we find, respectively, \( \gamma_c \approx 10^{-9} \, N_H \) and \( \gamma_c \approx 5 \times 10^{-6} \, N_+ \). These values are to be compared with \( \gamma_n \approx 10^8 \) for a resonance line. Thomas and Athay's (1960) model of the low chromosphere gives \( N_H \approx 10^{14} \) at \( \tau_e \approx 3 \times 10^{-6} \), while Pagel (1956) gives a similar result. For less than 1 per cent collisionally introduced non-coherency in the region of formation of the line at three Doppler widths from the line center, we thus require \( a_0 / a_c \approx 3 \times 10^8 \), even adopting Zanstra's result. This corresponds to a very strong line like \( H \) or \( K \) of Ca II. For Lyman-\( \alpha \) of hydrogen linear Stark broadening by ions gives \( \gamma_c \approx 10^{-4} \, N_+ \), which seems too strong to allow a high proportion of coherency in the frame of the atom. The helium resonance line seems better suited for observing any coherent scattering in the line wings.

While redistribution occurs through radiative or collisional transitions upward from the excited state with subsequent return, it does not seem reasonable that these interlocking processes should be important compared to transitions within the state itself. Transitions within the state are effectively the collisional transitions considered above.

b) The Solution of the Transfer Equation

The Eddington approximation, applied to equation (1) with continuum emission, gives (as in Paper II)

\[
\frac{d^2 I_v}{d \tau_0^2} = \frac{\pi}{\tau_0} \left( I_v - \frac{S_L + \tau_r S_C}{1 + \tau_r} \right),
\]

where \( S_L \) and \( S_C \) are the source functions in the line and continuum, the latter being assumed Planckian; \( \tau_r \) is the ratio of line to continuum absorption coefficient; \( \tau_0 \) is the optical depth in the line center; and

\[
x_v = \phi_v (1 + \tau_r) \sqrt{3},
\]

where \( \phi_v \) is the ratio of the absorption coefficient at \( v \) to that at line center. We shall restrict attention to collisionally controlled lines for which—as in Paper I—

\[
(1 + \epsilon) \, S_L (v) = \int I_v' \, g' (v'; v) \, dv' + \epsilon B_v (T_e),
\]
where
\[ g'(v'; \nu) = b_v g(v'; \nu) + (1 - b_v) \phi_v', \]

in which \( b_v \) represents the proportion of scattering which is coherent in the atom's frame of reference. To the approximation of \( g(v'; \nu) \) by a sum of two components in the observer's frame of reference, equation (12) becomes
\[ g'(v'; \nu) = \beta_v \delta(v' - \nu) + (1 - \beta_v) \phi_v', \]

where \( \beta_v = b \alpha_v \). The transfer equation then becomes
\[ \frac{d^2 I_v}{d \tau_0^2} = x_v^2 [ I_v - \beta_v \omega_v I_v - (1 - \beta_v) \omega_v \int I_{v'} \phi_v' dv' - \lambda_v B_v (T_e) ], \]

where
\[ \omega_v = \frac{1}{(1 + \epsilon)(1 + r_v)} \]
and
\[ \lambda_v = 1 - \omega_v. \]

Equation (14) has been solved as before by replacing the integral by a three-point Gaussian quadrature,
\[ \int I_{v'} \phi_v' dv' = \sum_i a_i I_{i}, \]

where the \( a_i \)'s are the weights associated with the division points \( v_i \). The solutions are applied to determining \( S_L(\nu) \) and hence the line profile from the equation
\[ I_v (0; \mu) = \int_0^\infty (S_L + r_v S_C) e^{-\tau_v (1 + r_v)/\mu} \frac{d \tau_v}{\mu}, \]

with \( \tau_v \) the line optical depth at frequency \( \nu \). In the particular case where
\[ B_v (T_e) = S_1, \]
we find, with \( \gamma = \epsilon/(1 + \epsilon) \),
\[ S_L (\nu) = S_1 \{ 1 + (1 - \gamma) \beta_v \omega_v \exp - [x_v \sqrt{(1 - \beta_v \omega_v)} \tau_0] \]
\[ + (1 - \gamma) (1 - \beta_v) \sum \alpha \frac{(1 - k_v^2/x_v^2) L_a e^{-k_a \tau}}{1 - \beta \omega - k_v^2/x_v^2} \}, \]

with
\[ 1 = \sum_i \frac{a_i (1 - \beta_i) \omega_i}{1 - \beta_i \omega_i - k_v^2/x_v^2} \]
as the determining equation for the \( k_v \). The \( L_a \) and \( \omega_v \) are found from
\[ 1 + \sum \alpha \frac{(1 - \beta_i) \omega_i L_a (1 + k_v/x_i)}{1 - \beta_i \omega_i - k_v^2/x_i} = 0 \]
and
\[ \omega [1 + \sqrt{(1 - \beta_v \omega_v)}] + 1 + (1 - \beta_v) \omega_v \sum \alpha \frac{L_a (1 + k_v/x_v)}{1 - \beta \omega_v - k_v^2/x_v} = 0, \]
which express the Eddington boundary condition at \( \tau_0 = 0 \),

\[
\tilde{I}_v = \frac{1}{\tau_0} \frac{dI_v}{dt}.
\]

Emergent profiles computed from equations (15) and (17) have been obtained for a variety of cases. Figure 3 shows the results for an isothermal atmosphere. To allow for the possibility that collisional damping may be important, we have limited the maximum value of \( \beta_v \) to the values indicated. This also shows the sensitivity of the line profiles to the introduction of a degree of non-coherency. We are interested only in the form of the results, which show that, unless there is appreciable collision damping, a line maximum or point of inflection can occur. Since there are very general reasons why the coherent profile lies below the non-coherent one, this conclusion is not a product of the specific numerical values used. For solar lines formed against a background continuum—in the visible, say—collision damping, i.e., non-coherent scattering, will become important somewhere in the wings, so that in practice the profile ultimately merges with the pure non-coherent case.

**IV. DISCUSSION**

These results may be compared with those obtained for a chromospheric temperature rise and complete redistribution obtained in earlier papers of this series. The same emission characteristic is obtained with the position of the maxima depending on the depth of the position of temperature rise. We then have two possible mechanisms for explaining...
the appearance of the H and K lines and the "bell"-shaped lines. There are points in favor of both mechanisms.

First, the solar Ca lines are observed (Smith 1960) to show emission peaks at various separations from zero up over sunspots and plages. These are readily explained in terms of the temperature gradient and complete redistribution, but they seem incapable of explanation as a consequence of a coherent to non-coherent transition which gives peaks at about 3\(\Delta T\). There is no evidence of a second set of emission peaks. This would seem to indicate non-coherency over the whole core; consequently, at least for Ca, we have underestimated the role of collision damping.

On the other hand the "bell"-shaped profiles are found in the neutral metal lines—and only in resonance lines—which, as shown in Paper I, are controlled by photoelectric processes in continua that could not be influenced by the chromospheric temperature rise. Furthermore, these profiles are only very slightly affected, if at all, by plages and sunspots (Smith 1960). Since the Ca lines are presumably stronger, the role of collision broadening should be less for them. It is, however, a little difficult to see how the high opacity necessary for the wings to be formed free of collision damping, i.e., in the low chromosphere, can be obtained in neutral atoms like Fe, Mg, and Ca, since such atoms should be predominantly ionized in this region.

By assuming non-coherency in the core and coherency in the wings, Miyamoto (1953) has obtained theoretical profiles resembling the H and K profiles and the "bell"-shaped profiles in strong resonance lines of Ca and the \(b\) group. He also shows that the center-to-limb variations in these lines do not resolve the problem as to their mode of formation.

In summary, we may say that coherent scattering over the whole line is demonstrably a wrong assumption for any spectral line formed under solar atmospheric conditions, and this is particularly true in the line core. Whether or not some lines will show some wing coherency can be decided when an adequate theory of the effect of collisional perturbation is available or suitable laboratory experiments made. It seems rather doubtful that the effect is large, especially if Edmond's criticism is valid, but in some lines it may be observable. A crucial test may be possible on the neutral helium resonance lines when spectroscopic data are available. In the meantime, lacking adequate theoretical or laboratory guidance, it seems to us desirable to explore, first, the consequences of complete redistribution over the entire line: inadequacies in this approach have at least still to be found.

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