LIMITATIONS ON PHYSICAL THEORIES OF H AND K EMISSION LINES

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ABSTRACT

A satisfactory physical theory of the H and K emission lines in the sun and stars has not yet been developed. By consideration of existing observations, certain requirements of and limitations upon such theories are established. If \( \tau_0 \) is the optical thickness in the line center, \( N \) the number of \( \text{Ca} \ II \) ions per square centimeter vertical column, \( L \) the luminosity (visual), and \( \Delta \lambda_D \) the Doppler width, the following items must be accounted for:

A. Emission produced in optically thin layers (\( \tau_0 > 1 \)). (1) \( \Delta \lambda_D \propto L^{1/6} \). (2) \( N \) unrestricted except for upper limit on \( \tau_0 \). (3) Mechanism for adequate emission from thin layers.

B. Emission produced in optically thick layers (\( 10 < \tau_0 < 10^4 \)). (1) Both \( \Delta \lambda_D \) and \( N \) must vary appropriately with \( L \). (2) The rate of variation of \( \Delta \lambda_D \) with \( L \) must lie within the maximum and minimum rates, which, together with the corresponding variations of \( N \), are, respectively, \( \Delta \lambda_D \propto L^{1/3} \) and \( \Delta \lambda_D \propto L^{1/2} \).

The recent discovery (Wilson 1954; Wilson and Bappu 1957) of a linear relationship between the logarithms of the widths of the H and K emission lines in late-type stars and the absolute visual magnitudes raises some interesting questions. Apart from the obvious desirability of understanding why the correlation exists, a theoretical derivation of it would be of practical importance. At present the use of the measured line widths as a tool for the determination of reliable absolute luminosities rests entirely upon an empirical calibration, and it is not clear how this calibration can be greatly improved in the immediate future. A theoretical derivation would, if it could be obtained, place the method on a sound basis and free it of empirical uncertainties.

The first step toward a solution must be to determine by what mechanism the lines are widened. Following this, the second and much more difficult step will be to relate the mechanism in the proper fashion with the luminosity. At present there is uncertainty as to the widening mechanism itself. The writer has tended to favor pure Doppler widening in optically thin layers, for reasons set forth in a previous paper (Wilson and Bappu 1957). It must be conceded that there is a serious drawback to this viewpoint, namely, whether enough emission could be obtained from thin layers to produce observable emission. Presumably this could be done only by making the pseudo-Planck emission function rather large, which, in turn, might require unusual physical conditions.

Goldberg (1957) has pointed out that the assumption of optically thick emitting layers reduces considerably the turbulence required to yield the observed line widths. This assumption will also remove much of the difficulty with the emission function encountered with thin layers and has much to recommend it. Eventually, observation should be able to distinguish between optically thick or thin layers. In the meantime, however, it is possible to make use of the existing observations to show what must be accounted for by a successful physical theory based on either of the widening mechanisms.

A. OPTICALLY THIN LAYERS

In optically thin layers the measured line width, \( \Delta \lambda \), is virtually equal to the Doppler width, \( \Delta \lambda_D \). Hence the observed relationship, \( \Delta \lambda \propto L^{1/6} \), means that the turbulent velocities in the emitting layers must vary with luminosity in the same manner: \( \Delta \lambda_D \propto L^{1/6} \).
There is no restriction on $N$ (number of ions in vertical cm$^2$ column) other than that $N$ must not be so large that $\tau_0$, the opacity in the line center, exceeds approximately the value 1.

## B. OPTICALLY THICK LAYERS

In the case of optically thick layers the equation for the half-width at half-intensity is (Goldberg 1957)

$$\Delta \lambda = 1.52 \Delta \lambda_D [\log (1.44 \, \tau_0)]^{1/2},$$

and, for the K line,

$$\tau_0 = 7.7 \times 10^{-14} \frac{N}{\Delta \lambda_D}.$$

Thus $N$ and $\Delta \lambda_D$ are both involved in the expression for $\Delta \lambda$ and cannot be allowed to vary independently.

The observations indicate that the intrinsic scatter in the line widths for stars of the same luminosity is probably small. For instance, for the Hyades stars it is less than $\pm 10$ per cent. Whether this is true in general cannot be said as yet, though it is a not unreasonable estimate. It is clear from equations (1) and (2) that the principal dependence of $\Delta \lambda$ on $\Delta \lambda_D$ is through the factor outside the brackets. Small changes in $\Delta \lambda_D$ will have an inappreciable effect on the other factor, and we may therefore safely assume, without making any significant error, that $\Delta \lambda_D$ is also constant to within $\pm 10$ per cent for stars of the same luminosity.

To find the range in $N$ which must not be exceeded, we may differentiate equation (1), treating $\Delta \lambda_D$ as a constant. We find

$$\frac{d (\Delta \lambda)}{\Delta \lambda} = \frac{1}{2} \left( \frac{\Delta \lambda_D}{\Delta \lambda} \right)^2 \frac{d \tau_0}{\tau_0}.$$  

For a 10 per cent change in $\Delta \lambda$, $d(\Delta \lambda)/\Delta \lambda = 0.1$,

$$\frac{d \tau_0}{\tau_0} = \frac{1}{5} \left( \frac{\Delta \lambda}{\Delta \lambda_D} \right)^2,$$

or $d\tau_0/\tau_0 \approx 1$ for $\tau_0$ in the range $10^{-10}$ to $10^4$. In other words, $N$ must be constant within a factor of about 4 in order that the intrinsic scatter of line widths for stars of the same luminosity may not exceed $\pm 10$ per cent. This means that the assumption of optically thick layers requires both $\Delta \lambda_D$ and $N$ to vary with the luminosity in a specified manner.

We can go one step further in setting forth the requirements of a physical theory by making use of the fact that $\Delta \lambda \propto L^{1/6}$ according to the observations. More precisely, a main-sequence star with $\Delta \lambda$ (half-width) of 10 km/sec and a supergiant with $\Delta \lambda = 100$ km/sec differ in brightness by 15.6 mag., corresponding to a luminosity ratio of $10^6$. Let us suppose that the range of optically thick layers, in the sense we are using the term here, is between $\tau_0 \approx 10$ and $\tau_0 \approx 10^4$. Above $\tau_0 = 10^4$, damping wings appear quickly, and there is nothing in the observations to suggest their presence. Then we can find the maximum and minimum ranges of $\Delta \lambda_D$ and the corresponding variations in $N$, between the main sequence and the supergiants, in the following manner.

From equation (1) we see that the minimum range of $\Delta \lambda_D$ will occur if the term in brackets changes in the same sense as $\Delta \lambda_D$ does. This requires the main-sequence star to be at the lower extreme of the optically thick zone and the supergiant to be at the upper extreme. For simplicity, we take $1.44 \tau_0$ to be equal to 10 for the main-sequence star and equal to $10^4$ for the supergiant, and $N$ therefore increases with luminosity. Then, using equation (1), we find

$$\Delta \lambda_D \text{ (m.s.)} = 6.6 \text{ km/sec}, \quad \Delta \lambda_D \text{ (sg.)} = 32.9 \text{ km/sec}.$$

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The maximum range of $\Delta \lambda_D$ will be found by assuming the main-sequence star to be at the upper extreme of the optically thick zone and the supergiant at the lower; $N$ therefore decreases with luminosity, and we find

$$\Delta \lambda_D \text{ (m.s.)} = 3.3 \text{ km/sec,} \quad \Delta \lambda_D \text{ (sg.)} = 65.9 \text{ km/sec.}$$

We may now calculate the corresponding values of $N$ by means of equation (2).

**Case 1. $N$ Increasing with Luminosity**

$$\Delta \lambda_D \text{ (m.s.)} = 6.6 \text{ km/sec} = 0.0865 \text{ Å,} \quad 1.44 \tau_0 = 10, \quad N = 7.8 \times 10^{12};$$

$$\Delta \lambda_D \text{ (sg.)} = 32.9 \text{ km/sec} = 0.431 \text{ Å,} \quad 1.44 \tau_0 = 10^4, \quad N = 3.9 \times 10^{16}.$$

**Case 2. $N$ Decreasing with Luminosity**

$$\Delta \lambda_D \text{ (m.s.)} = 3.3 \text{ km/sec} = 0.0432 \text{ Å,} \quad 1.44 \tau_0 = 10^4, \quad N = 3.9 \times 10^{15};$$

$$\Delta \lambda_D \text{ (sg.)} = 65.9 \text{ km/sec} = 0.862 \text{ Å,} \quad 1.44 \tau_0 = 10, \quad N = 7.8 \times 10^{13}.$$

It is now possible to determine how $\Delta \lambda_D$ and $N$ each must vary with the luminosity. Thus, for

$$\frac{L \text{ (sg.)}}{L \text{ (m.s.)}} = 10^6 24;$$

we have, for case 1,

$$\frac{\Delta \lambda_D \text{ (sg.)}}{\Delta \lambda_D \text{ (m.s.)}} = 5.0; \quad \therefore \Delta \lambda_D \propto L^{1/6};$$

$$\frac{N \text{ (sg.)}}{N \text{ (m.s.)}} = 5 \times 10^8; \quad \therefore N \propto L^{1/7};$$

and, for case 2,

$$\frac{\Delta \lambda_D \text{ (sg.)}}{\Delta \lambda_D \text{ (m.s.)}} = 20; \quad \therefore \Delta \lambda_D \propto L^{1/4};$$

$$\frac{N \text{ (sg.)}}{N \text{ (m.s.)}} = 2 \times 10^{-2}; \quad \therefore N \propto L^{-1/3}.$$

If the decimal fractions in the exponents of the foregoing expressions are replaced by the nearest whole numbers, the resulting variations can be shown by computation to satisfy the observed relationship $\Delta \lambda \propto L^{1/6}$ with adequate accuracy.

Therefore, while we are no closer to a physical theory of the phenomenon than before, we can now specify fairly closely the requirements which such a theory must meet. These requirements may be summarized thus:

**A. OPTICALLY THIN LAYERS**

The theory must account for the variation of $\Delta \lambda_D$ with luminosity according to the approximate relationship $\Delta \lambda_D \propto L^{1/6}$ and must also explain the large magnitude of the turbulence in the supergiants; $N$ is unrestricted, except that it must not be so large that $\tau_0$ rises much above 1. The theory must give an explanation for this upper limit of $N$, and, finally, it must enable optically thin layers to produce observable emission.
B. OPTICALLY THICK LAYERS

A theory based on optically thick layers must provide for a variation in both $\Delta \lambda_D$ and $N$ with luminosity in the proper manner. The foregoing discussion has enabled us to set the approximate limits of this variation, namely,

$$\Delta \lambda_D \propto L^{1/9}, \quad N \propto L^{1/2} \text{ for minimum range of } \Delta \lambda_D,$$

and

$$\Delta \lambda_D \propto L^{1/8}, \quad N \propto L^{-1/4} \text{ for maximum range}.$$

A theory giving any rate of change of $\Delta \lambda_D$ with $L$ between these extremes would be formally satisfactory, provided that $N$ is also required to vary at the appropriate corresponding rate. If, for example, it should be theoretically necessary that $N/\Delta \lambda_D$ be always constant, then the same treatment must require that $\Delta \lambda_D \propto L^{1/6}$.

REFERENCES