RELATIVE [O II] INTENSITIES IN GASEOUS NEBULAE

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ABSTRACT

The intensity ratio \( r = I(3729)/I(3726) \) tends to a value \( r(\infty) \) at high densities determined by radiative transition probabilities and a value \( r(0) \) at low densities determined by collision strengths, proportional to collision cross-sections. This ratio has been measured in nebulae for which these two limiting values should be approached. The transition probability calculations are discussed; the best-calculated value of \( r(\infty) \) obtained is 0.43. From observations of IC 4997 it is concluded that the correct value is 0.35 ± 0.04. For future applications, values of the transition probabilities are adopted consistent with \( r(\infty) = 0.35 \). Results of improved collision-strength calculations are presented, including those for transitions between all individual \( \ell \)-levels. The quantities required in these calculations are used to calculate quantum defects in various \( \text{O I} \) spectral series. Comparison with observed quantum defects shows that the collision strengths should be correct to within 30-40 per cent. The calculations give \( r(0) \) to be 1.50 at low temperatures and 1.42 at high temperatures. Electron-density estimates of 18 cm\(^{-3}\) and 10 cm\(^{-3}\) are obtained from the surface brightnesses of two nebulae, NGC 281 and NGC 7000, for which the low-density limit should be approached. Assuming \( T_e = 10^4 \) K, these densities would be consistent with \( r = 1.47-1.48 \). The observed ratios are 1.37 and 1.48, respectively, suggesting densities of the order of 100 cm\(^{-3}\). The discrepancy is probably real and due to an inhomogeneous density distribution, together with selection effects. The ratio \( r' = I(7320)/I(7330) \), calculated to be between 1.24 and 1.31 for all values of \( T_e \) and \( N_e \), is in satisfactory agreement with available measurements. Expressions are obtained for the intensity ratios \( r \) and \( r'' = I(\lambda 729) + I(\lambda 726)/I(\lambda 7320) - j I(\lambda 7330) \) as functions of \( T_e \) and \( N_e \).

For IC 418 the measured value of \( r \), 0.37 ± 0.03, is consistent with the density \( N_e = 2.5 \times 10^4 \) cm\(^{-3}\) obtained from the surface brightness, interpreted assuming the hydrogen emission to be concentrated in a hollow spherical shell. For NGC 7027, \( r = 0.47 \), measured by Aller and Minkowski, gives \( N_e = 0.85 \times 10^4 \) cm\(^{-3}\), which is considerably smaller than values previously obtained from other forbidden-line ratios. The latter values are confirmed by using \( r'' = 1.38 \), measured by Aller, Bowen, and Minkowski, which gives \( N_e = 3 \times 10^4 \) cm\(^{-3}\). It is considered that the discrepancy, which is too large to be attributed to errors in observations or in atomic parameters, is due to the occurrence of local density fluctuations. The measured values of \( r \) and \( r'' \) are consistent with a model having a background density of the order of \( 5 \times 10^3 \) cm\(^{-3}\) and in which dense clouds or filaments, with densities of the order of \( 7 \times 10^4 \) cm\(^{-3}\), occupy 1 or 2 per cent of the total volume. There is similar evidence for density fluctuations in a number of other bright planetary nebulae. In many cases direct photographs reveal the presence of filamentary structure. It is pointed out that for IC 418 there is evidence of large-scale density and possibly temperature variations, but the spectrophotometric evidence does not suggest the existence of local density fluctuations. This is consistent with the particularly uniform appearance of this object.

I. INTRODUCTION

Bowen (1928) identified the "nebulium" lines in gaseous nebulae with forbidden transitions in various atoms and ions and suggested the mechanism of excitation by electron impact. Later work has confirmed the correctness of this mechanism and, from quantitative comparisons of theory and observation for the relative line intensities, has enabled valuable information to be obtained about the physical conditions in gaseous nebulae.

In the limit of high electron density, detailed balance between collisional excitation and deactivation leads to a Boltzmann distribution among the various ionic levels. For a given ion the relative line intensities are then determined by the radiative transition
probabilities and the kinetic temperature, $T_e$, but do not depend on the electron density, $N_e$. In the limit of low electron density, on the other hand, collisional deactivation is negligible; the population of a given level is then determined by the equilibrium between the rate of radiative transitions from the level and the rate of collisional excitations from the ground state to the level (including excitations to higher excited states which populate the level considered by cascade transitions). At low densities the intensity ratios are again independent of $N_e$ but may be very different from the ratios in the high-density limit. For intermediate densities estimates of both $T_e$ and $N_e$ may be obtained from observations of two or more intensity ratios. This method has been used by Seaton (1954a; see also Aller and White 1949) and the results obtained found to be in fair accord with those obtained from the absolute surface brightness (Menzel and Aller 1941; Liller and Aller 1954; Seaton 1954b) and from the Balmer discontinuity (Seaton 1954c, 1955c).

The present paper is concerned with the [O II] lines listed in Table 1 (Bowen 1955).

### TABLE 1

**WAVE LENGTHS OF [O II] LINES**

<table>
<thead>
<tr>
<th>Transition</th>
<th>$\lambda$</th>
<th>Transition</th>
<th>$\lambda$</th>
<th>Transition</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^2D_{5/2} \rightarrow ^4S_{3/2}$</td>
<td>3728 80</td>
<td>$^2P_{3/2} \rightarrow ^2D_{5/2}$</td>
<td>7319 92</td>
<td>$^2P_{3/2} \rightarrow ^4D_{5/2}$</td>
<td>7330 19</td>
</tr>
<tr>
<td>$^2D_{3/2} \rightarrow ^4S_{3/2}$</td>
<td>3726 05</td>
<td>$^2P_{1/2} \rightarrow ^4D_{5/2}$</td>
<td></td>
<td>$^2P_{1/2} \rightarrow ^4D_{5/2}$</td>
<td></td>
</tr>
</tbody>
</table>

The ratio $I(3729)/I(3726)$ will be denoted by $r$. The high-density limit is

$$r(\infty) = \frac{3}{2} \frac{A(\overline{2D_{5/2} \rightarrow 4S_{3/2}})}{A(\overline{2D_{3/2} \rightarrow 4S_{3/2}})},$$

(1)

where $\frac{3}{2}$ is the statistical weight ratio and each $A$ is a transition probability between the levels indicated. Neglecting a small cascade contribution from $^2P$, the low-density limit is

$$r(0) = \frac{\Omega(\overline{2D_{5/2}, 4S_{3/2}})}{\Omega(\overline{2D_{3/2}, 4S_{3/2}})},$$

(2)

each $\Omega$ being a collision strength, proportional to the collision cross-section (Hebb and Menzel 1940; Seaton 1953, 1955a).

Calculations of Pasternack (1940), allowing for second-order terms in the spin-orbit interactions, gave $r(\infty) = 1.9$; similar calculations of Aller and Menzel (1945) gave $r(\infty) = 1.64$. Aller and Menzel drew attention to the discrepancy between these results and the observed values ($\sim 0.5$) for several bright planetaries in which they would have expected the high-density limit to be approached. Considerable improvement in the theory of such calculations was made by Aller, Ufford, and Van Vleck (1949), who included first-order spin-spin and spin-other-orbit interactions; they obtained $r(\infty) = 0.58$. Similar calculations have also been made by Naqvi (1951). Garstang (1952) later noted that non-exchange wave functions had been used to calculate the quadrupole integral $s_q$, and, on substituting exchange wave functions, obtained $r(\infty) = 0.47$, in close agreement with the observed values for most planetary nebulae.

The effect of departures from the high-density limit was estimated by Seaton (1954d), using a calculated value of

$$\Omega(\overline{2D, 4S}) = \Omega(\overline{2D_{5/2}, 4S_{3/2}}) + \Omega(\overline{3D_{3/2}, 4S_{3/2}})$$

(3)
and assuming each individual \( \Omega \) to be proportional to the product of the initial and final statistical weights. This gave \( r(0) = 1.5 \) for the low-density limit. Unusually large values of \( r \) for one or two planetaries were found to be in good accord with theory when approximate density estimates obtained from the surface brightness were adopted. With the best available density estimates for the brighter planetaries, the theory indicated that the high-density limit should be approached in these objects, in apparent agreement with observation. Confirmation of the predicted trend of the variation of \( r \) with \( N_e \) was obtained by Osterbrock (1955), who found that in the Orion Nebula \( r \) increased as the surface brightness decreased.

One disquieting observation was that Aller et al. (1949) had obtained two plates for IC 4997 giving \( r = 0.36 \) and 0.40, values inconsistent with the theoretical result, \( 0.47 \leq r \leq 1.50 \). Although this is a "stellar" planetary for which the observations were difficult and the results noted as uncertain, a check appeared particularly desirable in view of the fact that Menzel, Aller, and Hebb (1941) had found the \([\text{O iii}]\) ratio \( I(4959) + I(5007)/I(4363) \) to be abnormal in this object and suggested that this was due to an unusually high electron density.

### Table 2

<table>
<thead>
<tr>
<th>Object</th>
<th>Intensity Ratio ( r )</th>
<th>( n )</th>
<th>A D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC 4997</td>
<td>0.34</td>
<td>3</td>
<td>0.02</td>
</tr>
<tr>
<td>MHa 78(1)</td>
<td>0.44</td>
<td>3</td>
<td>0.02</td>
</tr>
<tr>
<td>MW 319</td>
<td>0.46</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Cannon 5</td>
<td>0.54</td>
<td>3</td>
<td>0.03</td>
</tr>
<tr>
<td>IC 418</td>
<td>0.37</td>
<td>7</td>
<td>0.03</td>
</tr>
<tr>
<td>IC 4593</td>
<td>0.65</td>
<td>3</td>
<td>0.01</td>
</tr>
<tr>
<td>NGC 281*</td>
<td>1.37</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>NGC 7000†</td>
<td>1.38</td>
<td>2</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Further measurements of the \( 3729/3726 \) ratio made by Osterbrock for both low-density and high-density objects are reported in Section II. The results obtained show that the high-density limit is close to \( r(\infty) = 0.35 \), and it therefore appears that further improvements are still required in the transition probability calculations; this is discussed in Section III. The results of improved collision-strength calculations made by Seaton are reported in Section IV; and the interpretation of the observed intensity ratios for various nebulae is discussed in Section V.

### II. Observations

The observations described in this section were made in an effort to measure the intensity ratio \( r = I(3729)/I(3726) \) in objects having values of this ratio as close as possible to the high- and low-density limits. In addition, the planetary IC 418 was included as an interesting object for which accurate measurements of \( r \) have not previously been made, and IC 4593 was observed in order to compare with a previous measurement of Aller et al. (1949).

The first four objects in the first column of Table 2 are the nebulae which were observed in an attempt to measure the high-density limit. In addition to IC 4997, men-
tioned in the introduction, three other objects were selected for their small apparent size, as it may be expected that there will be a statistical tendency for the smallest objects to be the densest. "Cannon 5" refers to the fifth object in the list of peculiar spectra published by Cannon (1926), while "MHa 78(1)" is the number of the planetary discovered by Humason (1922). A direct plate taken with the 200-inch Hale telescope shows Cannon 5 to be an elliptically shaped planetary with major and minor axes approximately 4.5 and 3", respectively. It has a bright outer ring and looks more or less similar to, although much smaller than, IC 418. On the other hand, IC 4997 appears stellar on a 200-inch plate taken in fairly good seeing. The other two semistellar objects in Table 2 have not been photographed with the Hale telescope, but Humason (1922) has given the diameter of MHa 78(1) as approximately 3", while Merrill (1941) has described MW 319 as "minute."

The observations were made with the equipment previously described in the study of the variation of the \( \lambda 3727 \) intensity ratio \( r \) across the Orion Nebula (Osterbrock 1955). However, as all these nebulae are considerably smaller than the slit length of the spectrograph, 3 mm or 48" on the sky, the observations were made by trailing them on the slit. In the case of IC 418 the bright outer ring was trailed on the slit, while in the other cases it was the center of the nebula which was trailed. The calibration plate was in each case exposed for a time not differing by a factor greater than 3 from the effective exposure time received by any point on the corresponding nebular plate.

The measured intensity ratios are tabulated in Table 2, in which the second column gives the mean measured ratio \( r \), the third column gives the number of plates measured, and the fourth column gives the average deviation, without regard to sign, of a single plate from the mean. It is seen that the value of \( r \) for IC 4997 is somewhat smaller than the mean value given by Aller et al. (1949) and that IC 418 also has a low value of this ratio, in good agreement with the visual estimate of Wyse (1942). MHa 78(1), MW 319, and Cannon 5 have values of \( r \) in the range covered by other bright planetaries measured by Aller et al. (1949).

It is important to inquire into the possible errors in the observed value of the ratio. Besides the accidental error, there is the possibility of a systematic error in the photometric calibration; the two lines are so close together in wave length that other systematic errors due to differences in plate sensitivity, spectrograph performance, atmospheric transmission, etc., cannot be important. It is difficult to estimate this error, but an upper limit can be based on the agreement between plates reduced using different parts of the calibration-curve. In this way it is estimated that the measured intensity ratios for the planetaries listed in Table 2 are correct to within \( \pm 0.03 \). The value \( r = 0.65 \) measured for IC 4593 is in good agreement with the value \( r = 0.67 \) obtained by Aller et al. (1949) from one plate; this suggests that serious systematic errors have not crept into either set of observations.

The objects selected to be observed for the low-density limit were amorphous regions in the diffuse nebulae NGC 281 and NGC 7000. It is, of course, easy to find objects of lower density than these by picking faint but unreddened nebulae; the limitation, however, is not in finding but in observing these faint H \( \pi \) regions. The exposure times for the plates of NGC 281 and NGC 7000 average 4 hours each, and much longer exposures are not practicable. These plates were taken with the same equipment as the planetary plates, except that the slit length of the spectrograph was 8 mm or 128" on the sky, and the objects were held stationary on the slit rather than being trailed. Three independent tracings, of the top, middle, and bottom of the slit image, were made for each plate and reduced separately, using the common calibration for the plate. The three resulting intensity ratios were averaged to get the mean intensity ratio for the plate, which is treated as a single observation in Table 2. For these three plates the average deviation of a single intensity ratio from the mean for the plate was between 0.03 and 0.05.
III. O II Transition Probabilities

a) $^3D-^4S$ Transitions

From the work of Aller et al. (1949) we obtain, in the usual notation, the following expressions for the line strengths:

$$S_p (^2D_{5/2}, ^4S_{3/2}) = \frac{14 \xi^2}{(PS)^2} s_q^2,$$

$$S_q (^2D_{5/2}, ^4S_{3/2}) = \frac{6 \xi^2}{(PS)^2} \left[ 1 + \frac{5 \xi^2}{4 (PD) (DS)} + \frac{3 \eta}{(PD) (DS)} \right]^2 s_q^2,$$

$$S_m (^2D_{5/2}, ^4S_{3/2}) = \frac{17.28}{(DS)^2} \left[ \eta + \frac{5 \xi^2}{12 (PS)^2} \right]^2,$$

$$S_m (^2D_{5/2}, ^4S_{3/2}) = \frac{155.52}{(DS)^2} \left[ \eta + \frac{5 \xi^2}{108 (PD) (PS)} \right]^2.$$

The transition probabilities are given by $A = A_q + A_m$ with

$$A_q (i \rightarrow j) = \frac{2674}{\omega_i} \left( \frac{v}{R} \right)^6 S_q \sec^{-1},$$

$$A_m (i \rightarrow j) = \frac{35652}{\omega_i} \left( \frac{v}{R} \right)^4 S_m \sec^{-1},$$

where $v$ is the wave number of the emitted radiation in cm$^{-1}$ and $R$ is the Rydberg wave number (109737 cm$^{-1}$). Expressions for the doublet intervals are

$$\Delta (^2P) \equiv E (^2P_{3/2}) - E (^2P_{1/2}) = -7.5 \eta + \xi^2 \left[ \frac{1.25}{(PD)} + \frac{1}{(PS)} \right],$$

$$\Delta (^2D) \equiv E (^2D_{5/2}) - E (^2D_{3/2}) = -18.5 \eta + \xi^2 \frac{1.25}{(PD)}.$$

It should be noted that it is scarcely practicable to determine the parameters $\eta$ and $\xi$ empirically from the doublet intervals, since the latter, being very small, are likely to be appreciably perturbed by the interaction of configurations other than $2p^3$. We discuss the determination of the various quantities entering the expressions for the transition probabilities.

1. The term intervals $(PS), (PD), (DS).$—We adopt the experimental values (Moore 1949) in preference to theoretical estimates consistent with the Slater ratio $(PD)/(DS) = \frac{3}{2}$. The recent work of Garstang (1956) leads us to expect that in doing so we are making some allowance for configuration interaction. It should be noted, however, that the results obtained are not sensitive to the adopted term intervals.

2. The spin-orbit integral $\xi$.—Aller et al. (1949) adopted $\xi(O \ II) = 174$ cm$^{-1}$ interpolated by Robinson and Shortley (1937), neglecting terms in $\eta$. For hydrogenic ions $\xi$ varies as $Z^4$, and we may therefore put

$$\xi \left( \frac{Z_1 + Z_2}{2} \right) = \left\{ \frac{1}{2} \left[ \xi^{1/4} (Z_1) + \xi^{1/4} (Z_2) \right] \right\}^4,$$

which may be used for interpolation. Using empirical results for $O \ I$ and $O \ III$, calculated allowing for $\eta$, Naqvi (1951) obtained $\xi(O \ I) = 181$ cm$^{-1}$. Interpolating in the same way
from the empirical results of Garstang, we obtain \( \chi(\text{O} \text{ II}) = 181 \text{ cm}^{-1} \) from O I and O III and \( \chi(\text{O} \text{ II}) = 184 \text{ cm}^{-1} \) from N II and F II. We adopt \( \chi(\text{O} \text{ II}) = 181 \text{ cm}^{-1} \).

3. The spin-spin and spin-other-orbit integral \( \eta \).—Using non-exchange wave functions, Aller et al. (1949) obtained \( \eta(\text{O} \text{ II}) = 1.33 \text{ cm}^{-1} \), while from exchange wave functions (for \( ^4S \)) (Hartree, Hartree, and Swirles 1939) we obtain \( \eta(\text{O} \text{ II}) = 1.54 \text{ cm}^{-1} \). Table 3 gives a comparison of values of \( \eta \) determined empirically with those calculated from exchange wave functions for \( 2p^2 \) and \( 2p^4 \) ions. These are all quoted from Garstang (1951), with the exception of the calculated result for N II, which is obtained from the wave functions of Seaton (1953). For hydrogenic wave functions \( \eta \) varies as \( Z^3 \), and we may therefore put

\[
\eta \left( \frac{Z_1 + Z_2}{2} \right) = \left\{ \frac{1}{2} \left[ \eta^{1/3} (Z_1) + \eta^{1/3} (Z_2) \right] \right\}^3.
\]

Using Garstang's empirical estimates, we obtain \( \eta(\text{O} \text{ II}) = 1.55 \text{ cm}^{-1} \) from N II and F II and \( \eta(\text{O} \text{ II}) = 1.60 \text{ cm}^{-1} \) from O I and O III, in good agreement with the calculated value. We adopt \( \eta(\text{O} \text{ II}) = 1.55 \text{ cm}^{-1} \).

Interpolating between empirical estimates for O I and O III, Naqvi (1951) obtained \( \eta(\text{O} \text{ I}) = 1.765 \text{ cm}^{-1} \). The difference between this result and that obtained above is due to the difference between Naqvi's \( \eta(\text{O} \text{ I}) = 1.50 \) and Garstang's \( \eta(\text{O} \text{ I}) = 1.33 \) and to the fact that Naqvi smoothed the results for \( 2p^2 \) and \( 2p^4 \) sequences before interpolating for \( 2p^3 \). We are indebted to Dr. A. M. Naqvi for informing us of the procedure which he used.

4. The quadrupole integral \( s_q \).—From non-exchange wave functions Aller et al. (1949) obtained \( s_q = 0.70 \), while from exchange wave functions (\( ^4D \) and \( ^4S \)) Garstang (1952) obtained \( s_q = 0.596 \). In LS coupling the only allowed quadrupole transitions are for \( ^2P \rightarrow ^2D \); but when departures from LS coupling are taken into account, other quadrupole transitions occur owing to admixture of the \( ^2P \rightarrow ^2D \) transitions. It therefore appears preferable, for all transitions, to calculate the quadrupole integral from the \( ^2D \) and \( ^2P \) radial functions. This gives \( s_q = 0.616 \), which is the value we adopt.

The attempt has been made to obtain empirical estimates of \( s_q \) in \( p^2 \) configurations from a study of the \( p^2 \) nd spectral series (details of this method will be reported elsewhere). The method cannot be used for \( p^4 \) and can be used in practice for only a limited number of ions in \( p^2 \) and \( p^4 \) configurations. For N II it has been possible to obtain the empirical estimate \( s_q = 0.74 \), which does not compare unfavorably with the value 0.797 from exchange wave functions or the value 0.785 interpolated by Garstang (1951).

Results for the \( ^2D \rightarrow ^4S \) transition probabilities and for the doublet splitting are given in Table 4. The observed value for the ratio \( r(\varepsilon) \) is obtained in Section Vc(1). It is seen that the use of exchange in place of non-exchange wave functions gives improved agreement between the calculated and interpolated empirical values of \( \eta \) and between the ob-

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
        & \( \eta(\text{Calc}) \) & \( \eta(\text{Emp}) \) & \( \eta(\text{Calc}) \) & \( \eta(\text{Emp}) \) \\
\hline
\( 2p^2 \) & \( \text{C I} \) & 0 40 & 0 38 & \( \text{O I} \) & 1 15 & 1 33 \\
         & \( \text{N II} \) & 0 96 & 1 01 & \( \text{F II} \) & 1 15 & 2 26 \\
         & \( \text{O III} \) & 1 94 & 1 90 & \( \text{Ne III} \) & 3 7 & 4 3 \\
\hline
\end{tabular}
\caption{Calculated and Empirical Values of \( \eta \) (CM\(^{-1}\))}
\end{table}
served and calculated ratio \( r(\infty) \) but leads to a slightly worse agreement with observation for the doublet intervals. It would seem that further improvements in the theory could be made only by taking configuration interaction explicitly into account.\(^2\) It is fortunate that the absolute values of the transition probabilities are not sensitive to the approximations considered. For future applications we adopt values of the transition probabilities (Table 5) such that the mean, \[ \frac{3A(2D_{5/2} \rightarrow 4S_{3/2}) + 2A(2D_{3/2} \rightarrow 4S_{3/2})}{5}, \] is that given by the adopted parameters and the ratio is consistent with \( r(\infty) = 0.35 \).

### TABLE 4

<table>
<thead>
<tr>
<th>([\text{O ii}] 2D-4S) Transition Probabilities (Sec(^{-1}))</th>
<th>(A(2D_{5/2} - 4S_{3/2}))</th>
<th>(A(2D_{3/2} - 4S_{3/2}))</th>
<th>(\Delta(\text{P})) (cm(^{-1}))</th>
<th>(\Delta(\text{D})) (cm(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>0.35 (\pm) 0.04</td>
<td>0.58</td>
<td>(-1.5)</td>
<td>(-19.8) (\pm) 0.2*</td>
</tr>
<tr>
<td>Aller et al. (1949)</td>
<td>5.41 (\times) 10(^{-5})</td>
<td>14.01 (\times) 10(^{-5})</td>
<td>47</td>
<td>21.8</td>
</tr>
<tr>
<td>Garstang (1952)</td>
<td>4.08 (\times) 10(^{-5})</td>
<td>13.15 (\times) 10(^{-5})</td>
<td>64</td>
<td>21.8</td>
</tr>
<tr>
<td>Seaton and Osterbrock</td>
<td>4.88 (\times) 10(^{-6})</td>
<td>16.99 (\times) 10(^{-6})</td>
<td>43</td>
<td>25.7</td>
</tr>
</tbody>
</table>

* From Bowen (1955); the \(2\)\(P\) splitting is from Moore (1949).
† Owing to a misprint, this was given as \(-16.4\) cm\(^{-1}\) by Aller et al.

### TABLE 5*

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n')</th>
<th>(A_q(n \rightarrow n'))</th>
<th>(A_m(n \rightarrow n'))</th>
<th>(A(n \rightarrow n'))</th>
<th>(\Omega(n, n'))</th>
<th>(\pi_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2P_{1/2})</td>
<td>(2P_{3/2})</td>
<td>(3 \times 10^{-25})</td>
<td>(6 \times 10^{-11})</td>
<td>(6 \times 10^{-11})</td>
<td>0.33</td>
<td>4 (\times) 10(^{-7})</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(2D_{5/2})</td>
<td>(0.090)</td>
<td>(0.0103)</td>
<td>0.100</td>
<td>0.33</td>
<td>700</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(2D_{3/2})</td>
<td>0.061</td>
<td>0</td>
<td>0.061</td>
<td>38</td>
<td>370</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(4S_{2/2})</td>
<td>7 (\times) 10(^{-7})</td>
<td>0.0238</td>
<td>0.024</td>
<td>19</td>
<td>290</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(4D_{3/2})</td>
<td>0.045</td>
<td>0.0160</td>
<td>0.061</td>
<td>52</td>
<td>540</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(4D_{5/2})</td>
<td>0.106</td>
<td>0.0091</td>
<td>0.115</td>
<td>89</td>
<td>600</td>
</tr>
<tr>
<td>(2P_{1/2})</td>
<td>(4S_{3/2})</td>
<td>1 (\times) 10(^{-7})</td>
<td>0.0597</td>
<td>0.060</td>
<td>39</td>
<td>710</td>
</tr>
<tr>
<td>(2D_{3/2})</td>
<td>(2D_{5/2})</td>
<td>1 (\times) 10(^{-7})</td>
<td>1.3 (\times) 10(^{-7})</td>
<td>1 (\times) 10(^{-7})</td>
<td>85</td>
<td>7 (\times) 10(^{-4})</td>
</tr>
<tr>
<td>(4S_{2/2})</td>
<td>(4S_{3/2})</td>
<td>2 (\times) 10(^{-4})</td>
<td>14 (\times) 10(^{-6})</td>
<td>18 (\times) 10(^{-6})</td>
<td>51</td>
<td>1.64</td>
</tr>
<tr>
<td>(4S_{3/2})</td>
<td>(4S_{5/2})</td>
<td>4 (\times) 10(^{-6})</td>
<td>0.74 (\times) 10(^{-5})</td>
<td>2 (\times) 10(^{-5})</td>
<td>0.77</td>
<td>0.38</td>
</tr>
</tbody>
</table>

* Note that the values of \(A_q\) and \(A_m\) for \(2D \rightarrow 4S\) transitions are as calculated, while the final values of \(A\) have been adjusted to be consistent with the observed high-density limit \(r(\infty)\).

#### b) Other Transitions

To calculate the other transition probabilities in the approximation of Aller et al. (1949), we use equations (11) and (12) of Shortley, Aller, Baker, and Menzel (1941) together with equations (18) of Aller et al. supplemented by \(a'' = 1\), \(b'' = \xi/(PS)\), and \(c'' = (\sqrt{5})\xi/(2PD)\). For transitions other than \(2D \rightarrow 4S\) the difference between our results, given in Table 5, and the results of Pasternack (1940) is almost entirely due to the different values of \(s_q\) adopted.

\(^1\) It may be noted that use of \(s_q(2D, 4S) = 0.596\) in place of \(s_q(2P, 2D) = 0.616\) would have given \(r(\infty) = 0.41\), in better agreement with observation; this illustrates the sensitivity of the calculated results to the adopted parameters.

\(^2\) It may also be noted that with Naqvi's \(\eta(\text{O ii}) = 1.76\) cm\(^{-1}\) we would obtain better agreement between theory and observation for \(r(\infty)\) but worse agreement for the doublet intervals.
IV. CALCULATION OF COLLISION STRENGTHS

a) Definitions

The collision strengths \( \Omega(n, m) \) are dimensionless parameters, symmetrical in \( n \) and \( m \), related to the collision cross-sections by

\[
Q(n \rightarrow m) = \frac{\pi \Omega(n, m)}{k_n^2 \omega_n},
\]

where

\[
k_n = \frac{2\pi m v_n}{h},
\]

\( v_n \) is the initial electron velocity, and \( \omega_n \) is the statistical weight of level \( n \). For slow collisions between electrons and positive ions the collision strengths vary slowly with \( v_n \) and may be treated as constants.

We denote the excitation energies by \( E_n \) and consider two levels such that \( E_n > E_{n'} \). Assuming a Maxwell distribution of electron velocities with temperature \( T_e \), the probability per second of a positive ion in level \( n \) undergoing collisional deactivation to \( n' \) is

\[
q_{nn'} = \frac{8.63 \times 10^{-4} \Omega(n, n') \chi}{\omega_n},
\]

where \( \chi = 10^{-4} N_e/\nu_f^2 \), \( \nu_f = 10^{-4} T_e \), and \( T_e \) is in \( ^\circ \)K. The corresponding excitation probability is

\[
q_{n'n} = \frac{\omega_{n'}}{\omega_n} q_{nn'} \exp \left[ -\frac{(E_n - E_{n'})}{kT_e} \right].
\]

b) Method of Calculation

In previous work (Seaton 1953, 1955a) two approximate methods were considered: (1) the distorted wave (DW) approximation, for which the wave functions were calculated by neglecting all coupling terms and all exchange terms, and (2) the exact resonance (ER) approximation for which the wave functions were calculated by neglecting all nonspherically symmetric terms. The dominant exchange terms are included in calculating the ER functions, which should therefore be a good deal more accurate than the DW functions. The collision strengths previously adopted were obtained on estimating the coupling terms neglected in the ER approximation, using perturbation expressions involving both the ER and the DW functions.

An improved method of calculation has since been developed in which the ER results are corrected by the use of expressions, derived from a variation principle, which involve only the more accurate ER functions. Corrections are also made for other effects, such as exchange distortion by the 1s^22s^2 core, which were previously neglected. Details of these calculations will be described elsewhere.

A further improvement has been to obtain the collision strengths for transitions between the individual \( J \)-levels of the various spectral terms. To do this, use is made of the transformation amplitudes between the schemes \( SL, sl, S^2L^2 \), and \( SLJ, slj, J^T \), where \( SLJ \) refer to the ion, \( slj \) to the colliding electron, and \( S^2L^2J^2 \) to the whole system. It has been shown that, in the ER approximation, all \( \Omega(p^3 SLJ, p^3S^2L^2J^2) \) are proportional to \( (2J + 1)(2J' + 1) \); this provides some justification for previous estimates (Seaton 1954d). It is found that the ratios

\[
\frac{\Omega(^3D_{5/2}, ^4S_{3/2})}{\Omega(^3D_{3/2}, ^4S_{3/2})} = \frac{3}{2}
\]
and
\[
\frac{\Omega (^4P_{3/2}, ^4S_{3/2})}{\Omega (^2P_{1/2}, ^2S_{1/2})} = 2
\]
are obtained directly from the properties of the transformation amplitudes. A consequence of this is that there should be little error in the calculated low-density limit for the [O II] 3729/3726 ratio. However, the adopted values of the other ratios and the numerical values of the individual collision strengths depend on the results of detailed numerical calculations.

Most of the new results, given in Table 5, are in fair agreement with those obtained previously, but the value of \(\Omega (^2P, ^4S)\) is increased by a factor of 2.7; this is because all the corrections are large and of the same sign.

c) Accuracy of the Calculations

A measure of the non-hydrogenic character of the continuum wave functions for atomic oxygen is provided by the phase difference \(\delta(k^2)\) between the asymptotic form of the radial wave functions and that of the corresponding hydrogen functions. A similar measure of the non-hydrogenic character of the excited atomic oxygen states is provided by the quantum defects \(\mu(n)\). Continuity of the wave functions at the spectral limit requires that \(\delta(0) = \pi \mu(\infty)\) (Seaton 1955b). This relation may be used to check the accuracy of the collision-strength calculations.

The dominant contributions to the collision strengths come from the \(2p^3kp S^T L^T\) continuum states. The quantities required in the calculation of the collision strengths may also be used to calculate the quantum defects \(\mu(\infty)\) at the limits of various \(2p^3(SL)np S^T L^T\) series. We consider two cases: (1) those series \((S^T L^T = ^1S, ^1F, ^3F, \text{ and } ^3P)\) for which the states may be specified completely in terms of the quantum numbers \(2p^3np S^T L^T\) and (2) those for which there is weak coupling between the states of \(2p^3(SL)np S^T L^T\) and the states of \(2p^3(S' L')n'p S^T L^T\) for \(SL \neq S' L'\). For case 2 the coupling terms have been neglected, and the calculations are therefore less accurate. Quantum defects obtained from observed energy levels (Moore 1949) are compared with the calculated values of \(\mu(\infty)\) in Table 6. Since the quantum defects \(\mu(n)\) vary slowly with \(n\), some significant information may be obtained on comparing \(\mu(3)_{obs}\) with \(\mu(\infty)_{calc}\), but it should be noted that the available results for \(\pi = 4\) suggest that the agreement between \(\mu(n)_{obs}\) and \(\mu(\infty)_{calc}\) would be closer for larger values of \(n\).

If we put
\[
\Delta \mu (n | S^T L^T) = \mu (n | S^T L^T) - \mu (n | ^1S),
\]
then it may be shown that the errors in the calculated collision strengths should be comparable to the errors in \(\sin^2 [\pi \Delta \mu(n)]\). From this it would appear that the results obtained should be correct to within 30–40 per cent.

V. THE INTERPRETATION OF OBSERVED INTENSITY RATIOS

a) The Equations of Equilibrium

Denoting by \(N_n\) the number of \(O^+\) ions per cubic centimeter in level \(n\), the equations defining a steady state are
\[
N_n \left\{ \sum_{n'} [A (n \rightarrow n') + q_{nn'}] + \sum_{n''} q_{nn''} \right\} = \sum_{n'} N_{n'} q_{n'n} + \sum_{n''} N_{n''} [A (n'' \rightarrow n) + q_{n'n''}],
\]
where
\[
\Omega (\alpha, \beta) \equiv \frac{\sum_{n} N_n \sum_{n'} [A (n \rightarrow n') + q_{nn'}] \sum_{n''} N_{n''} [A (n'' \rightarrow n) + q_{n'n''}]}{\sum_{n} N_n \sum_{n'} q_{nn'}}
\]
where $E_{n''} > E_n > E_{n'}$. For each transition $n \rightarrow n'$ we introduce a critical value $x_c$ of $x$ such that $q_{nn'} = A(n \rightarrow n')$ for $x = x_c$. From the values of $x_c$ given in Table 5 it is evident that (1) $A(^3P_{1/2} \rightarrow ^2P_{3/2})$ and $A(^3D_{3/2} \rightarrow ^2D_{5/2})$ may be neglected in all cases of practical importance; (2) for $x \gg 710$ all the $A$'s may be neglected, and with equation (12) it may then be shown that equation (16) gives a Boltzmann distribution for $N_n$; and (3) for $x \ll 290$, which may be assumed for most planetary nebulae, all $q_{nn'}$, may be neglected for $n = ^2P_{1/2}$ and $^2P_{3/2}$.

The solutions of the equilibrium equations may be written

$$N_n = \frac{NP(n)}{S}, \quad \text{(17)}$$

where

$$S = \sum_n P(n), \quad \text{(18)}$$

### TABLE 6

**Quantum Defects for 2p($^3S$)np $^3P^L$ Series in O I**

<table>
<thead>
<tr>
<th></th>
<th>$\mu(n)_{\text{obs}}$</th>
<th>$\mu(\infty)_{\text{calc}}$</th>
<th>$\Delta\mu(3)_{\text{obs}}$</th>
<th>$\Delta\mu(\infty)_{\text{calc}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^3S$</td>
<td>0.619</td>
<td>0.520</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$^1F$</td>
<td>0.799</td>
<td>0.739</td>
<td>0.180</td>
<td>0.219</td>
</tr>
<tr>
<td>$^3F$</td>
<td>0.813</td>
<td>0.751</td>
<td>0.193</td>
<td>0.231</td>
</tr>
<tr>
<td>$^3P$</td>
<td>0.826</td>
<td>0.804</td>
<td>0.206</td>
<td>0.242</td>
</tr>
</tbody>
</table>

**Case 2**

<table>
<thead>
<tr>
<th>$^3P$</th>
<th>$^1P$</th>
<th>$^3P$</th>
<th>$^3D$</th>
<th>$^3D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.798</td>
<td>0.748</td>
<td>0.179</td>
<td>0.228</td>
<td></td>
</tr>
<tr>
<td>0.817</td>
<td>0.754</td>
<td>0.197</td>
<td>0.234</td>
<td></td>
</tr>
<tr>
<td>0.833</td>
<td>0.824</td>
<td>0.213</td>
<td>0.239</td>
<td></td>
</tr>
</tbody>
</table>

and $N$ is the total number of O$^+$ ions per cubic centimeter. For $x \ll 290$ we obtain

- $P(^1S_{3/2}) = 1 + 7.2x(1 + 0.43x) + 6.1x^2(1 + 0.78x + 0.15x^2),$
- $P(^3S_{3/2}) = 4.0x[1 + 0.33x + 2.30x(1 + 0.75x + 0.14x^2)] e^{-3x};$
- $P(^3D_{3/2}) = 0.61x[1 + 0.40x + 9.9x(1 + 0.84x + 0.17x^2)] e^{-3x};$
- $P(^3P_{3/2}) = 3.60 \times 10^{-4}x[1 + 14x(1 + 0.38x)]$
  + $28.1x^2(1 + 0.78x + 0.15x^2)] e^{-5x};$
- $P(^3P_{1/2}) = 2.22 \times 10^{-4}x[1 + 13.5x(1 + 0.39x)]$
  + $28.8x^2(1 + 0.79x + 0.15x^2)] e^{-5x},$
where \( e = \exp(-1.96/t) \). When \( T_e \) is small, we may put \( e = 0 \); this corresponds to the approximation previously considered (Seaton 1954a, d) of neglecting the \( ^2P \) levels when calculating the \( ^2D \) populations.

b) Expressions for the Intensity Ratios

The observed intensity of a given line is proportional to

\[
I_{n,n'} = \int \left( E_n - E_{n'} \right) A \left( n \rightarrow n' \right) N_n \, dV,
\]

where the integration is to be extended over that part of a nebula which is in the line of sight.

1. The intensity ratio \( r = I(3729)/I(3726) \).—We obtain, for \( x \ll 290 \),

\[
r = 1.5 \frac{\int \left[ 1 + 0.33e + 2.30x \left( 1 + 0.75e + 0.14e^2 \right) \right] \left( N_x/S \right) e^{-3.86/t} \, dV}{\int \left[ 1 + 0.40e + 9.9x \left( 1 + 0.84e + 0.17e^2 \right) \right] \left( N_x/S \right) e^{-3.86/t} \, dV},
\]

which simplifies to

\[
r = 1.5 \left[ 1 + 0.33e + 2.30x \left( 1 + 0.75e + 0.14e^2 \right) \right] \left[ 1 + 0.40e + 9.9x \left( 1 + 0.84e + 0.17e^2 \right) \right]^{-1}
\]

if it is assumed that \( N_e \) and \( T_e \) are constant.

2. The ratio \( r' = I(7320)/I(7330) \).—We obtain \( r' = 1.24 \) for \( x \ll 290 \), \( r' = 1.31 \) for \( x \gg 700 \). For intermediate densities \( r' \) will vary monotonically between these two limiting values, and it may therefore be concluded that \( r' \) is insensitive to \( T_e \) and \( N_e \) under all conditions. Swings and Jose (1949) have obtained visual estimates giving \( r' = 1.0 \) for BD+30°3639, NGC 6572, and NGC 6543, while Merrill (1928) estimates the approximate ratio 1.5 for NGC 7027 and NGC 6572. The agreement with theory thus appears satisfactory, but a more precise observational determination would be of interest.

3. The ratio \( r'' = \left[ I(3729) + I(3726) \right] / \left[ I(7320) + I(7330) \right] \).—We obtain, for \( x \ll 290 \),

\[
r'' = 5.5 \frac{\int \left[ 1 + 0.36e + 5.3x \left( 1 + 0.82e + 0.16e^2 \right) \right] \left( N_x/S \right) e^{-3.86/t} \, dV}{\int \left[ 1 + 13.8x \left( 1 + 0.38e \right) + 28.4x^2 \left( 1 + 0.78e + 0.15e^2 \right) \right] \left( N_x/S \right) e^{-5.82/t} \, dV},
\]

which reduces to

\[
r'' = 5.5 \frac{\left[ 1 + 0.36e + 5.3x \left( 1 + 0.82e + 0.16e^2 \right) \right]}{\epsilon \left[ 1 + 13.8x \left( 1 + 0.38e \right) + 28.4x^2 \left( 1 + 0.78e + 0.15e^2 \right) \right]}
\]

if \( N_e \) and \( T_e \) are constant.

c) Results for Individual Nebulae

1. IC 4997.—The observed [O III] ratio \( I(4959) + I(5007) \)/\( I(4363) \) is 13.1 (Aller 1941) or 9.2 when corrected for reddening, assuming the calculated \( \text{H}\beta/\text{H}\gamma \) ratio of 2.0. The theoretical expression for the [O III] ratio is

\[
\frac{I(4959) + I(5007)}{I(4363)} = 8.74 \left( \frac{1 + 3.8 \times 10^{-4}x}{1 + 4.4 \times 10^{-2}x} \right) e^{3 \, 30/t}
\]

(Seaton 1954a). Some values of \( T_e \) and \( N_e \) consistent with an observed ratio of 9.2 are given in Table 7. Since \( T_e \) is unlikely to be much greater than \( 2 \times 10^4 \) K (Aller 1953), we conclude that it is improbable that \( N_e \) is much less than \( 10^6 \) cm\(^{-3}\). The last column of Table 7 gives values of \( r(\infty) \) calculated from the equilibrium equations, together with \( r(\text{obs.}) = 0.34 \). We adopt \( r(\infty) = 0.35 \pm 0.04 \), where the estimate of maximum probable
error is based on consideration of the observational procedure employed and on the possible small differences between $r$(obs.) and $r(\infty)$ for IC 4997.

2. NGC 281 and NGC 7000.—The intensity ratios in these nebulae were measured as a check at as low a density as it is feasible to reach with the available equipment. The approximate densities in these nebulae can be estimated from their surface brightnesses. Under the assumption of case B of Baker and Menzel (1938) and with $T_e = 10^4° K$, the density of a uniform spherically symmetric nebula is given by

$$\log N_e = 5.95 + \frac{1}{2} \log \frac{S_a}{AR},$$

where $S_a$ is the Hα surface brightness in ergs per square centimeter per second, $R$ is the distance in parsecs, and $A$ is the angular radius in seconds of arc. The average Hα magnitudes per square minute of arc of a number of diffuse nebulae, including NGC 281 and NGC 7000, have been measured by Shajn, Hase, and Pickelner (1955). Their magnitudes may be converted into energy units by comparing their measured values for NGC 6514, 6523, 6611, and 6618 with the surface brightnesses of these same nebulae as measured by Boggess (1954). It is not clear just how much of the area in the nebula the average magnitude per square minute of arc measured by Shajn et al. (1955) refers to, but we have compared in each their measure with the mean surface brightness as measured by Boggess inside an isophote that contains the main body of the nebula as seen on relatively long exposures. The result of this approximate calibration is that an $m_{Ha}$ magnitude of 9.9 per square minute of arc in Table 1 of Shajn et al. (1955) corresponds to a surface brightness of $1.0 \times 10^{-3}$ ergs/cm$^2$/sec in Hα, and the logarithms of the mean surface brightnesses of NGC 281 and NGC 7000 are, in these same units, $-3.4$ and $-3.5$, respectively.

The distances of these nebulae can be determined from the spectroscopic parallaxes of the O stars involved, HD 5005 br in NGC 281, and HD 199579 in NGC 7000. The spectral types and color indices were taken from the catalogue of Morgan, Code, and Whitford (1955); the normal colors (extrapolated slightly to $C_1 = -0.30$ at O6) and the conversion of color excess to absorption were taken from the paper by Morgan, Harris, and Johnson (1953); and the absolute magnitude of an O6 star was taken as $M_v = -5.0$ (Roman 1951). The resulting distances are $R = 2.3$ kpc for NGC 281 and $R = 1.0$ kpc for NGC 7000. The color excesses may also be used together with the interstellar extinction-curve published by Whitford (1948) to estimate the absorptions at Hα. This method gives an absorption of 0.3 in the logarithm of the surface brightness for each nebula, and thus the logarithms of the intrinsic surface brightnesses are $3$ $-3.1$ for NGC 281 and $-3.2$ for NGC 7000. The approximate radii as estimated on 48-inch Schmidt plates are $A = 15' = 900''$ for NGC 281 and $A = 80' = 4800''$ for NGC 7000. This latter figure is the mean radius of the large elliptical H II region, heavily overlaid by foreground absorp-

It may be noted that almost identical results are obtained on calibrating the measurements of Shajn et al. against the absolute measurements of Liller and Aller (1954) for the bright planetaries NGC 7027 and NGC 6572.

<table>
<thead>
<tr>
<th>$T_e$ (° K)</th>
<th>$N_e$ (cm$^{-3}$)</th>
<th>$r(\infty)$</th>
<th>$T_e$ (° K)</th>
<th>$N_e$ (cm$^{-3}$)</th>
<th>$r(\infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sim 7.5 \times 10^5$</td>
<td>$2 \times 10^6$</td>
<td>0.36</td>
<td>$2 \times 10^4$</td>
<td>$1.3 \times 10^6$</td>
<td>0.36</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$1 \times 10^6$</td>
<td>0.34</td>
<td>$7 \times 10^5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 7

Possible Temperatures and Densities in IC 4997
tion, which includes as its brightest parts NGC 7000, IC 5067, IC 5068, and IC 5070 (Morgan, Strömgren, and Johnson 1955). Substituting these figures in equation (26), we obtain the mean density estimates $N_e = 18$ cm$^{-3}$ for NGC 281 and $N_e = 10$ cm$^{-3}$ for NGC 7000.

According to theory the low-density limit of the intensity ratio is $r(0) = 1.50$ for $T_e$ small ($e = 0$) and $r(0) = 1.42$ for $T_e$ large ($e = 1$). Assuming the densities obtained above, together with a temperature of $10^4$° K, we see from Table 8 that theory would predict values of $r = 1.47$ and 1.48. The observed ratios are 1.37 for NGC 281 and 1.38 for NGC 7000. Since the probable observational errors are of order ±0.06, there appears to be a small but real discrepancy. This may be due to the fact that the points at which the spectroscopic observations were made are not average points in the nebulae but are selected points having especially high surface brightnesses and hence, presumably, especially high densities. Furthermore, when the density is low, any inhomogeneities tend to make the mean determined from equation (26) smaller than the mean determined from the $\lambda$ 3727 ratio. The observed ratios suggest densities of the order of 100 cm$^{-3}$, but it should be noted that when the density is low, precise estimates cannot be made from the ratio, since $r$ is then insensitive to $N_e$.

### Table 8

<table>
<thead>
<tr>
<th>$N_e$...</th>
<th>1000</th>
<th>300</th>
<th>100</th>
<th>30</th>
<th>→0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$...</td>
<td>0.90</td>
<td>1.22</td>
<td>1.38</td>
<td>1.46</td>
<td>1.49</td>
</tr>
</tbody>
</table>

It would seem reasonable to conclude that the observations are consistent with the calculated low-density limit. A more critical test will be possible only when the ratio $r$ can be measured in H II regions of still lower density.

3. **IC 418**.—This object, which is remarkable for its uniformity and for the absence of apparent fine structure (Wilson and Aller 1951), has been studied in some detail by various workers. From observations of the brightness variation across the disk, Wilson and Aller (1951) estimated emission rates and derived particle densities, assuming spherical symmetry, as functions of the radial distance from the central star. They did not claim these results to be of high accuracy but considered them schematically correct in major outline. The results indicated a density $N_e$ which falls just inside the bright ring but rises again toward the central star. However, on careful reconsideration of the observational material, Wilson (1953) doubted the correctness of this central rise in $N_e$ and indicated that it is more nearly correct to consider that all the hydrogen emission comes from the region of the bright outer shell.

The earlier absolute density estimates were made by using estimates of surface brightness obtained from photographic magnitudes. Greatly improved surface-brightness measurements have been made photoelectrically by Liller and Aller (1954) and by Liller (1955). Using these measurements, corrected for reddening, and assuming the Wilson and Aller density distribution function, Seaton (1954b) found that the electron densities of Wilson and Aller should be multiplied by a factor of 2.5, giving a maximum density of $N_e = 1.6 \times 10^4$ cm$^{-3}$ in the outer ring. If, however, there is no central rise in $N_e$, then the best density to adopt for the ring, as deduced from the surface-brightness measurements, is $N_e = 2.5 \times 10^4$ cm$^{-3}$ (Seaton 1954b).

The relative forbidden-line intensities were considered by Seaton (1954a) who obtained an electron temperature of $T_e = 1.8 \times 10^4$° K from the [O III] ratio and an electron density of $0.8 \times 10^4$ cm$^{-3}$. Independent temperature estimates have been obtained...
by Aller (1953) from a consideration of the thermal balance and estimated central star temperature; using the Wilson and Aller density distribution, he obtained a minimum of $1.8 \times 10^4$ °K at radial distances a good deal smaller than those of the bright ring and a maximum of $2.3 \times 10^4$ °K in the region of the bright ring. Despite uncertainties in the density distribution, these results give an interesting indication of the extent of temperature variations which may occur. It may be noted that the [O III] emission is strongest for the smaller radial distances (Wilson and Aller 1951; Wilson 1953) and that it is therefore possible that the temperature within the bright ring may be significantly different from that derived from the [O III] ratio. From measurements of the Balmer discontinuity Minkowski (1953) estimated that $T_e \geq 1.5 \times 10^4$ °K in the ring.

Since the densities indicated for IC 418 are no greater than those for a number of other bright planetaries, it is remarkable that the measured 3729/3726 ratio for the bright ring, $r = 0.37 \pm 0.03$, is unusually small, typical values for bright planetaries being about 0.5. Putting $T_e = 2 \times 10^4$ °K we obtain from equation (22) the variations of $r$ as a function of $N_e$ shown in Table 9. It is seen that the results for $r$ are in good agree-

<table>
<thead>
<tr>
<th>$10^{-4} N_e$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.46</td>
<td>0.41</td>
<td>0.38</td>
<td>0.37</td>
</tr>
</tbody>
</table>

TABLE 9

VARIATION OF 3729/3726 INTENSITY RATIO WITH ELECTRON DENSITY AT TEMPERATURE 2X10^4 °K

ment with the density obtained from the surface brightness, assuming no central density rise, but do not agree so well with the density obtained using the Wilson and Aller distribution. They appear to be in definite disagreement with the density of $0.8 \times 10^4$ cm$^{-3}$ previously obtained from the forbidden-line ratios. The latter was obtained by using intensity ratios for [S II] and a combination of intensities for [O II], [N II], [O I], and [N I] (the “O/N” method). It seems very probable that the [O II] emission is greatest in the region of greatest electron density but that the [S II] and [O I] emissions are strongest farther out. If this is so for [O I], the same would be expected for [N I]. Some evidence is provided by the measurements of Wilson and Aller (1951), but much stronger evidence is provided by the observations of Wilson (1950, 1953) that the [S II] and [O I] lines show expansion velocities considerably in excess of those shown for [O II]. The general correlation between expansion velocity and distance from the central star strongly suggests, as pointed out by Wilson, that these radiations are emitted at the greatest radial distances. Since the density is apparently falling rapidly outside the spherical shell, this might well explain the low densities previously obtained from the forbidden lines.

If the density in the spherical shell is greater than had previously been assumed, the Balmer discontinuity calculated for a given temperature will be reduced because of collisional deactivation of the 2s state of hydrogen (Seaton 1954c, 1955c); for $T_e = 1.8 \times 10^4$ °K and $N_e \leq 1 \times 10^4$ cm$^{-3}$, theory and observation are in good accord; but, with the same temperature and $N_e = 3 \times 10^4$ cm$^{-3}$, the calculated logarithmic discontinuity is 0.59, as compared with $0.48 \pm 0.07$ measured by Minkowski (1953). With
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\[ n_e = 3 \times 10^4 \text{ cm}^{-3}, \]
the calculated value is 0.48 for \( T_e = 2.3 \times 10^4 \text{ cm}^{-3} \), which suggests that this higher temperature may be more nearly correct for the bright ring.

4. NGC 7027.—From the forbidden-line intensity ratios and the Balmer discontinuity, Seaton (1954a, 1955c) obtained \( T_e = 1.55 \times 10^4 \) and \( n_e = 4.5 \times 10^4 \). The photoelectric surface-brightness measurements of Liller and Aller (1954), corrected for reddening, give \( n_e = 1.7 \times 10^4 \) (Aller 1954). There has been some discussion as to whether such differences in the densities obtained from the surface brightness and from the forbidden lines can be explained in terms of local density condensations, but, in view of the various uncertainties entering the two methods, it is not easy to be certain that the differences are really significant.

The 3729/3726 ratio has been measured by Aller and Minkowski (Aller et al. 1949; Aller, Bowen, and Minkowski 1955) who obtained values of \( r \) ranging from 0.42 to 0.50; we adopt the mean, \( r = 0.47 \). With \( T_e = 1.55 \times 10^4 \), this gives \( x = 0.68 \) and \( n_e = 0.85 \times 10^4 \). To examine further the discrepancy between this and previous density estimates, we consider the [O III] ratio \( r'' \). Aller, Bowen, and Minkowski (1955) obtain \( r'' = 1.38 \), which is not too different from \( r'' = 1.1 \) previously used by Seaton (1954a).

With \( T_e = 1.55 \times 10^4 \), \( r'' = 1.38 \) gives \( x = 2.5 \) and \( n_e = 3.1 \times 10^4 \). The difference between the densities obtained from \( r \) and \( r'' \) cannot be explained in terms of different distributions of different ions and is too large to be attributable to errors in observations or in atomic parameters.

We have been informed by Dr. R. Minkowski that he has recently obtained a photograph of NGC 7027 with the 200-inch Hale telescope, in conditions of excellent seeing, which shows that this object has a pronounced filamentary structure. This suggests that local variations in density may be important. When such density variations occur, the highly forbidden \( \lambda 3729 \) and \( \lambda 3726 \) lines will be strongest, relative to other features, in those regions where the density is smallest, and, in consequence, the density calculated from the ratio \( r \) will be smaller than that calculated by most other means. To make a quantitative estimate of the extent of density fluctuations, we consider that the volume \( V \) in which the [O III] lines are strong may be divided into two volumes, \( V_1 \) and \( V_2 \), such that \( n_e = n_{e1} \) in \( V_1 \) and \( n_e = n_{e2} \) in \( V_2 \). It is necessary to consider how the ratio \( N(O^+)/N_e \) will vary in those two regions. If \( N(O^+) > N(O^+) \), then we may expect that, because of increased recombination, \( N(O^+)/N_e \) will increase with \( n_e \). However, although \( \int N(O^+)dV \) is greater than \( \int N(O^+)dV \), where the integrations are extended over the whole nebula, \( N(O^+) \) may not be greater than \( N(O^+) \) in the regions giving the maximum [O III] emission. It is also possible that \( T_e \) may vary between the low- and high-density regions, the most probable variation being such that \( T_e \) is less when \( n_e \) is greater (Zanstra 1955). To make a first estimate of the extent of density fluctuations we will make the two assumptions that \( N(O^+)/N_e \) is independent of \( n_e \) and that \( T_e \) is equal to \( 1.55 \times 10^4 \), also independent of \( n_e \). We then obtain, for the intensity ratios,

\[
\begin{align*}
\frac{r}{r''} & = 0.47 = 1.47 \frac{1 + 2 \cdot 5.8 x_1}{1 + 11.2 x_2} \frac{x_1^2 V_1 / S(x_1)}{x_2^2 V_2 / S(x_2)} + (1 + 2 \cdot 5.8 x_2) \frac{x_2^3 V_2 / S(x_2)}{x_1^2 V_1 / S(x_1)} + (1 + 11.2 x_2) \frac{x_2^2 V_1 / S(x_2)}{x_2^2 V_2 / S(x_2)} \ ,
\end{align*}
\]

\[ r'' = 1.38 = 21.4 \]

\[
\times \left[ \frac{(1 + 6 \cdot 0 x_1) x_1^2 V_1 / S(x_1) + (1 + 6 \cdot 0 x_2) x_2^2 V_2 / S(x_2)}{(1 + 15.3 x_1 + 35.0 x_1^2) x_1^2 V_1 / S(x_1) + (1 + 15.3 x_2 + 35.0 x_2^2) x_2^2 V_2 / S(x_2)} \right] \ ,
\]

\[ \text{The Balmer decrements measured by Aller, Bowen, and Minkowski (1955) are less steep than those obtained earlier, and in consequence the calculated absorption correction is reduced. The same surface-brightness measurements corrected using the old Balmer decrements gave } n_e = 3 \times 10^4 \text{ (Seaton 1954).} \]

\[ \text{It may be noted that if } N(O^+)/N_e \text{ increases and } T_e \text{ decreases in the denser regions, then there will be some tendency for the errors introduced by these two assumptions to cancel each other to some extent.} \]
where \( S(x) = 1 + 8.5x + 9.1x^2 \). Substituting values of \( x \), we solve for \( x_2 \) and \( V_2/V_1 \). The results are given in Table 10. This table includes values of

\[
N_e(\text{rms}) = \left( \frac{N_{e1}^2 V_1 + N_{e2}^2 V_2}{V_1 + V_2} \right)^{1/2},
\]

which should be compared with the density of \( 1.7 \times 10^4 \text{ cm}^{-3} \) obtained from the surface brightness. It is seen that the values of \( N_e(\text{rms}) \) are of the order of \( 1.0 \times 10^4 \text{ cm}^{-3} \). Although the difference between these two figures is not excessive, it should be borne in mind that the surface-brightness determination refers to the whole nebula, while the estimate from the \([\text{O II}]\) lines refers to the regions in the line of sight where the \([\text{O II}]\) emission is greatest.

### Table 10

**Possible Density Fluctuations in NGC 7027**

(Densities \( N_e \) in \( 10^4 \text{ cm}^{-3} \))

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( N_{e1} )</th>
<th>( N_{e2} )</th>
<th>( V_2/V_1 )</th>
<th>( N_e(\text{rms}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3.5</td>
<td>—</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>4.0</td>
<td>014</td>
<td>0.49</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>4.6</td>
<td>024</td>
<td>0.75</td>
</tr>
<tr>
<td>3</td>
<td>37</td>
<td>5.6</td>
<td>022</td>
<td>0.91</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>7.3</td>
<td>0155</td>
<td>1.03</td>
</tr>
<tr>
<td>5</td>
<td>62</td>
<td>11.0</td>
<td>0074</td>
<td>1.13</td>
</tr>
<tr>
<td>6</td>
<td>75</td>
<td>23.5</td>
<td>0016</td>
<td>1.20</td>
</tr>
<tr>
<td>0.68</td>
<td>0.85</td>
<td>—</td>
<td>—</td>
<td>1.26</td>
</tr>
</tbody>
</table>

### Table 11

**Measured Densities in Planetary Nebulae**

<table>
<thead>
<tr>
<th>NGC (1)</th>
<th>( r ) (2)</th>
<th>( 10^4 T_e ) (3)</th>
<th>( 10^4 N_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (4)^* )</td>
<td>( (5)^† )</td>
<td>( (6)^‡ )</td>
<td></td>
</tr>
<tr>
<td>6543</td>
<td>0.50 (3)</td>
<td>0.91</td>
<td>2.4</td>
</tr>
<tr>
<td>6572</td>
<td>0.45 (1)</td>
<td>1.25</td>
<td>4.9</td>
</tr>
<tr>
<td>7009</td>
<td>0.30 (8)</td>
<td>1.21</td>
<td>2.8</td>
</tr>
<tr>
<td>7027</td>
<td>0.47 (7)</td>
<td>1.55</td>
<td>4.5</td>
</tr>
<tr>
<td>7062</td>
<td>1.30</td>
<td>5.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

* Density from Balmer discontinuity and forbidden-line ratios other than \( r \)
† Density from surface brightness
‡ Density from \( r \)

The results obtained indicate that the density throughout most of the volume is of the order of \( 5 \times 10^3 \text{ cm}^{-3} \) but that about 1 or 2 per cent of the total volume is occupied by dense clouds or filaments having densities of the order of \( 7 \times 10^4 \text{ cm}^{-3} \).

5. **Other nebulae.**—Table 11 gives the following results for a number of planetaries: (1) catalogue number; (2) values of \( r \) measured by Aller et al. (1949), with the number of points measured in parentheses; (3) and (4) values of \( T_e \) and \( N_e \), respectively, obtained by Seaton (1955c) from the Balmer discontinuity and the forbidden-line ratios other than \( r \); (5) values of \( N_e \) obtained from the surface brightness, assuming case B of Baker and Menzel (1938; Liller and Aller 1954; Seaton 1955c); (6) values of \( N_e \) obtained from \( r \). It
is seen that in all these objects the differences between the various density estimates are similar to those for NGC 7027. It would appear that NGC 7027 is in no way exceptional and that considerable local density fluctuations occur in many bright planetaries.

In Table 12 we give the densities deduced from $r$ for various other nebulae; in this table all calculations have been made by assuming a temperature $T_e = 1.2 \times 10^4 \, ^\circ K$.

VI. DISCUSSION

The present results show that the high-density limit for the ratio $r$ is considerably smaller than the value previously adopted, which was in close agreement with the observed value for a number of brighter planetaries. It has been shown that, if density variations occur, the effective density for the ratio $r$ will be smaller than the effective densities for other intensity ratios and for the surface brightness. The occurrence of such density variations explains why the previous argument that $r$ should be close to $r(\infty)$ in a number of bright planetaries was incorrect. The importance of this result is that quantitative information about the extent of density fluctuations may be obtained from a study of the relative line intensities in the spectra of gaseous nebulae.

<table>
<thead>
<tr>
<th>Object</th>
<th>$r$</th>
<th>Reference</th>
<th>$N_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 40</td>
<td>0.85</td>
<td>Aller et al. (1949)</td>
<td>$1 \times 10^6$</td>
</tr>
<tr>
<td>IC 4593</td>
<td>0.65</td>
<td>[Aller et al. (1949)]</td>
<td>$2 \times 10^6$</td>
</tr>
<tr>
<td>Cannon 5</td>
<td>0.54</td>
<td>Sec II</td>
<td>$5 \times 10^6$</td>
</tr>
<tr>
<td>MW 319</td>
<td>0.46</td>
<td>Sec II</td>
<td>$9 \times 10^6$</td>
</tr>
<tr>
<td>MHα 78(1)</td>
<td>0.44</td>
<td>Sec II</td>
<td>$1 \times 10^6$</td>
</tr>
<tr>
<td>Orion, greatest $N_e$</td>
<td>0.50</td>
<td>Osterbrock (1955)</td>
<td>$3 \times 10^6$</td>
</tr>
<tr>
<td>Orion, smallest $N_e$</td>
<td>1.26</td>
<td>Osterbrock (1955)</td>
<td>$2 \times 10^6$</td>
</tr>
</tbody>
</table>

Independent evidence for such fluctuations is provided in many cases by direct photographs of nebulae which show cloudy or filamentary structure. Frequently, however, such structure can be seen only in the outermost regions; it would not be expected for the main body of a nebula due to superposition in the line of sight of many regions of different emissivity. For IC 418 there are certain discrepancies between the various temperature and density estimates which may be explained in terms of large-scale density and temperature variations and different spatial distributions of different ions, but it is to be noted that these discrepancies are not such as to suggest the existence of local condensations. This is consistent with the particularly uniform appearance of this object.

It would be of interest to measure the ratio $r''$ for IC 418 in order to determine whether the densities obtained from $r$ and $r''$ are in accord. It would also be of interest to measure the ratio $r$ for various points in NGC 7662, a nebula for which density fluctuations may be considerable. It may be noted that Andrillat (1955) has obtained abnormally small values of the [O III] ratio for various points in the bright outer ring of this object.

In order to confirm and extend the present results, further calculations and observations are being carried out which will be required in the interpretation of the ratios $r = I(D_{5/2} \rightarrow S_{3/2})/I(D_{3/2} \rightarrow S_{3/2})$ for [S II], [N i], and [A iv].

We are indebted to Dr. R. Minkowski for permission to refer to his photograph of NGC 7027, to Dr. R. H. Garstang for his comments on the transition probability calculations, and to Mrs. Silvia Marcus for assistance in reducing some of the spectrophotometric measurements presented in this paper.
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