I

Most of what I shall say may be considered as properly belonging to applied mathematics; indeed, in large part they will be based on the works of the great applied mathematicians: Lord Rayleigh of the past and Sir Geoffrey Taylor and Sir Harold Jeffreys of the present. And applied mathematics though it be, I hope I can succeed in bringing some measure of conviction to you that the considerations I shall present have some bearing on and some relevance to the problems of astronomy.

Let me begin then by stating in general terms what we may expect to learn from studies on stability.

When we know that an object has existed in nearly the same state for a long time we generally infer that it is stable; and by this we mean that there is something in its construction and in its constitution which enables it to withstand small perturbations to which any system in Nature must be subject. More precisely, what we have in mind when we say that an object is stable is this: if the object experiences an external perturbation which produces small displacements among its component parts, then it reacts in such a way that it eventually restores itself to its original state. In practice this definition of the meaning of stability requires some amplification: for, though an object may be strictly unstable, it may still be that the time which an external perturbation needs to manifest itself is so long compared to the total duration of its existence that its potential instability has not had time to reveal itself. All that we can conclude, then, from the knowledge that an object has endured for a certain period of time is that its time of instability is long compared to that period. If this is the case, what, one may ask, can we learn from an investigation of the stability of an object? It appears that an investigation of the stability of an object can, under favourable conditions, provide information of two sorts. First, we can often infer some significant facts concerning the construction and the constitution of an object merely from the knowledge that it has successfully endured the effects of small perturbations for a known period of time; I shall presently give illustrations of this. Second, an investigation of stability may disclose the relative importance and sometimes the conflicting tendencies of the different forces and constraints which are kept dormant in the undisturbed state of the object. Thus, when we are confronted with a novel object—and most astronomical objects are novel—a study of its stability may provide a basis for a first comprehension. And I may confess that for an applied mathematician problems of stability present a particular attraction: by their very nature, these problems lead to linear equations and linear equations are always more pleasant to deal with than nonlinear ones. While to emphasize problems of stability for this last reason may appear as an undue concession to one's limitations, I think it will be admitted that an attempt to unravel the wide range of astronomical phenomena in which turbulence,
rotation and magnetism all play their parts, is not likely to meet with success if one started with the most general equations of hydrodynamics and electromagnetism without discrimination. In any event, I should like to illustrate by means of examples as to what we might learn about some of the newer problems of astronomy from studies of stability.

II

I said that we can often infer some significant facts concerning the constitution of an object from the mere knowledge that it has existed for a long period of time. Let me illustrate this by a classical example from the theory of stellar structure and then by a more modern version of the same.

Consider a sphere of perfect gas—a "star" let us say—in equilibrium under its own gravitation. Then a simple calculation shows that its total energy, $E$, including its internal heat energy ($U$) and its gravitational potential energy ($\Omega$), is given by

$$E = \frac{3\gamma - 4}{3(\gamma - 1)} |\Omega|,$$  \hspace{1cm} (1)

where $\gamma$ denotes the ratio of the specific heats. Now it is clearly necessary that $E$ be negative; otherwise the star will expand to infinity releasing energy in a time measurable in days. The condition $E < 0$ requires that $\gamma > \frac{4}{3}$. This conclusion regarding a physical parameter describing the star is clearly a significant piece of information.

The meaning of the condition $\gamma > \frac{4}{3}$ becomes clearer when we examine the stability of the gas sphere to its own natural modes of oscillation. By considering adiabatic radial oscillations, Ledoux (1) showed by a very simple argument based on the virial theorem that the formula

$$\sigma^2 = (3\gamma - 4) |\Omega| / I,$$  \hspace{1cm} (2)

where

$$I = \int_0^M r^2 dM(r)$$

denotes the moment of inertia, gives $\sigma$ the frequency of the oscillation to a sufficient degree of accuracy under most conditions. The meaning of the condition $\gamma > \frac{4}{3}$ is apparent from this formula; if this condition is not met, the frequency of the oscillation will become imaginary and the star will be dynamically unstable, i.e. unstable with respect to one of its own natural modes of oscillation. All this, of course, is well known. But let us apply similar considerations to the magnetic variables discovered by Horace Babcock.

It can be shown that for a gas sphere of the dimensions of a star in which there is a prevalent magnetic field, $H$, the total energy, $E$, including the heat ($U$), the gravitational ($\Omega$) and the magnetic ($\mathcal{M}$) energies is given by (cf. Chandrasekhar and Fermi (2))

$$E = \frac{3\gamma - 4}{3(\gamma - 1)} (|\Omega| - \mathcal{M}),$$  \hspace{1cm} (3)

where

$$\mathcal{M} = \frac{1}{8\pi} \iiint |H|^2 dx_1 dx_2 dx_3.$$  \hspace{1cm} (4)

From equation (3) it follows that one of the conditions for equilibrium is

$$\mathcal{M} < |\Omega|.$$  \hspace{1cm} (5)
This condition clearly sets an upper limit to the strength of a magnetic field which may prevail in a star. Since for a spherical configuration of uniform density \( \Omega = -3GM^2/5R \) (where \( G \) denotes the constant of gravitation, \( M \) the mass and \( R \) the radius of the configuration), we can estimate the limit on the magnetic field set by (5) by using this expression for \( \Omega \) in it. In this manner we find

\[
\sqrt{|H|^2_{av.}} < 2 \cdot 10^8 M/R^4 \text{ gauss,}
\]

where \( M \) and \( R \) are now expressed in solar units. While for most of the magnetic variables discovered by Babcock the surface fields measured are a hundred to a thousand times smaller than the limit for the root mean square field set by (6), there are a few stars—three to be exact—for which the surface fields already approach the limit set by (6) to within a factor ten.

Now the meaning of this upper limit to the magnetic field which may prevail in a star can be understood by considering the period of radial adiabatic pulsation. By following the method of Ledoux, we can show that in this case equation (2) is replaced by (cf. Chandrasekhar and Limber (3))

\[
\sigma^2 = (3\gamma - 4)(|\Omega| - \mathcal{M})/I;
\]

the meaning of the condition \( \mathcal{M} < |\Omega| \) is apparent. From formula (7) it would appear that we can make \( \sigma^2 \rightarrow 0 \) (i.e. make the period tend to infinity) by letting \( \mathcal{M} \rightarrow |\Omega| \). It is possible that this is the explanation of the known relatively long periods of the magnetic variables. I am here referring to the fact that while the observed periods of the magnetic variables are of the order of 6 to 10 days, the period of radial pulsation which we should predict for these stars, were they normal, is of the order of \( \frac{1}{3} \) day. If the suggested explanation of the long periods is correct, then the peculiar A-stars for which surface fields of the order of \( 10^4 \) gauss have been measured must have internal fields which are a thousand times larger; and this is perhaps not impossible.

III

I should like to consider next a problem in gravitational stability which may have a bearing on galactic structure (cf. Chandrasekhar and Fermi (2)).

Consider an infinite cylinder of an incompressible fluid of uniform circular cross-section of radius \( R_0 \). Is this gravitationally stable? We can answer this question by considering a perturbation of the cylinder which deforms the boundary into

\[
r = R + a \cos kz
\]

(where \( a \ll R \), \( z \) is measured along the axis of the cylinder and \( k \) is a constant) and asking whether the resulting change in the gravitational potential energy, \( \Delta \Omega \), per unit length is positive or negative. The fact that the potential energy per unit length of an infinite cylinder is infinite requires that some care be exercised in the evaluation of \( \Delta \Omega \).

When \( \Delta \Omega \) is evaluated it is found that

\[
\Delta \Omega > 0 \text{ for } kR_0 > 1.07
\]

and

\[
\Delta \Omega < 0 \text{ for } kR_0 < 1.07.
\]

The cylinder is, therefore, gravitationally unstable for all symmetrical deformations of the boundary with wave-lengths exceeding \( \lambda_\ast = 2\pi R_0/1.07 \). This recalls to mind a well-known result of Rayleigh’s (4) that a liquid jet is unstable, on
account of surface tension, for all symmetrical deformations with wave-lengths exceeding the circumference of the cylinder. And as in Rayleigh's discussion of the stability of the liquid jet, we may ask whether there is a mode of maximum instability. We find that there is one whose wave-length is approximately 1.84 times the minimum wave-length, \( \lambda_\ast \), for instability; and the time of instability as judged by this most unstable mode is

\[
\tau = 4.07(4\pi G \rho)^{-1/2}. \tag{10}
\]

If we substitute for \( \rho \) in this formula the value of the mean density of the interstellar matter in the spiral arm in which we are located (namely \( 2 \times 10^{-24} \text{ g/cm}^3 \)) we obtain \( \tau = 10^8 \) years. This is so short a time that a structure like the spiral arm, if we can idealize it as an infinite cylinder, would have disintegrated a long time ago.

Now the spiral arms of a galaxy show every sign of instability; but we should be happier if the time of instability were of the order of \( 5 \times 10^9 \) years. It appears that we can make an infinite cylinder gravitationally stable for periods of this order by supposing that along the axis of the cylinder there is a magnetic field of the order of

\[
H_s = 4\pi \rho R_0 \sqrt{G}. \tag{11}
\]

Thus a field \( H = H_s \) increases the wave-length of maximum instability by a factor ten and the time of instability by a factor nine. Table I gives more data relating to this problem. From this table we may conclude that if the spiral arms can be idealized in the manner we have done, the requirement that they be unstable but with times of instability of the order of \( 5 \times 10^9 \) years will imply that there are magnetic fields of the order of \( 7 \times 10^{-6} \) gauss along the arms. It is gratifying that a number of other independent evidences lead one to postulate galactic magnetic fields of this same order.

### Table I

<table>
<thead>
<tr>
<th>( H ) (gauss)</th>
<th>( \lambda_\ast ) (parsecs)</th>
<th>( \lambda_m ) (parsecs)</th>
<th>( \tau ) (years)</th>
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<tr>
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<td>3.5 \times 10^{11}</td>
</tr>
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</table>

**IV**

In a general way one may expect that a conducting fluid in the presence of a magnetic field and under the influence of Coriolis' force is subject to conflicting tendencies and that it may exhibit unexpected patterns of behaviour. I should like to illustrate this by considering a classical problem in the theory of thermal instability. The problem I have in mind is the instability of a layer of fluid heated below which has been investigated by Rayleigh (5), Jeffreys (6) and others.
When a horizontal layer of an incompressible fluid is heated below—"underside" as Rayleigh calls it—then on account of thermal expansion the liquid above, being colder, will be heavier. This is a top-heavy arrangement and is potentially unstable. Consequently, if the temperature gradient is made sufficiently adverse, we should expect the instability to manifest itself.

The manner of the onset of convection under these circumstances has been the subject of both experimental and theoretical investigations since Bénard's first experiments on the subject in 1900 and 1901. Those early experiments of Bénard established that instability sets in only when the temperature gradient exceeds a certain critical value and that when it does set in, it does so as a pattern of cellular convection. The correct interpretation of Bénard's experiments was given by Rayleigh in 1916. Rayleigh showed that what decides the stability of a layer of fluid heated below is the numerical value of the non-dimensional quantity

\[ R = \frac{g \alpha \beta}{\kappa \nu} d^4, \]  

where \( g \) denotes the value of gravity, \( d \) the depth of the layer, \( \beta = |dT/dz| \), the constant adverse temperature gradient which is maintained, and \( \alpha, \kappa \) and \( \nu \) are the coefficients of volume expansion, thermometric conductivity and kinematic viscosity respectively. We shall call \( R \) the Rayleigh number. Both experiments and theory show that instability must set in when the Rayleigh number exceeds a certain determinate critical value: for example, it is 1708 when it is confined between two rigid plane boundaries. Accordingly, higher temperature gradients can be maintained, before instability sets in, in a liquid of higher viscosity and/or higher thermal conductivity; and the reason for this is quite evident.

Before passing on to an examination of the effects of Coriolis' force and magnetic field on the onset of convection by thermal instability we may first consider very briefly the manner in which one determines the critical Rayleigh number, theoretically. One determines this by supposing that an initial constant state is slightly disturbed and asking whether there exists any type of disturbance which will not be damped. Since any disturbance in the horizontal plane can be expressed as a superposition of two-dimensional periodic waves, we can investigate this problem by asking what the lowest Rayleigh number, \( R(a) \), is at which a mode of disturbance in the horizontal plane of a given wave number, \( a \), when excited, does not get damped. On solving this problem one finds that the resulting function \( R(a) \) has in general a single minimum which it attains for a particular value of \( a \). It is clear that this minimum of the function \( R(a) \) specifies the critical Rayleigh number at which instability will first set in; at the same time the value of \( a \) at which the minimum occurs determines the dimensions of the cell which will appear at marginal stability.

We shall now suppose that in addition to gravity the fluid is subject to the Coriolis acceleration \( 2u \times \Omega \) resulting from the fluid partaking in a rotation with angular velocity \( \Omega \). The discussion of the stability of fluid heated below under these circumstances can be carried out and it is found (Chandrasekhar (7)) that the effect of the Coriolis acceleration is to inhibit the onset of convection, the extent of the inhibition depending on the value of the non-dimensional parameter

\[ T = 4 \frac{\Omega^2 \cos^2 \vartheta}{\nu^2} d^4, \]
where \( \vartheta \) denotes the angle which the direction of \( \Omega \) makes with the vertical and \( \Omega = |\Omega| \). The inhibition factors are of the order of ten for \( T \) of the order of \( 10^6 \) (see Fig. 1). It is further found that

\[
R \to \text{constant} \, T^{2/3} \quad \text{and} \quad \alpha \to \text{constant} \, T^{1/6} \quad \text{as} \quad T \to \infty ; \tag{14}
\]

here \( \alpha \) denotes the wave number (in units of \( 1/d \)) of the cells which appear at marginal stability. I might mention here that experiments carried out by Dr D. Fultz at Chicago confirm the predicted dependence of \( R \) on \( T \).

In a general way the reason for the inhibiting effect of \( \Omega \) is clear: according to a theorem of Kelvin and Helmholtz, the vortex lines have a tendency to be dragged with the fluid, the attachment of the fluid to the vortex lines being stronger the lower the viscosity and the higher the angular velocity; and in the limit of zero viscosity the attachment is a permanent one. Consequently as \( \Omega \cos \vartheta \) increases and/or \( \nu \) decreases, motions at right angles to the vertical will become increasingly difficult; and this prevents an easy closing-in of the streamlines required for convection. This also explains why instability, when it sets in, does so for a value of \( \alpha \) which is increasingly large: for, a large value of \( \alpha \) means that the cells are elongated in the direction of the vertical and motions in this direction are not hindered by the component of \( \Omega \) in this direction.

According to equations (13) and (14), in the limit \( T \to \infty \), the relation giving the critical temperature gradient at which convection sets in changes from

\[
g\alpha \beta_c = \text{constant} \, \kappa \nu \, d^{-4} \tag{15}
\]

to

\[
g\alpha \beta_c = \text{constant} \, \kappa (\Omega^4 \cos^4 \vartheta/d^4 \nu)^{1/3}. \tag{16}
\]

This changed dependence of \( \beta_c \) on \( d \) from a \( d^{-4} \)-law to a \( d^{-4/3} \)-law is likely to be a decisive factor in determining the character of the convection when Coriolis'
force is acting. If the indications of this theory may be taken as a guide as to what may happen in reality, then we should conclude that the effects of Coriolis' force must control thermally induced convection in all atmospheric layers exceeding ten metres on the Earth and a hundred kilometres on the Sun.

We turn next to the effect of a magnetic field on the onset of convection. On general grounds, we may expect that the effect of a magnetic field will also be to inhibit convection and that this inhibiting effect will be greater the stronger the magnetic field ($H$) and the higher the electrical conductivity ($\sigma$): for, when the field is strong (or the conductivity high), the lines of force tend to be glued to the material and this will make motions at right angles to the vertical increasingly difficult. Moreover, when cellular convection does set in, we should expect that the cells are elongated in the direction of the vertical, the elongation being greater the stronger the vertical component of the magnetic field. In the limit of infinite electrical conductivity, when the lines of force are permanently attached to the fluid, convection in the usual sense will become impossible. A detailed theoretical treatment of the problem (cf. Chandrasekhar (8)) confirms these anticipations. In particular, it is found that the critical Rayleigh number for the onset of instability in the presence of a magnetic field depends on the strength of the field and the electrical conductivity $\sigma$ through the non-dimensional parameter

$$Q = \frac{\sigma H^2 \cos^2 \theta}{\rho \nu} d^2,$$  

(17)

where $\theta$ now denotes the inclination of the direction of the impressed magnetic field to the vertical. It is further found that we have inhibition factors of the order of ten for $Q$ of the order of a thousand (see Fig. 2) and that in the limit $Q \to \infty$, $R \to \text{constant } Q$ and $a \to \text{constant } Q^{\frac{1}{16}}$. (18)

All that I have said so far regarding the effect of $\Omega$ and $H$ on the onset of thermal instability might possibly have been predicted on general grounds. But let us now consider what happens when both $\Omega$ and $H$ are simultaneously present. The results I shall describe have been obtained for the case when $g$, $H$ and $\Omega$ are all co-planar and both the confining boundaries are free surfaces; however, the general character of the solution to be described does not depend on these latter restrictions.

The solution of the problem presents some unexpected features which result from the curve $R(a)$ (which represents the lowest Rayleigh number at which a horizontal disturbance having a wave number $a$ when excited is undamped) having two minima, the one occurring at the larger $a$ giving the lower Rayleigh number when $Q$ is small and the higher Rayleigh number when $Q$ is large (see Figs. 3 and 4). Thus for $T_1 (= T/\pi^4) = 100000$ and $Q_1 (= Q/\pi^8) = 50$ the two minima occur at $a$ (the wave number measured in the unit $1/d$) = 18.6 and 3.48 where $R = 4.03 \times 10^6$ and $7.26 \times 10^6$, respectively. On the other hand for $Q_1 = 100$ (and the same value of $T_1$) the two minima occur at $a_1 = 18.2$ and 3.37 where $R = 3.98 \times 10^6$ and $3.93 \times 10^5$, respectively. Consequently, for a value of $Q_1$ slightly less than 100 the wave number of the cells which appear at marginal stability will suddenly decrease from $a = 18.2$ to $a = 3.4$. In other words, if we start with an initial situation in which $T_1$ has the value $10^6$ and no magnetic field is present and gradually increase the strength of the magnetic field, then at first the cells which appear at marginal stability will be elongated; but when the
magnetic field has increased to a value corresponding to \( Q_1 = 100 \), cells of two very different sizes will appear simultaneously: one set which will be highly elongated and another set which will be much less elongated. As the magnetic field increases beyond this value, the critical Rayleigh number will actually begin to decrease. However, for sufficiently large \( Q \) the inhibition due to the magnetic field will predominate and will take control of the situation. This is an unexpected sequence of events and I do not know if one could have predicted it.

![Graph showing the variation of the critical Rayleigh number, \( R_c \), for the onset of instability as a function of \( Q \) for the three cases: (i) both bounding surfaces free (curve marked I), (ii) both bounding surfaces rigid (curve marked II) and (iii) one bounding surface free and the other rigid (curve marked III). The points marked I, II and III on the \( R_c \)-axis are the limiting values for the three cases considered in the absence of a field.](image)

I shall turn now from these problems in thermal instability to one of a somewhat different kind: the problem of the gravitational instability of an infinite homogeneous medium. This was first considered by Jeans (9) more than fifty years ago. In this problem we start from an initial state of homogeneity and rest and consider the velocity of propagation of a density fluctuation through the medium. If the gravitational effects of the density fluctuation are ignored, the problem is the classical one of the propagation of sound; and as is well known, the velocity of sound in a gaseous medium is independent of wave-length and is given by

\[
c = \sqrt{\gamma \rho / \rho},
\]

(19)
Fig. 3.—The dependence of the wave number $a$ of the cells which appear at marginal stability under the simultaneous influence of a Coriolis acceleration and a magnetic field. The different curves refer to the various assigned values of $T_1$ and varying values of $Q_1$.

Fig. 4.—The variation of the critical Rayleigh number $R$ for the onset of instability as a function of $Q_1$ for various assigned values of $T_1$.  

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where \( p \) denotes the pressure, \( \rho \) the density and \( \gamma \) the ratio of the specific heats. On the other hand, if the change in the gravitational potential, \( \delta V \), consequent on the density fluctuation, \( \delta \rho \), is taken into account in the equation of motion, with \( \delta V \) related to \( \delta \rho \) through Poisson's equation

\[
\nabla^2 \delta V = -4\pi G \delta \rho, \tag{20}
\]

then, as Jeans showed, the velocity of wave propagation is no longer independent of the wave-length. It is given by

\[
V_j^2 = c^2 - 4\pi G \rho / k^2, \tag{21}
\]

where \( k(=2\pi/\lambda) \) is the wave number. The velocity of wave propagation, therefore, becomes imaginary if

\[
k^2 < 4\pi G \rho / c^2 \quad \text{or} \quad \lambda^2 > \frac{\pi c^2}{G \rho} \quad \text{(say)}. \tag{22}
\]

But an imaginary velocity of wave propagation only means that the amplitude of the wave will increase exponentially with time. Accordingly, if an arbitrary initial perturbation in density is represented by a Fourier integral, then the amplitudes of those components in the Fourier representation which have wave-lengths exceeding \( \lambda_j \) will increase exponentially with time; \( \lambda_j \) is therefore a measure of the linear dimensions of the condensations which will form in the medium on account of this instability; this is Jeans's result.

Now suppose that the medium considered is partaking in rotation and that there is a term \( 2u \times \mathbf{\Omega} \) representing the Coriolis acceleration in the equation of motion. Then a reconsideration of Jeans's problem taking account of this additional term shows that there are now two modes of wave propagation. If \( V_1 \) and \( V_2 \) are the velocities of propagation of these two modes, then one finds that

\[
V_1^2 V_2^2 = 4 \left( \frac{\Omega \cos \delta}{k} \right)^2 V_j^2, \tag{23}
\]

where \( \delta \) denotes the angle between the direction of \( \mathbf{\Omega} \) and the direction of wave propagation considered. From this relation between \( V_1 \) and \( V_2 \), it follows that if \( V_j^2 < 0 \), then either \( V_1^2 \) or \( V_2^2 \) must be negative. In other words if Jeans's condition for gravitational instability is satisfied, then the propagation by one of the two modes must be unstable. Jeans's condition is, therefore, unaffected by the inclusion of Coriolis' force. There is only one exception to this rule and this occurs when we consider a wave propagated in a direction at right angles to \( \mathbf{\Omega} \). In this latter case, there is only one mode in which a wave can be propagated and its velocity is given by

\[
V^2 = \frac{\Omega^2}{k^2} + c^2 - \frac{4\pi G \rho}{k^2}. \tag{24}
\]

Consequently, in this plane the propagation of a wave will always be stable provided

\[
\Omega^2 > \pi G \rho. \tag{25}
\]

But even when this last condition is satisfied the propagation of waves in other directions will exhibit gravitational instability.

Similarly, we find (cf. Chandrasekhar and Fermi (2)) that Jeans's condition is unaffected also when a uniform magnetic field is present. In this case it is found that there are three modes of wave propagation. One of these is
unaffected by gravity and compressibility and represents the usual hydromagnetic wave of Alfvén. But the other two modes are affected and their velocities are related by

\[ V_1^2 V_2^2 = \frac{H^2 \cos^2 \vartheta}{4\pi\rho} V_f^2, \quad (26) \]

where \( \vartheta \) now denotes the angle between the direction of \( \mathbf{H} \) and the direction of wave propagation considered. From this relation it is apparent that if \( V_f^2 < 0 \) then either \( V_1^2 \) or \( V_2^2 \) must be negative; and Jeans's condition follows.

Finally, if we consider the case when a magnetic field is present and Coriolis' force is also acting, we find that Jeans's condition is still unaffected. In this case there are three modes of wave propagation all coupled in such a way that their velocities \( V_1, V_2 \) and \( V_3 \) are related by

\[ V_1^2 V_2^2 V_3^2 = \left( \frac{H^2 \cos^2 \vartheta}{4\pi\rho} \right)^2 V_f^2, \quad (27) \]

where \( \vartheta \) has the same meaning as in (26). And again if \( V_f^2 < 0 \), one of these modes will be unstable and gravitational instability will ensue.

VI

The foregoing discussion of the various problems of stability draws attention to an aspect of theoretical investigations in astronomy and geophysics which is not often recognized: it is that matter and motions in the large may exhibit patterns of behaviour which one might never suspect if one restricted oneself to matter and motions in the small. Several examples could be given to underline this. Thus, in most hydromagnetic problems which occur in astronomy one may treat the matter (be it in interstellar space, stellar atmospheres or stellar interiors) as an infinitely good electrical conductor with the lines of magnetic force permanently attached to it; and one may do this even though the electrical conductivity of stellar material judged by terrestrial standards is by no means extraordinary. And the reason why we may do this is simply the very large linear dimensions of the systems one deals with, the large dimensions leading to times of decay of a prevalent magnetic field which are very long compared to the times in which one is normally interested. This applies even to certain aspects of geomagnetism. For example, in considering the secular variation of the Earth's magnetic field we may treat the fluid core of the Earth as an infinitely good conductor; for, while the time of decay is of the order of 50,000 years, the periods in which the secular variations take place are of the order of a hundred to two hundred years. This reference to the Earth's magnetic field reminds us that its existence is the best illustration of what a large mass of electrically conducting fluid, in rotation, can exhibit. Indeed, rotation must influence fluid behaviour in the large in a manner which we are still very far from comprehending. Thus, all the evidence we have tends to support the view that solar rotation plays a decisive role in the entire complex of phenomena associated with the sunspots and the solar cycle; and yet the magnitude of the solar rotation is, by all conventional standards, very small.

It would appear then that the scope for applied mathematics in astronomy and geophysics is a very large one. But in order to make progress it may often be necessary to make the most severe idealizations, as will have been evident from
the examples I have considered. In defence I may quote from Sir George Darwin’s address of welcome to the Fifteenth International Congress of Mathematicians when it met in Cambridge in 1909:

“I appeal, then, for mercy to the applied mathematician and would ask you to consider in a kindly spirit the difficulties under which he labours. . . . yet they are honest attempts to unravel the secrets of the Universe in which we live.”

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