A STATISTICAL BASIS FOR THE THEORY OF STELLAR SCINTILLATION

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Summary

Current theories of stellar scintillation and astronomical seeing suppose that there is a disturbed region in the atmosphere at a height of about 4 km which corrugates a plane wave-front passing through it; and that the observed phenomena are to be accounted for in terms of such a corrugated wave-front. On the assumption that the disturbed region is a turbulent layer in which the refractive index, \( \mu \), is subject to irregular fluctuations, the auto-correlation, \( \delta \mu(r_1)\delta \mu(r_2)/\mu_0^2 \), of the instantaneous fluctuations in the refractive index at two different points \( r_1 \) and \( r_2 \) is introduced; on the further assumption that the turbulence which prevails is homogeneous and isotropic, the auto-correlation so defined can be a function, \( M(r) \) (say), only of the distance \( r \) between the two points considered. It is then shown how the statistical properties of the corrugated emergent wave-front, such as the angular dispersion in the wave normals, can be expressed in terms of \( M(r) \). From the known facts concerning astronomical seeing it is concluded that we can satisfactorily account for the observed phenomena by postulating a turbulent layer of a thickness of the order of a hundred metres, a micro-scale of turbulence of the order of ten centimetres and a root mean square fluctuation in refractive index of the order of \( 4 \times 10^{-8} \).

1. Introduction.—On examining the recent literature on the subject of stellar scintillation and astronomical seeing, it appeared to the writer that a proper statistical basis for discussing certain aspects of the phenomenon was lacking. In this paper an attempt will be made to provide such a basis. But it may be useful, first, to clarify the nature of the problem and to indicate the need for a theory along the lines outlined in this paper.

It would appear that in discussing the general problem of stellar scintillation we should distinguish between the phenomena which are observed near the horizon and at low altitudes and the phenomena which are observed near the zenith and at high altitudes. That such a distinction should be made is clear already from Lord Rayleigh's (1) discussion of the problem in 1893. For, as Rayleigh showed, at low altitudes the rays of different colours coming from the same star and reaching the same point on the Earth are separated by quite appreciable distances in the upper atmosphere. Thus, the separation, \( \Delta \eta \), between two rays of different colours at a height \( z \) above the Earth is given by (Rayleigh (1), eq. (10))

\[
\Delta \eta = \frac{\sin \theta}{\cos^2 \theta} \int_0^z \Delta \mu(z') dz' = \frac{\sin \theta}{\cos^2 \theta} \Delta \mu z,
\]

where \( \theta \) is the angle which the direction of observation makes with the vertical and \( \Delta \mu \) is a mean of the difference in refractive index, \( \Delta \mu(z') \) (at height \( z' \)) in the two colours. Over the visual spectral range \( \Delta \mu \sim 7 \times 10^{-5} \) and

\[
\Delta \eta \approx 0.7 \frac{\sin \theta}{\cos^2 \theta} z \quad (z \text{ in km}),
\]

(2)
From this formula Rayleigh computed that at a height of 8 km, $\Delta \eta = 183$ cm for $\theta = 80^\circ$, while it is 45 cm for $\theta = 70^\circ$; and separations of this amount are sufficient to account for the movements of the bands in the spectra of stars observed near the horizon as described by Montigny (2) and especially by Respighi (3). Indeed, Rayleigh was able to give satisfactory explanations for all the observations known at that time in terms of the separation of the rays due to chromatic dispersion in the atmosphere. On the other hand, for $\theta < 40^\circ$ the largest separations to be expected over heights of the order 10 km is less than 5 cm; it is evident that we must trace the phenomena observed at these higher altitudes to a different cause. And, in fact, it has been generally agreed (cf. Anderson (4), Strömgren (5) and Hansson, Kristenson, Nettelblad and Reiz (6)) that at these higher altitudes we must attribute seeing and scintillation to the corrugation of an incident plane wave-front by its passage through a "disturbed" region in the atmosphere. Expecting that the disturbed region (which is probably to be identified with the inversion layer) will be in a state of turbulence, one supposes that the wave-front in its passage through will be subject to an irregular and variable refractive index and become, in consequence, corrugated. It is precisely in terms of such a corrugated wave-front that one has sought to interpret the phenomenon of astronomical seeing. It is to be noted that the variable refractive index has been introduced only to account for the corrugated wave-front. While the manner of deducing the properties of seeing and scintillation from the assumed existence of a corrugated wave-front has not always been rigorous or quantitative, it, nevertheless, seems to the writer that Little's (7) recent description of the current ideas under the heading of a "refraction theory" which postulates a linear array of cylindrical lenses at a definite height is an unfair representation. In saying this, it is not the intention of the writer to criticize Little's application of Booker, Ratcliffe and Shinn's (8) notions of angular power spectrum and Fresnel diffraction by an irregular screen. The application is obviously sound and Little's use of the terminology of Booker, Ratcliffe and Shinn represents a distinct advance in the subject. But where Little's discussion is lacking is also the place where the discussion of the other writers has been lacking, namely, the manner in which the incident plane wave-front gets corrugated in the first place. It is to account for the corrugated wave-front (even if only qualitatively) that the earlier writers on the subject had invoked the variable refractive index; and Little's discussion of the problem hardly discredits this. Indeed, in the part of the paper where Little tries to deduce the variation in the electron concentration in the $F_2$ layers from the variable intensity of the radio stars, he implicitly recognizes the need for a "refraction basis" for the whole theory.

From the discussion in the preceding paragraph it would appear that we must trace to a disturbed layer the initial cause for the corrugation of the incident wave-front; and as has been further suggested, the corrugation may be considered as resulting from the fluctuating refractive index in the disturbed layer. On the assumption that the conditions in the disturbed layer can be approximated by those in homogeneous, isotropic turbulence, we shall outline a theory in terms of which the statistical features of the emergent corrugated wave-front can be described. Once this has been done, we can apply the notions of angular power spectrum and Fresnel diffraction as Little has done.

In large measure, the theory presented in this paper derives from a method which Münch and the writer (9) have recently developed in another connection.
2. The correlation functions to describe the fluctuating refractive index in the turbulent layer.—Let $\delta \mu(r)$ denote the instantaneous departure of the refractive index from the mean at the point $r$. On account of turbulence $\delta \mu(r)$ will be subject to fluctuations, and to describe this we shall introduce the correlation between the fluctuations $\delta \mu(r_1)$ and $\delta \mu(r_2)$ at two points $r_1$ and $r_2$ in the medium. Under conditions of homogeneity and isotropy (which we shall assume) the correlation will depend only on the distance, $r = |r_1 - r_2|$, between the two points considered. Thus, we shall let

$$\delta \mu(r_1) \delta \mu(r_2) = \overline{\delta^2 \mu} M(r),$$

(3)

where $\overline{\delta^2 \mu}$ denotes the mean square fluctuation in the refractive index and $M(r)$ is some (even) function of $r$.

We shall presently see that for following a ray, statistically, through a medium of variable refractive index we need the tensor

$$R_{ij} = \left( \frac{\partial}{\partial x_i} \delta \mu \right) \left( \frac{\partial}{\partial x_j} \delta \mu \right),$$

(4)

giving the correlation of a component of $\nabla \delta \mu$ at the point $x_i$ with a component at $x'_j$. The definition of this tensor does not require the introduction of any additional scalar function; for, clearly,

$$R_{ij} = -\overline{\delta^2 \mu} \frac{\partial^2 M}{\partial \xi_i \partial \xi_j} \quad (\xi_i = x'_j - x_i).$$

(5)

We therefore have

$$R_{ij} = -\overline{\delta^2 \mu} \left\{ \left( \frac{M'}{r^2} - \frac{M'}{r^3} \right) \xi_i \xi_j + \frac{M'}{r} \delta_{ij} \right\},$$

(6)

where primes denote differentiation with respect to $r$. From (6) it follows in particular that

$$\overline{\left( \frac{\partial}{\partial x_i} \delta \mu \right) \left( \frac{\partial}{\partial x_j} \delta \mu \right)}_{(0,0,0;0,0,x'_i-r)} = -\overline{\delta^2 \mu} \frac{M'}{r}.$$  

(7)

We shall denote this transverse correlation by

$$\overline{\delta^2 \mu} R(r), \text{ where } R(r) = - \frac{1}{r} \frac{dM}{dr}. $$

(8)

The correlation function $M(r)$ defines a "micro-scale", $r_0$, such that $M(r)$ becomes negligible for $r \gg r_0$. When we wish to illustrate the general formulae by a particular case, we shall assume that

$$M(r) = \exp(-r^2/r_0^2).$$

(9)

3. The equations governing a ray traversing a medium of variable refractive index.—The general equations governing a ray traversing a medium of variable refractive index can be written in the form (cf. Synge (10), p. 101)

$$\frac{d}{du} \left\{ \frac{\mu \dot{x}_i}{(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)^{1/2}} \right\} = \left( \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right)^{1/2} \frac{\partial \mu}{\partial x_i},$$

(10)

where

$$\dot{x}_i = \frac{dx_i}{du},$$

(11)

and $u$ is a single-valued parameter along the ray.
For the problem on hand, we may clearly regard the variations in \( \mu \) from its
mean value as small, so that the departures from linearity of the rays traversing
the medium may also be considered small. Under these conditions the general
equations governing the rays can be greatly simplified. Thus, letting \( u = s \) where
\( s \) is the distance measured along the undeflected linear trajectory and denoting
the displacements (considered small) along two fixed directions, \( x_1 \) and \( x_2 \) normal
to \( s \), by \( \xi(s) \) and \( \eta(s) \), and ignoring all quantities of order higher than the first in
equation (10), we find that

\[
\frac{d^2 \xi}{ds^2} = \left( \frac{\partial}{\partial x_1} \delta \mu \right)_s \quad \text{and} \quad \frac{d^2 \eta}{ds^2} = \left( \frac{\partial}{\partial x_2} \delta \mu \right)_s. \tag{12}
\]

It may be readily verified that equations (12) represent the proper generalizations
of the equation which Rayleigh has made the basis of his discussion ((1), eq. (7)).

One of the principal quantities in which we shall be interested is the angular
deflection of the ray from the linear trajectory. Let \( \psi_1 \) and \( \psi_2 \) denote the angles
which the ray makes with the direction \( s \) in planes containing \( x_1 \) and \( x_2 \),
respectively. Then in the approximation in which equations (12) are valid,

\[
\psi_1 = \frac{d \xi}{ds} \quad \text{and} \quad \psi_2 = \frac{d \eta}{ds}. \tag{13}
\]

The equations governing \( \psi_1 \) and \( \psi_2 \) are therefore

\[
\frac{d \psi_1}{ds} = \left( \frac{\partial}{\partial x_1} \delta \mu \right)_s \quad \text{and} \quad \frac{d \psi_2}{ds} = \left( \frac{\partial}{\partial x_2} \delta \mu \right)_s. \tag{14}
\]

4. The mean square deflection and the mean square displacement of a ray after
traversing a known distance in the medium.—Using equations (12) and (14) we can
readily express the mean square deflection and the mean square displacement of
a ray, after it has traversed a known distance in the medium, in terms of \( \bar{M}(r) \).
Thus considering the deflection, \( \psi_1(s) \) after the ray has traversed a distance \( s \), we
have (cf. eq. (14))

\[
\psi_1(s) = \int_0^s \left( \frac{\partial}{\partial x_1} \delta \mu \right)_s ds_1. \tag{15}
\]

From this equation it follows that

\[
\overline{\psi_1(s)} = 0. \tag{16}
\]

On the other hand,

\[
\overline{\psi_1^2(s)} = \int_0^s \int_0^s \left( \frac{\partial}{\partial x_1} \delta \mu \right)_s \left( \frac{\partial}{\partial x_1} \delta \mu \right)_s ds_1 ds_2. \tag{17}
\]

The averaged quantity under the integral sign is clearly the transverse correlation
(7). We can therefore write

\[
\overline{\psi_1^2(s)} = \overline{\delta^2 \mu} \int_0^s R(|s_1 - s_2|) ds_1 ds_2. \tag{18}
\]

By a known technique (cf. Chandrasekhar and Münch (9), eq. (18)) the double
integral in (18) can be reduced to the single integral

\[
\overline{\psi_1^2(s)} = 2 \overline{\delta^2 \mu} \int_0^s R(r)(s-r) dr. \tag{19}
\]

It is, of course, evident that \( \overline{\psi_2^2(s)} \) will be given by the same formula.
In order to obtain the mean square displacement, \( \bar{\xi}^2(s) \), we must integrate equation (15) once more. Thus

\[
\xi(s) = \int_0^s ds_1 \int_0^{s_1} dt_1 \left( \frac{\partial}{\partial x_1} \delta \mu \right)_{t_1}.
\]  

(20)

From this equation it follows that

\[
\bar{\xi}(s) = 0.
\]  

(21)

But

\[
\bar{\xi}^2(s) = \delta^2 s \int_0^s ds_1 \int_0^{s_1} dt_1 \int_0^{s_1} dt_2 R(\mid t_1 - t_2 \mid).
\]  

(22)

The quadruple integral representing \( \bar{\xi}^2(s) \) can also be reduced to a single integral by reductions analogous to those by which a similar quadruple integral was reduced by Chandrasekhar and Münch ((9), eq. (60)). We find:

\[
\bar{\xi}^2(s) = 2\delta s \int_0^s R(r) \left[ \frac{1}{3}(s^3 - r^3) - \frac{1}{3} r(s^3 - r^2) \right] dr.
\]  

(23)

Since there will be no correlation between the deflections \( \psi_1(s) \) and \( \psi_2(s) \) in the two directions \( x_1 \) and \( x_2 \), the mean square of the total deflection, \( \bar{\psi}^2(s) \), will be given by

\[
\bar{\psi}^2(s) = \psi_1^2(s) + \psi_2^2(s) = 2\bar{\psi}_1^2(s).
\]  

(24)

Similarly, the mean square of the radial displacement, \( \rho \), will be given by

\[
\bar{\rho}^2(s) = \bar{\xi}^2(s) + \bar{\eta}^2(s) = 2\bar{\xi}^2(s).
\]  

(25)

Finally, substituting for \( R(r) \) in terms of \( M(r) \) (eq. (8)) in equations (19) and (23), we obtain

\[
\bar{\psi}^2(s) = -4\delta^2 s \int_0^s \frac{M(r)}{r} (s - r) dr
\]  

(26)

and

\[
\bar{\rho}^2(s) = -4\delta^2 s \int_0^s \frac{M(r)}{r} \left[ \frac{1}{3}(s^3 - r^3) - \frac{1}{3} r(s^3 - r^2) \right] dr.
\]  

(27)

As we have already pointed out, the function \( M(r) \) introduces a micro-scale \( r_0 \) such that \( M(r) \) becomes negligible for \( r \gg r_0 \). Now in the applications, we shall be interested, principally, only in values of \( s \) which are large compared to \( r_0 \): thus, while \( s \) will be measured in units of 100 metres, \( r_0 \) is expected to be of the order of 10 cm. Under these circumstances we may approximate equations (26) and (27) to a high degree of accuracy by

\[
\bar{\psi}^2(s) = -4\delta^2 s \int_0^s \frac{M(r)}{r} dr
\]  

(28)

and

\[
\bar{\rho}^2(s) = -\frac{4}{3} \delta^2 s^3 \int_0^s \frac{M(r)}{r} dr.
\]  

(29)

Writing

\[
M(r) = M \left( \frac{r}{r_0} \right),
\]  

(30)

we can express equations (28) and (29) alternatively in the forms

\[
\bar{\psi}^2(s) = 4\alpha \delta^2 s \left( \frac{s}{r_0} \right)
\]  

(31)
and

\[ \bar{\rho}^2(s) = \frac{3}{8\pi^2} \delta^2 \mu \left( \frac{s^3}{\sigma_0} \right), \tag{32} \]

where

\[ \sigma = - \int_{0}^{\infty} \frac{dM(x)}{x} \left( x = \frac{r}{\sigma_0} \right). \tag{33} \]

For the case (9),

\[ M(x) = \exp(-x^2) \quad \text{and} \quad \alpha = \sqrt{\pi}. \tag{34} \]

Using the value \( \alpha = \sqrt{\pi} \), measuring \( s \) in \( 10^4 \) cm and \( \sigma_0 \) in \( 10 \) cm, we find that equations (31) and (32) take the numerical forms

\[ \sqrt{\psi^2(s)} = 1.74 \times 10^7 \left( \frac{s}{\sigma_0} \right)^{1/2} \sqrt{\delta^2 \mu} \text{ seconds of arc} \tag{35} \]

and

\[ \sqrt{\rho^2(s)} = 4.86 \times 10^5 \left( \frac{s}{\sigma_0} \right)^{1/2} \sqrt{\delta^2 \mu} \text{ cm}. \tag{36} \]

We shall return to the application of these formulae in Section 7.

5. The correlation in the deflections experienced by two parallel rays.—In Section 4 we obtained an expression (eq. (19)) for the mean square deflection, \( \psi^2(s) \), in a direction normal to the ray. We shall now consider the correlation in the deflections experienced by two parallel rays incident on the atmosphere and separated by a distance \( X \). There is no loss of generality in supposing that this separation, \( X \), is in the direction of \( x_1 \).

Let \( s_1 \) and \( s_2 \) measure the distances along the two (undeflected) linear trajectories. Then, clearly,

\[ \bar{\psi}_1(0; s)\psi_1(X; s) = \int_{0}^{s} \int_{s}^{s} \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial \mu} \right)_{s_1} \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial \mu} \right)_{s_2} ds_1 ds_2. \tag{37} \]

[It may be noted here that the validity of this formula requires that \( \sqrt{\psi^2(s)} \ll \sigma_0 \); we shall see that this is the case in the applications we shall consider.] Since \( s_1 \) and \( s_2 \) are measured along two lines parallel to the \( x_1 \)-axis and separated by a distance \( X \), the value of the correlation which occurs under the integral sign in (37) is not given by (7); instead, it is given by (cf. eq. (6))

\[ \left( \frac{M''}{r^2} - \frac{M'}{r^3} \right) X^2 + \frac{M'}{r} = \delta^2 \mu Q(r) \text{ (say)}, \tag{38} \]

where, now,

\[ r = [X^2 + (s_2 - s_1)^2]^{1/2}. \tag{39} \]

Hence,

\[ \bar{\psi}_1(0; s)\psi_1(X; s) = 2\delta^2 \mu \int_{0}^{s} \int_{s}^{s} ds_1 \int_{s_1}^{s} ds_2 Q(r). \tag{40} \]

Using \( r \) as the variable of integration in place of \( s_2 \), we can rewrite equation (40) in the form

\[ \bar{\psi}_1(0; s)\psi_1(X; s) = 2\delta^2 \mu \int_{0}^{s} \int_{X}^{\sqrt{(X^2 + (s - s_1)^2)}} dr Q(r) \frac{r}{(r^2 - X^2)^{1/2}}. \tag{41} \]

Letting \( y = s - s_1 \) and inverting the order of the integrations, we find that the integration over \( y \) can be performed and we are left with

\[ \bar{\psi}_1(0; s)\psi_1(X; s) = 2\delta^2 \mu \int_{X}^{\sqrt{(X^2 + y^2)}} Q(r) \frac{r dr}{(r^2 - X^2)^{1/2}}. \tag{42} \]
For \( s \gg r_0 \), equation (42) reduces to
\[
\psi_1(0; s) = \psi_1(X; s) = 2\delta^2 \mu r_0^2 \int_{r_0^2}^{r_1^2} \frac{rQ(r)}{(r^2 - X^2)^{1/2}} dr.
\]
(43)

The corresponding correlation coefficient is given by (cf. eq. (28))
\[
\frac{\psi_1(0; s)\psi_1(X; s)}{\psi_1^2(s)} = \int_X^{\infty} \left[ \frac{M'}{r} + X^2 \frac{d}{dr} \left( \frac{M'}{r} \right) \right] \frac{r dr}{(r^2 - X^2)^{1/2}}.
\]
(44)

For the case (9), formula (44) reduces to
\[
\frac{\psi_1(0; s)\psi_1(X; s)}{\psi_1^2(s)} = \left( 1 - 2 \frac{X^2}{r_0^2} \right) \exp \left( -\frac{X^2}{r_0^2} \right);
\]
(45)

this is the same as the longitudinal correlation,
\[
\left( \frac{\partial}{\partial x_1} \delta \mu \right) \left( \frac{\partial}{\partial x_1} \delta \mu \right)_{(0, 0, 0; x_1 = r_0, 0)} = -\delta^2 \mu \left[ \frac{M'}{r} + r \frac{d}{dr} \left( \frac{M'}{r} \right) \right],
\]
(46)

for \( M \) given by equation (9).

6. The retardation in phase on traversing the turbulent layer.—The change in phase as the ray traverses the medium will be given by \( 2\pi L(s)/\lambda \), where
\[
L(s) = \int_0^s \delta \mu(s_1) ds_1
\]
(47)

and \( \lambda \) is the wave-length of light considered. The mean square change in phase will, therefore, be determined by
\[
L^2(s) = \int_0^s \int_0^s \delta \mu(s_1) \delta \mu(s_2) ds_1 ds_2 = \delta^2 \mu \int_0^s \int_0^s M(s_1 - s_2) ds_1 ds_2.
\]
(48)

On reducing this integral over \( M \) in the usual fashion, we have (cf. eq. (19))
\[
\overline{L^2(s)} = 2\delta^2 \mu \int_0^s M(r)(s - r) dr;
\]
(49)

and for \( s \gg r_0 \), we have
\[
\overline{L^2(s)} = 2\delta^2 \mu s \int_0^s M(r) dr.
\]
(50)

For the case (9), the numerical form of equation (50) is
\[
\sqrt{\overline{L^2(s)}} = 4 \cdot 2 \times 10^3 \sqrt{\delta^2 \mu \text{ cm}},
\]
(51)

if \( s \) is measured in \( 10^4 \text{ cm} \) and \( r_0 \) in \( 10 \text{ cm} \). Also, in this case (cf. eqs. (31) and (34))
\[
\sqrt{\overline{L^2(s)}} = \frac{2}{r_0} \sqrt{L^2(s)};
\]
(52)

i.e. the root mean square change in path length is therefore one-half of the root mean square angular deflection times the micro-scale. This is a physically understandable result.

The correlation in the changes in phase experienced by two parallel rays incident on the turbulent layer and separated by a distance \( X \) can also be found. Thus, it will be determined by
\[
\overline{L(0; s)L(X; s)} = \int_0^s \int_0^s \delta \mu(s_1) \delta \mu(s_2) ds_1 ds_2,
\]
(53)
where \( s_1 \) and \( s_2 \) are measured along the two (undeflected) rays. Therefore

\[
\frac{L(0; s)L(X; s)}{L(0; s)L(X; s)} = \delta^2 \mu \int_0^s M(r) ds_1 ds_2,
\]

(54)

where

\[
r = \sqrt{X^2 + (s_2 - s_1)^2}.
\]

(55)

Reducing the integral on the right-hand side of equation (53) in the manner the integral in equation (40) was reduced in Section 5 (eqs. (40) to (43)), we find

\[
\frac{L(0; s)L(X; s)}{L(0; s)L(X; s)} = 2\delta^2 \mu \int_0^\infty \frac{r M(r)}{(r^2 - X^2)^{1/2}} dr.
\]

(56)

For \( s \gg r_0 \), this tends to

\[
\frac{L(0; s)L(X; s)}{L(0; s)L(X; s)} = 2\delta^2 \mu \int_0^\infty \frac{r M(r)}{(r^2 - X^2)^{1/2}} dr;
\]

(57)

and for the case (9), this gives (cf. eq. (50))

\[
\frac{L(0; s)L(X; s)}{L(0; s)L(X; s)} = \exp \left( -\frac{X^2}{r_0^2} \right) = M(X).
\]

(58)

Thus, in this case, the random changes in phase experienced by the different rays in passing through the medium directly reflect the variable refractive index in the medium.

7. Numerical illustrations.—For the sake of definiteness we shall suppose that the disturbed region in the atmosphere responsible for astronomical seeing is a horizontal turbulent layer of thickness \( d \). For a star observed at an angle \( \theta \) from the zenith, the length of the path traversed by a ray in its passage through the layer is

\[
s = d \sec \theta.
\]

(59)

Inserting this value of \( s \) in equations (35), (36) and (51), we have

\[
\sqrt{\rho^2} = 1.74 \times 10^7 \sqrt{\delta^2 \mu} \left( \frac{d}{r_0} \right)^{1/2} \text{ (sec } \theta \text{)}^{1/2} \text{ seconds of arc,}
\]

(60)

\[
\sqrt{\rho^2} = 4.86 \times 10^5 \sqrt{\delta^2 \mu} \left( \frac{d^2}{r_0^3} \right)^{1/2} \text{ (sec } \theta \text{)}^{3/2} \text{ cm,}
\]

(61)

and

\[
\sqrt{L^2} = 4.21 \times 10^8 \sqrt{\delta^2 \mu} (dr_0)^{1/2} \text{ (sec } \theta \text{)}^{1/2} \text{ cm,}
\]

(62)

where it may be recalled that \( s \) is measured in \( 10^4 \) cm and \( r_0 \) in 10 cm.

Now according to the measures of Strömgren (5) and Hansson and others (6), the angular radius of the “tremor disk” is about 0.75 seconds of arc under conditions of average seeing and near the zenith. Inserting this value for \( \sqrt{\rho^2} \) in (60), we find:

\[
\left( \frac{d}{r_0} \right)^{1/2} \sqrt{\delta^2 \mu} \approx 4.3 \times 10^{-8}.
\]

(63)

On the other hand, a number of independent lines of evidence lead one to suppose that

\[
r_0 \sim 10 \text{ cm}, \quad \text{i.e. I in the units adopted.}
\]

(64)

Hence, combining (62), (63) and (64), we infer that

\[
\sqrt{L^2} \approx 4.2 \times 10^2 \times 4.3 \times 10^{-8} = 1.8 \times 10^{-5} = 1800 \text{ A.}
\]

(65)

A change in path length of this order is entirely sufficient to account for scintillation in colour.
Again, for $r_0 = 1$ (i.e. 10 cm) equation (63) gives
\[
d^{1/2} \sqrt{\frac{\delta^2 \mu}{\nu}} \approx 4 \times 10^{-8}.
\] (66)

If we assume that the thickness of the turbulent layer is 100 metres (i.e. $d = 1$ in the units adopted) then we should conclude that
\[
\sqrt{\frac{\delta^2 \mu}{\nu}} \approx 4 \times 10^{-8}.
\] (67)

This corresponds to a root mean square fluctuation in density of amount $10^{-4}$ times the mean density; since a variation in density of this amount may be expected to occur over a distance of the order of $r_0$ (10 cm), we are postulating gradients of density of the order of $10^{-5}$ cm$^{-1}$ in units of the mean density; the assumption that density gradients of this order of magnitude occur in the turbulent layer does not seem unreasonable. Finally, we may note that for $d = r_0 = 1$ and $\sqrt{\frac{\delta^2 \mu}{\nu}} \approx 4 \times 10^{-8}$, equation (61) leads to $\sqrt{\rho^2} \approx 2 \times 10^{-4}$ cm; and this is indeed small compared to $r_0$ ($\sim 10$ cm).

From the preceding discussion it would appear that we can satisfactorily account for normal astronomical seeing by postulating a turbulent layer of a thickness of the order of 100 metres, a micro-scale of turbulence of the order of 10 cm, and a root mean square fluctuation in refractive index of the order of $4 \times 10^{-8}$. And it would follow that to explain bad seeing when $\sqrt{\rho^2}$ can become as large as 10 seconds of arc (or even larger, occasionally), we should suppose that $r_0$ is smaller and/or $d$ is larger. And finally, it would also appear that in astronomical seeing we have a means of exploring the spectrum of turbulence in the atmosphere at heights of the order of 3 or 4 km.

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References
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