THE THEORY OF THE FLUCTUATIONS IN BRIGHTNESS OF THE MILKY WAY. V

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ABSTRACT

In this paper a new picture of the distribution of the interstellar matter is proposed which may be considered as an alternative to (or a refinement of) the current picture, which visualizes the interstellar medium as consisting of a distribution of discrete clouds. On this new picture the distribution of density is considered to be continuous but exhibiting fluctuations of a statistical character. More particularly, it is assumed that the volume absorption coefficient \( \kappa(r) \) at any point in the medium can be written in the form \( \kappa(r) = \kappa_0[1 + \delta(r)] \), where \( \delta(r) \) is a chance variable whose mean expectancy is zero. The following additional assumptions regarding \( \delta(r) \) are made:

\[ \overline{\delta^2(r)} = \alpha^2 \quad \text{and} \quad \delta(r_1) \delta(r_2) = \alpha^2 R(|r_1 - r_2|) \]

where \( \alpha^2 \) is a constant throughout the medium and \( R \) represents the correlation function of the fluctuations \( \delta(r) \) at two different points; according to our assumption, this latter correlation depends only on the distance \( r \) between the two points considered. The correlation function \( R(r) \) defines a micro-scale, \( \tau_0 \), such that for \( r > \tau_0 \) the correlation rapidly becomes negligible. On the picture of continuous distribution, the parameters \( \alpha^2 \) and \( \tau_0 \) replace the parameters \( v \) and \( \tau_* \) (giving the number of clouds per unit distance and the optical thickness per cloud) on the discrete picture.

The various problems of stellar statistics, such as the fluctuations in the counts of extragalactic nebulae and in the brightness of the Milky Way, are rediscussed in terms of this new picture, and it is shown that the observations can be interpreted equally in terms of it. Indeed, it appears that \( 2\alpha^2 \int_0^{\infty} R(\tau/\tau_0) d\tau = 2\alpha^2 \int_0^{\tau} R(\xi) d\xi \) (where \( \tau \) denotes the optical thickness in \( \kappa_0 \) and \( \tau_0 \) is the micro-scale measured in optical thickness) replaces the optical thickness \( \tau_* \) of a single cloud on the discrete picture. However, the particular advantage of the continuous picture over the discrete picture is that the angular correlations in the brightness of the Milky Way in two neighboring directions can be discussed without any additional assumptions. This latter problem is discussed in some detail, and it is shown that, by combining the results of the fluctuations in the brightness itself with the results of the angular correlations, it is possible to infer the order of magnitudes of both \( \alpha^2 \) and \( \tau_0 \). The results of Pannekoek's survey of the southern Milky Way are analyzed with this in view, and it is found that \( \tau_0 \approx 0.01 \) and that \( \alpha^2 \approx 14 \). A value of \( \alpha^2 \approx 14 \) implies that the root mean square of the deviations in the density is about three to four times the mean density itself.

1. Introduction.—In the earlier papers of this series\(^1\) we examined certain basic problems in stellar statistics arising from the irregular distribution of the interstellar absorbing matter. In these papers the assumption was made that interstellar matter occurs in the form of discrete clouds; that these clouds are characterized by a certain average optical thickness \( \tau_* \); and, finally, that they occur with a certain average frequency, \( v \), per unit distance. This idealized picture of the interstellar medium was first advanced by Ambarzumian\(^2\) and has since been generally accepted.\(^3\) And during recent years there

\(^1\) Ap. J., 112, 380, 393, 1950; 114, 110, 1951; 115, 94, 1952; these papers will be referred to as "Papers I, II, III, and IV," respectively. After these papers were written, a paper by G. E. Rusakov was received (Uchenie Zapiski, Ser. Math., 13, 53, 1949) in which the solution found in Paper II for the integral equation governing the fluctuations in brightness of the Milky Way for the case when the system extends to infinity in the direction of the line of sight and all the clouds are equally transparent is also given. However, the solution for the case (also given in Paper II) when the transparency factor of the clouds is governed by the frequency function \( \psi(q) = (n + 1)q^n \), as well as all the other results contained in Papers I, III, and IV, has not, so far as we know, been anticipated by the Russian writers.

\(^2\) V. A. Ambarzumian and S. G. Gordeladse, Bull. Abastumani Obs., No. 2, p. 37, 1938; also V. A. Ambarzumian, Bull. Abastumani Obs., No. 4, p. 17, 1940.

have been many attempts, particularly by the Russian writers, to determine the average optical thickness \( r_* \) of a cloud. But it would appear that the only direct determination is that which follows from ascribing to the clouds an optical thickness which will account for the known coefficient of interstellar absorption of 1.5 mag. per kiloparsec. Thus, on the assumption that the stars and the clouds are randomly distributed and that the reason why early-type stars are generally associated with diffuse nebulae is due to the larger volumes of space they illuminate, it can be deduced that a line of sight will, on the average, intersect about six to seven clouds per kiloparsec. On the picture of discrete clouds, we must therefore suppose that the average optical thickness of a cloud is about 0.25. We believe that it may be stated with fairness that all the other attempts to determine \( r_* \) have served only to show that the other data used for this purpose, such as the fluctuations in brightness of the Milky Way and the counts of stars and extragalactic nebulae, are consistent with this value of \( r_* = 0.25 \) and the picture of discrete clouds. We wish to emphasize this point, since a tendency to argue in circles can be noted in the literature, in that confirmation for the picture of interstellar matter as occurring in the form of discrete clouds is sought in the data analyzed. In saying this, we are not suggesting that the picture of interstellar matter as occurring in discrete clouds may not be true; but we are suggesting that it may be worth while to inquire whether the observed fluctuations in the brightness of the Milky Way and the associated phenomena cannot also be analyzed in terms of a picture of interstellar matter as continuously distributed with a fluctuating (albeit a "wildly" fluctuating) density distribution. From one point of view such a picture may indeed recommend itself as being more consistent with the picture which regards the irregular distribution of interstellar matter as an exemplification of large-scale turbulence. And it should be remembered in this connection that so far no satisfactory explanation has been given from gas dynamics for the relative permanence of the interstellar gas clouds which must be attributed to them on the picture of discrete clouds. In this paper we shall therefore present an alternative picture of the interstellar medium as a fluctuating continuous distribution of matter and shall show how all the phenomena, such as the fluctuations in brightness of the Milky Way, etc., can be interpreted equally well on this basis. Moreover, it will appear that the correlations in the brightness of the Milky Way in different directions can be analyzed much more readily in terms of the continuous picture than in terms of the discrete picture, as has recently been attempted by Rusakov on the basis of a formula due to Ambarzumian.

2. A picture of the interstellar medium as a fluctuating continuous distribution of matter.

—We shall suppose that the coefficient of general absorption, \( \kappa \), and the density, \( \rho \), of the interstellar medium are continuous functions of position \( (r) \) but that they are subject to fluctuations. We shall accordingly write

\[
\kappa \rho (r) = \overline{\kappa \rho} [1 + \delta (r)],
\]

where the mean coefficient of volume absorption, \( \overline{\kappa \rho} \), is assumed to be constant throughout the medium and \( \delta (r) \) is a chance variable whose expectation value is zero:

\[
\delta (r) = 0.
\]

\[\text{4 Cf. J. M. Burgers, Kon. Ned. Akad. Wetensch., 49, 588, 1946. A referee, commenting on the present paper, has written: "There can be no doubt about the reality and individuality of these large clouds and complexes of clouds [like those in Taurus and Ophiuchus] nor about the relative sharpness of their boundaries. . . . We must probably conclude that there is the small scale irregular structure the authors use as a starting point, as well as a large cloud formation." We are in substantial agreement with this comment.}

\[\text{6 Op. cit.}

\[\text{6 V. A. Ambarzumian, Doklady Acad. Nauk Armenian S.S.R., 1, 9, 1944.} \]
Regarding $\delta(r)$ we shall make the following additional assumptions:

$$\overline{\delta^2 (r)} = a^2,$$  \hspace{1cm} (3)

and

$$\delta (r_1) \delta (r_2) = a^2 R (|r_1 - r_2|),$$  \hspace{1cm} (4)

where $a^2$ is a constant independent of position and $R$ represents the correlation coefficient of the fluctuations $\delta(r)$ at two different points; according to equation (4), the correlation coefficient is assumed to depend only on the distance between the two points considered.

The foregoing assumptions concerning the fluctuations in $\kappa_0$ will be valid if the conditions in the interstellar medium are those expected to prevail in homogeneous isotropic turbulence.

It follows from definitions (3) and (4) that

$$R (0) = 1.$$  \hspace{1cm} (5)

It is also clear that we must require

$$R (r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty.$$  \hspace{1cm} (6)

Now the correlation function, $R(r)$, defines a scale of length; this may be defined, for example, as the distance $r_0$ at which $R$ decreases to the value $1/e$. On the picture of a continuously distributed fluctuating density of matter, the two basic parameters are, therefore, $a^2$, which is a measure of $\delta^2 \rho / \rho^2$ (cf. eq. [25] below) and $r_0$, which is a measure of the “micro-scale” of the turbulent eddies. The parameters $a^2$ and $r_0$ of the present picture replace the parameters $v$ and $r_*$ of the picture of discrete clouds.

3. The equivalent optical thickness of a cloud.—The two pictures of the interstellar medium described, respectively, in terms of the parameters $v$ and $r_*$ and $a^2$ and $r_0$ can be formally related by considering the optical thickness,

$$\tau (s) = \int_0^s \kappa \rho d s,$$  \hspace{1cm} (7)

on the two pictures.

On the picture of discrete clouds we have

$$\overline{\tau} (s) = \overline{n} (s) \tau_* \quad \text{and} \quad \overline{\tau^2} (s) = \overline{n^2} (s) \tau_*^2,$$  \hspace{1cm} (8)

where $\overline{n}(s)$ and $\overline{n^2}(s)$ are the mean and the mean square of the number of clouds to be expected in a distance $s$. For the assumed uniform Poisson distribution of the clouds, we have

$$\overline{n} (s) = \nu s \quad \text{and} \quad \overline{n^2} (s) = \nu^2 s^2 + \nu s.$$  \hspace{1cm} (9)

Hence

$$\overline{\tau} (s) = \nu s \tau_* \quad \text{and} \quad \overline{\tau^2} (s) = (\nu^2 s^2 + \nu s) \tau_*^2 = \overline{\tau^2} (s) + \overline{\tau} (s) \tau_*.$$  \hspace{1cm} (10)

Thus we have the relation

$$\overline{\tau} (s) \tau_* = [\overline{\tau^2} (s) - \overline{\tau^2} (s)] .$$  \hspace{1cm} (11)

Considering, next, $\overline{\tau}(s)$ and $\overline{\tau^2}(s)$ on the picture of continuous distribution, we have (cf. eqs. [1] and [2])

$$\overline{\tau} (s) = \kappa \rho \int_0^s [1 + \overline{\delta} (s)] d s = \kappa \rho s$$  \hspace{1cm} (12)

$^7$ The length $r_0$ we have defined is clearly equivalent to Taylor’s micro-scale in his theory of isotropic turbulence (Proc. R. Soc. London, A, 151, 421, 1935).
and
\[
\tau^2(s) = \kappa \rho^2 \int_0^s \int_0^s (1 + \delta(x))(1 + \delta(y)) \, dx \, dy = (\kappa \rho s)^2 + \kappa \rho^2 \int_0^s \int_0^s \delta(x) \delta(y) \, dx \, dy.
\] (13)

It is now convenient to introduce the optical thickness, \( \tau \), in the mean volume absorption coefficient \( \kappa \rho \). Thus, letting \( dt_1 = \kappa \rho dx \) and \( dt_2 = \kappa \rho dy \), we can rewrite the expression for \( \tau^2(s) \) in the form
\[
\tau^2(s) = \tau^2(s) + a^2 \int_0^s \int_0^s \delta(t_1) \delta(t_2) \, dt_1 \, dt_2,
\] (14)

where we have further substituted for \( \delta(t_1) \delta(t_2) \) in terms of the correlation coefficient \( R \) (cf. eq. [4]). Since the integrand is symmetrical in \( t_1 \) and \( t_2 \), an alternative form of equation (14) is
\[
\tau^2(s) = \tau^2(s) + 2a^2 \int_0^s \int_0^s \delta(t_1) \delta(t_2) R(t_2 - t_1).
\] (15)

Letting \( t = t_2 - t_1 \) as the variable of integration in place of \( t_2 \) and inverting the order of the integrations, we have
\[
\tau^2(s) - \tau^2(s) = 2a^2 \int_0^s \int_0^s \delta(t_1) \delta(t_2) R(t_2 - t_1) \, dt_1.
\] (16)

or
\[
\tau^2(s) - \tau^2(s) = 2a^2 \int_0^s \int_0^s \delta(t_1) \delta(t_2) R(t_2 - t_1) \, dt_1.
\] (17)

It may be noted here that in deriving equation (17) we have incidentally established the following relation:
\[
\int_0^s \int_0^s \delta(t_1) \delta(t_2) \, dt_1 \, dt_2 = 2a^2 \int_0^s \tau(t) R(t) \, dt.
\] (18)

We shall find this relation useful in our subsequent work.

As we have already remarked, the correlation function \( R(\tau) \) introduces a scale of length \( r_0 \). If \( \tau_0 \) denotes the corresponding optical thickness (i.e., \( \kappa \rho r_0 \)), we can write
\[
R(\tau) = R(\tau/\tau_0) = R(\xi) \quad \text{(say)}.
\] (19)

Expressing \( R \) in terms of \( \xi \), we can rewrite equation (17) in the form
\[
\tau^2(s) - \tau^2(s) = 2a^2 \tau_0 \int_0^s \tau(t) R(\xi) \, d\xi.
\] (20)

For \( \tau(s) \gg \tau_0 \), the foregoing equation becomes
\[
\tau^2(s) - \tau^2(s) \to 2a^2 \tau_0 \tau(s) \int_0^\infty R(\xi) \, d\xi \quad [\tau(s) \gg \tau_0].
\] (21)

Hence
\[
[\tau^2(s) - \tau^2(s)] \to 2a^2 \tau(s) \tau_0 R_0 \quad [\tau(s) \gg \tau_0],
\] (22)

where we have written
\[
R_0 = \int_0^\infty R(\xi) \, d\xi.
\] (23)
We shall see that, in the practical problems we shall consider, \( \tau_0 \sim 0.01 \) (cf. § 8); accordingly, in these cases the asymptotic relation (22) will provide a good approximation even for \( \tau(s) \sim 1 \).

From equations (11) and (22) it would appear that, on the picture of continuous distribution, the quantity \( 2a^2\tau_0R_0 \) will replace \( \tau* \) of the discrete picture. We shall see that this is actually the case. Hence

\[
\tau* = 2a^2\tau_0R_0
\]

provides a relation in terms of which we may pass from the picture of continuous distribution to the picture of discrete clouds. We may therefore say that \( 2a^2\tau_0R_0 \) gives the equivalent optical thickness of a cloud on the discrete picture. It is to be particularly noted that the equivalent optical thickness of a cloud is not \( \tau_0 \); in other words the micro-scale should not be confused with the macro-scale!

In practice we may suppose that the mass-absorption coefficient, \( \kappa \), is constant throughout the medium. In that case

\[
a^2 = \frac{\overline{\delta^2 \rho}}{\overline{\rho}^2},
\]

where \( \overline{\delta^2 \rho} \) is the mean-square deviation in the density and \( \overline{\rho} \) is the mean density. The expression for the equivalent optical thickness, then, is

\[
\tau* = 2 \frac{\overline{\delta^2 \rho}}{\overline{\rho}^2} \tau_0 R_0 .
\]

In our further work we shall sometimes find it convenient to specify \( R(\tau) \) more explicitly than by its micro-scale, \( \tau_0 \). When the need for this arises, we shall adopt either of the following laws:

\[
R(\tau) = e^{-\tau/\tau_0} \quad \text{or} \quad R(\tau) = e^{-\tau^2/\tau_0^2}.
\]

For these two cases,

\[
\tau* = 2a^2 \tau_0 \quad \text{and} \quad \tau* = a^2 \tau_0 \sqrt{\pi} ,
\]

respectively. However, it should be emphasized that it need not be required of the correlation function \( R \) that it be a monotonically decreasing function. Indeed, in the theory of turbulence, one often encounters correlation functions which change sign, pass through a minimum, and then tend to zero as the argument tends to infinity.

4. The fluctuations in the counts of extragalactic nebulae.—The earliest determination of \( \tau* \) on the picture of discrete clouds is that due to Ambarzumian\(^\text{10}\) based on the fluctuations exhibited by the counts of extragalactic nebulae. Ambarzumian showed that, on the picture of discrete clouds, the dispersion \( \sigma_m^2 \) in the number of nebulae \( N_m \) per square degree brighter than a given apparent magnitude \( m \) should exhibit the following dependence on the galactic latitude \( \beta \):

\[
\sigma_m^2 = \frac{N_m^2}{N_m^2} - 1 = 2.25 \tau* \tau_1 \cosec \beta ,
\]

where \( \tau_1 \) is the optical thickness of the interstellar absorbing layer perpendicular to the

\(^8\) This essentially means that in the interstellar medium it is the fluctuations in the density which are important and which outweigh any variations there may be in the composition.


\(^{10}\) \textit{Bull. Abastumani Obs.}, No. 4, p. 17, 1940; see also his later discussion in \textit{Trans. I.A.U.}, 7, 452–455, 1950.
galactic plane. Ambarzumian satisfied himself that the observed dispersion did show a linear dependence on cosec $\beta$ as required by equation (29) and, further, that its application led to a value of $\tau_*$ in the range

$$0.20 \leq \tau_* \leq 0.30 . \quad (30)$$

Some doubts have recently been expressed for believing that there is any real observational evidence for the linear dependence of $\sigma^2_m$ on cosec $\beta$; nevertheless, it seems desirable to derive an expression for $\sigma^2_m$ on the picture of continuous distribution.

In the absence of interstellar absorption, it is expected that the counted numbers of extragalactic nebulae, $N_m$, must show the proportionality

$$N_m = 10^{0.6 m} N_0 , \quad (31)$$

where $N_0$ is a constant. But if there is absorbing matter of total optical thickness $\tau(s)$ in the line of sight, then we should have, instead,

$$N_m = 10^{0.6 m} N_0 e^{-1.5 \tau(s)} . \quad (32)$$

On the picture of the continuous but fluctuating distribution of interstellar matter (cf. eq. [12])

$$\tau(s) = \int_0^s \{ 1 + \delta(t) \} dt = \tau + \int_0^s \delta(t) dt , \quad (33)$$

where $\tau = \tau(s)$. With the foregoing expression for $\tau(s)$, equation (32) becomes

$$N_m = \mathcal{R} \exp \left\{ -1.5 \int_0^s \delta(t) dt \right\} , \quad (34)$$

where

$$\mathcal{R} = 10^{0.6 m} N_0 e^{-1.5 \tau} . \quad (35)$$

We shall assume that the exponential term in equation (34) containing the chance variable $\delta(t)$ can be expanded in a power series and that it is sufficient to retain only the first three terms in the expansion. Thus

$$\exp \left\{ -1.5 \int_0^s \delta(t) dt \right\} = 1 - 1.5 \int_0^s \delta(t) dt + 0.25 \int_0^s \delta(t_1) \delta(t_2) dt_1 dt_2 . \quad (36)$$

We shall see that the validity of this approximation requires that the equivalent optical thickness $\tau_* \ll 1$ and that quantities of order $\tau_*^2$ can be neglected.

In the framework of approximation (36),

$$\overline{N}_m = \mathcal{R} \left[ 1 + 0.25 \int_0^s \int_0^s \delta(t_1) \delta(t_2) dt_1 dt_2 \right] , \quad (37)$$

and

$$\overline{N}_m^2 = \mathcal{R}^2 \left[ 1 + 4.5 \int_0^s \int_0^s \delta(t_1) \delta(t_2) dt_1 dt_2 \right] . \quad (38)$$

In the same approximation,

$$\frac{\overline{N}_m^2}{\overline{N}_m} = 1 + 2.25 \int_0^s \int_0^s \delta(t_1) \delta(t_2) dt_1 dt_2 ; \quad (39)$$

or, using the result expressed by equation (18), we have

$$\sigma_m^2 = \frac{N_m^2}{N_m^2} - 1 = 4.5 a^2 \int_0^r (\tau - t) R(t) \, dt.$$  \hfill (40)

Now if $\tau$ refers to the average optical thickness of the absorbing layer in the direction of galactic latitude $\beta$, then

$$\tau = \tau_1 \cosec \beta,$$  \hfill (41)

where $\tau_1$ is the corresponding optical thickness perpendicular to the galactic plane. Also, if $\tau_0$ is the micro-scale defined by the correlation function $R(\tau)$, we can rewrite equation (40) in the form

$$\sigma_m^2 = 4.5 a^2 \tau_0 \int_{\tau_1}^{\cosec \beta / \tau_0} (\tau_1 \cosec \beta - \tau_0 \xi) R(\xi) \, d\xi.$$  \hfill (42)

Now it is known that $\tau_1 \sim 0.25$; also we shall see that $\tau_0 \sim 0.01$. Hence, to a sufficient accuracy (cf. eqs. [20], [21], and [23]) we may write

$$\sigma_m^2 \approx 2.25 \left(2a^2 \tau_0 R_0\right) \tau_1 \cosec \beta.$$  \hfill (43)

Thus, on the picture of continuous distribution, the dispersion in the counts is given by a formula of exactly the same type as on the picture of discrete clouds. Indeed, we see that, in agreement with what was stated in § 3, $2a^2 \tau_0 R_0$ replaces $\tau_*$. The earlier estimates of $\tau_*$ now imply that, on our present picture,

$$0.1 \leq \frac{\delta \rho}{\rho^2} \tau_0 R_0 \leq 0.15.$$  \hfill (44)

5. The fluctuations in the brightness of the Milky Way.—We shall now return to the problem which is the principal concern of this series of papers, namely, the fluctuations in the brightness of the Milky Way. On the picture of discrete clouds, the brightness, $u$, of the Milky Way measured in a certain suitably chosen unit is given by (cf. Paper I, eq. [4])

$$u = \int_0^\infty \prod_{i=1}^n q_i r^i \, dr,$$  \hfill (45)

where $r$ is the linear distance also measured in a suitable unit. In writing equation (45), we have assumed that the system extends to infinity in the direction of the line of sight. (In this paper we shall restrict ourselves to this case.) From equation (45) it follows that (cf. Paper II, eq. [2])

$$\bar{u} = \frac{1}{1 - q_1} \quad \text{and} \quad \frac{\sigma^2}{u^2} = \frac{2}{(1 - q_1)(1 - q_2)},$$  \hfill (46)

where $q_1 = \bar{q}$ and $q_2 = \bar{q}^2$. If the clouds are all assumed to be equally transparent, then $q_1 = q$ and $q_2 = q^2$, and we deduce from equations (46) that

$$\bar{u}^2 = \frac{2}{1 + \bar{u}^2}.$$  \hfill (47)

Now $q$ is related to the optical thickness $\tau_*$ by

$$q = e^{-\tau_*} = 1 - \tau_* + O(\tau_*^2).$$  \hfill (48)
Hence to $O(\tau^2)$,

$$\frac{u^2}{u^2} = 1 + \frac{1}{2} \tau_s. \quad (49)$$

Returning to the picture of continuous distribution, we may first observe that the expression for the brightness analogous to equation (45) is now

$$u = \int_0^\infty e^{-\tau} d\tau,$$

where, according to equation (33),

$$\tau(s) = \tau + \int_0^s \delta(t) dt. \quad (50)$$

Hence

$$u = \int_0^\infty d\tau e^{-\tau} \exp \left\{ - \int_0^\tau \delta(t) dt \right\}. \quad (51)$$

On the scheme of approximation represented by equation (36), we have

$$u = \int_0^\infty d\tau e^{-\tau} \left[ 1 - \int_0^\tau \delta(t) dt + \frac{1}{2} \int_0^\tau \int_0^\tau \delta(t_1) \delta(t_2) dt_1 dt_2 \right]. \quad (52)$$

We shall adopt this approximation consistently in all our future work.

From equation (53) it readily follows that (cf. eq. [18])

$$\bar{u} = 1 + a^2 \int_0^\infty d\tau e^{-\tau} \int_0^\tau d\tau' (\tau - \tau') R(\tau'). \quad (53)$$

Inverting the order of the integrations in equation (54), we have

$$\bar{u} = 1 + a^2 \int_0^\infty dR(\tau') \int_0^\infty d\tau (\tau - \tau') e^{-\tau}, \quad (55)$$

or

$$\bar{u} = 1 + a^2 \int_0^\infty e^{-\tau} R(\tau') d\tau'. \quad (56)$$

Considering $u^2$ next, we have

$$u^2 = \int_0^\infty \int_0^\infty e^{-\tau_1 - \tau_2} \left[ 1 - \int_0^{\tau_1} \delta(t_1) dt_1 + \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_1} \delta(t_1) \delta(t_2) dt_1 dt_2 \right]$$

$$\times \left[ 1 - \int_0^{\tau_2} \delta(t_2) dt_2 + \frac{1}{2} \int_0^{\tau_2} \int_0^{\tau_2} \delta(t_1) \delta(t_2) dt_1 dt_2 \right] d\tau_1 d\tau_2. \quad (57)$$

Expanding the foregoing and retaining only the terms which are consistent with the adopted approximation, we have

$$\bar{u^2} = 1 + \int_0^\infty \int_0^\infty e^{-\tau_1 - \tau_2} \left[ \int_0^{\tau_1} \int_0^{\tau_1} \delta(t_1) \delta(t_2) dt_1 dt_2 \right]$$

$$+ \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_1} \delta(t_1) \delta(t_2) dt_1 dt_2 + \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_1} \delta(t_1) \delta(t_2) dt_1 dt_2 d\tau_1 d\tau_2 \quad (58)$$

Of the three double integrals which occur inside the brackets on the right-hand side of equation (58), only the first requires some consideration; the other two lead to terms of
the same type which we have already encountered in the evaluation of \( \bar{u} \). We thus have

\[
\bar{u}^2 = 1 + 2a^2 \int_0^\infty e^{-t} R(t) \, dt + a^2 \int_0^\infty \int_0^\infty d\tau_1 d\tau_2 e^{-\tau_1 - \tau_2} \int_{\tau_1}^{\tau_2} dt_1 R(\mid t_1 - t_2 \mid) .
\]  

(59)

The quadruple integral on the right-hand side of equation (59) can be reduced to a single integral by successively rearranging the orders of the integrations in the following manner:

\[
\int_0^\infty \int_0^\infty d\tau_1 d\tau_2 e^{-\tau_1 - \tau_2} \int_{\tau_1}^{\tau_2} dt_1 R(\mid t_1 - t_2 \mid) dt_2 d\tau_1 = \int_0^\infty dt_2 e^{-t_2} \int_0^{t_2} d\tau_1 e^{-\tau_1} \int_{t_1}^{\tau_2} dt_1 R(\mid t_1 - t_2 \mid) d\tau_1 e^{-\tau_1} = \int_0^\infty dt_1 e^{-t_1} \int_0^{t_1} d\tau_2 e^{-\tau_2} \int_{t_1}^{\tau_2} dt_2 R(\mid t_1 - t_2 \mid) dt_2 d\tau_2 = \int_0^\infty dt_1 e^{-t_1} \int_0^{t_1} d\tau_2 e^{-\tau_2} \int_{t_1}^{\tau_2} dt_2 e^{-t_2} R(t_2 - t_1) d\tau_2 dt_2 e^{-t_2} = 2 \int_0^\infty dt_1 e^{-t_1} \int_0^{t_1} d\tau_2 e^{-t_2} (t_2 - t_1) = \int_0^\infty e^{-t} R(t) \, dt.
\]

(60)

Using this result in equation (59), we have

\[
\bar{u}^2 = 1 + 3a^2 \int_0^\infty e^{-t} R(t) \, dt .
\]

(61)

Now, combining equations (56) and (61), we have, to the same order of accuracy,

\[
\frac{\bar{u}^2}{\bar{u}_0^2} = 1 + a^2 \int_0^\infty e^{-t} R(t) \, dt .
\]

(62)

Writing (cf. eq. [19])

\[
R(t) = R(t/\tau_0) = R(\xi) ,
\]

(63)

we have

\[
\frac{\bar{u}^2}{\bar{u}_0^2} = 1 + a^2 \tau_0 \int_0^\infty e^{-t_0 R(\xi)} d\xi .
\]

(64)

For \( \tau_0 \ll 1 \), equation (64) reduces to

\[
\frac{\bar{u}^2}{\bar{u}_0^2} = 1 + a^2 \tau_0 R_0 \quad (\tau_0 \ll 1) .
\]

(65)

Comparing this last equation with the corresponding equation on the picture of discrete clouds (eq. [49]), we again see that, on the continuous picture, \( 2a^2 \tau_0 R_0 \) replaces the
optical thickness $\tau_*$ of a single cloud on the discrete picture. It is also evident now that the adopted scheme of approximation is one which is correct to the first order in $\tau_*$.

6. The angular correlations in the brightness of the Milky Way.—A problem of considerable interest in the theory of the fluctuations in brightness of the Milky Way is the angular correlation, $\mu_{\varphi}$, of the brightness in two directions, $s_1$ and $s_2$, which are inclined at an angle $\varphi$ to each other. This problem has been considered by Ambarzumian on the picture of discrete clouds. In order to evaluate the required correlation, Ambarzumian introduced the probability $Q(s_i)$ that a cloud intersecting the line of sight in the direction $s_1$, somewhere between 0 and $s_1$, will also intersect the line of sight in the direction $s_2$ and contribute (by the full amount) to the absorption in that direction. In terms of this probability, he showed that

$$
\mu_{\varphi} = \frac{2}{1 - q} \int_0^\infty ds_1 \exp \left\{ - s_1 \left[ 1 - q \right] \left[ 2 - Q(s_1) \right] \left[ 1 - q \right] \right\}. \tag{66}
$$

We should, of course, expect that, as $\varphi \to 0$, $Q(s_1) \to 1$ independently of $s_1$; and Ambarzumian actually assumed that

$$
1 - Q(s_1) = \frac{\varphi s_1}{\nu D} + O(\varphi^2), \tag{67}
$$

where $D$ denotes the linear diameter of a single cloud. (The occurrence of the factor $\nu$ in eq. [67] is due to the fact that $s_1$ measures the linear distance in the line of sight in the unit $1/\nu$.) In using a formula of type (67), we are essentially assuming that the clouds are uniform plane disks with a diameter $D$; otherwise, the absorption by the cloud will not be the same for all normal intersections of the cloud by a line of sight.\footnote{In any case it is evident that, on the picture of discrete clouds, the angular correlation cannot be discussed without some explicit assumption concerning the shapes of the clouds.} It would appear, then, that the assumption underlining equation (67) is highly restrictive. In any case, with assumption (67), the integral defining $\mu_{\varphi}$ (eq. [66]) can be evaluated and we obtain

$$
\mu_{\varphi} = \frac{2}{(1 - q)(1 - q^2)} \left[ 1 - \frac{2\varphi}{\nu D (1 + q)^2} \right], \tag{68}
$$

or (cf. eq. [46])

$$
\mu_{\varphi} = u^2 \left[ 1 - \frac{2\varphi}{\nu D (1 + q)^2} \right]. \tag{69}
$$

On the basis of formula (69), Rusakov has recently analyzed the observed angular correlations in brightness of the Milky Way, with the object of estimating the linear diameter of the interstellar clouds. But, in view of the restrictive character of the assumptions which underlie the derivation of formula (69), it is difficult to assess the meaning we can attach to the derived diameters. Indeed, we shall see that the analysis of this problem of angular correlations on the picture of continuous distribution does not lead to a formula of type (69). And this is a serious objection, since the evaluation of $\mu_{\varphi}$ on the picture of continuous distribution requires no assumptions beyond those already made for evaluating $\mu$ and $u^2$.

On the picture of the continuous distribution, the brightnesses $u$ and $u_{\varphi}$ in two directions $s_1$ and $s_2$, inclined at an angle $\varphi$ to each other, are given by

$$
u = \int_0^\infty d\tau_1 e^{-\tau_1} \exp \left\{ - \int_0^{\tau_1} \delta(t_1) dt_1 \right\}
$$

and

$$
u_{\varphi} = \int_0^\infty d\tau_2 e^{-\tau_2} \exp \left\{ - \int_0^{\tau_2} \delta(t_2) dt_2 \right\}, \tag{70}
$$

In any case it is evident that, on the picture of discrete clouds, the angular correlation cannot be discussed without some explicit assumption concerning the shapes of the clouds.
where we have written \( t_1 \) and \( t_2 \) for the arguments of \( \delta \) to emphasize that the fluctuations in two different directions are considered and that therefore \(|t_1 - t_2| \neq |t_1 - t_2|\).

On the scheme of approximation represented by equation (36), we have

\[
\overline{uu}_\phi = \int_0^\infty \int_0^\infty e^{-\tau_1 - \tau_2} \left[ 1 - \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 \right] \\
\times \left[ 1 - \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 + \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 \right] \, d\tau_1 d\tau_2.
\]

(71)

On expanding and retaining only terms which are consistent with our approximation, we obtain

\[
\overline{uu}_\phi = 1 + \int_0^\infty \int_0^\infty \tau_1 d\tau_2 e^{-\tau_1 - \tau_2} \left[ \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 \right] \\
+ \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 + \frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 \right].
\]

(72)

The second and the third of the three double integrals which occur inside the brackets on the right-hand side of equation (72) are the same as those which occurred in the evaluation of \( \overline{u^2} \) (cf. eq. [58]). We can therefore write (cf. eq. [59])

\[
\overline{uu}_\phi = 1 + 2a^2 \int_0^\infty e^{-\tau} R (t) \, dt \\
+ a^2 \int_0^\infty \int_0^\infty \tau_1 d\tau_2 e^{-\tau_1 - \tau_2} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 R (|t_1 - t_2|),
\]

(73)

where it should be noted that now

\[
|t_1 - t_2| = \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos \varphi}.
\]

(74)

However, in spite of this last circumstance, the sequence of transformations (60), except for the last line, applies as well to the quadruple integral in equation (73). Hence we may write

\[
\overline{uu}_\phi = 1 + 2a^2 \int_0^\infty e^{-\tau} R (t) \, dt + 2a^2 \int_0^\infty \int_0^\infty d\tau_1 e^{-\tau_1} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 e^{-\tau_2} R (|t_1 - t_2|).
\]

(75)

Since (eq. [56])

\[
\overline{u} = 1 + a^2 \int_0^\infty e^{-\tau} R (t) \, dt,
\]

(76)

we can combine equations (75) and (76) (in our present approximation) to give

\[
\sigma^2_\phi = \frac{\overline{uu}_\phi}{\overline{u}^2} - 1 = 2a^2 \int_0^\infty dt_1 e^{-\tau_1} \int_0^{\tau_1} \int_0^{\tau_2} \delta (t_1) \delta (t_2) \, dt_1 dt_2 e^{-\tau_2} R (|t_1 - t_2|).
\]

(77)

Now, letting

\[
t = \sqrt{(t_1^2 + t_2^2 - 2t_1t_2 \cos \varphi)}
\]

(78)

as the variable of integration instead of \( t_2 \), we find

\[
\sigma^2_\phi = 2a^2 \int_0^\infty dt_1 \int_{2t_1 \sin (\varphi/2)}^{\infty} dR (t) \exp \{- 2t \cos^2 \frac{\varphi}{2} + \sqrt{(t_2^2 - t_1^2 \sin^2 \varphi)} \} \\
\times \frac{t}{\sqrt{(t_2^2 - t_1^2 \sin^2 \varphi)}}.
\]

(79)
Inverting the order of the integrations and introducing in place of \( t_1 \) a new variable of integration, \( \psi \), defined by the transformation

\[
t \sin \left( \psi - \frac{1}{2} \varphi \right) = t_1 \sin \varphi ,
\]

we obtain, after some further reductions,

\[
\sigma^2 = \frac{2a^2}{\sin \varphi} \int_0^\infty dt R(t) \int_{\varphi/2}^{\pi/2} d\psi \exp \left\{ -t \frac{\sin \psi}{\sin \frac{1}{2} \varphi} \right\} \cdot \sin \frac{\varphi}{2} \frac{d\psi}{\sin \varphi}.
\]

Alternative forms of this formula are:

\[
\sigma^2 = \frac{a^2}{\epsilon \sqrt{1 - \epsilon^2}} \int_0^\infty dt R(t) \int_0^1 \frac{dy}{\sqrt{1 - \epsilon^2}} e^{-t \psi / \epsilon},
\]

where

\[
\epsilon = \sin \frac{1}{2} \varphi.
\]

When \( \varphi \to 0 \),

\[
\frac{2}{\sin \varphi} \int_{\varphi/2}^{\pi/2} d\psi \exp \left\{ -t \frac{\sin \psi}{\sin \frac{1}{2} \varphi} \right\} - \frac{2}{\varphi} \int_{\varphi/2}^{\pi/2} e^{-2t \psi / \epsilon} d\psi = \frac{1}{t} e^{-t};
\]

and equation (81) reduces to

\[
\sigma^2_0 = a^2 \int_0^\infty e^{-t} R(t) dt,
\]

in agreement with the result obtained in § 5 (eq. [62]). For \( \varphi = \pi \) we can obtain a similar expression. For this purpose it is convenient to go back to equation (75), since, for \( \varphi = \pi, |t_2 - t_1| = t_1 + t_2 \), and the equation becomes

\[
\sigma^2 = 2a^2 \int_0^\infty dt_1 e^{-t_1} \int_{t_1}^\infty dt_2 e^{-t_2} R(t_2 + t_1).
\]

Letting \( t = t_1 + t_2 \) as the variable of integration in place of \( t_2 \) and inverting the order of the integrations, we have

\[
\sigma^2 = 2a^2 \int_0^\infty dt e^{-t} R(t) \int_0^{t/2} dt_1 = a^2 \int_0^\infty e^{-t} R(t) dt = a^2 \int_0^\infty R(\xi) \xi d\xi.
\]

Thus \( \sigma^2_0 \) is of a higher order of smallness than the quantities we have retained; accordingly, we must consider that it vanishes in the adopted scheme of approximation.

It does not appear that the integral over \( \psi \) in equation (81) (or over \( \gamma \) in the first form of eq. [82]) can be evaluated or expressed in terms of known functions. It does not also appear that a series expansion having a useful range of validity can be developed for \( \varphi \to 0 \). This will become apparent when we obtain explicit expressions for \( \sigma^2_0 \) for two special forms of \( R(t) \). But some light may be thrown on this peculiar behavior of \( \sigma^2_0 \) for \( \varphi \to 0 \) by the following analysis:

Considering the moment

\[
\int_0^\pi \sigma^2 \cos^2 \frac{1}{2} \varphi d\varphi = 2a^2 \int_0^\infty dt R(t) \int_0^\pi d\varphi \frac{\cos^2 \frac{1}{2} \varphi}{\sin \varphi} \int_{\varphi/2}^{\pi/2} d\psi \exp \left\{ -t \frac{\sin \psi}{\sin \frac{1}{2} \varphi} \right\},
\]

\[
\sigma^2_0 = \int_0^\infty dt R(t) \int_0^{t/2} dt_1 = \int_0^\infty dt R(t) \int_0^{t_2} dt_1 e^{-t_1}.
\]

\[
\sigma^2 = 2a^2 \int_0^\infty dt R(t) \int_0^{t/2} dt_1 = \int_0^\infty dt R(t) \int_0^{t_2} dt_1 e^{-t_1}.
\]

\[
\sigma^2_0 = \int_0^\infty dt R(t) \int_0^{t/2} dt_1 = \int_0^\infty dt R(t) \int_0^{t_2} dt_1 e^{-t_1}.
\]
BRIGHTNESS OF MILKY WAY. V

we can carry out the integrations over \( \psi \) and \( \varphi \). Thus

\[
2 \int_0^\pi d \varphi \frac{\cos^2 \frac{1}{2} \varphi}{\sin \varphi} \int_{\varphi/2}^{\pi/2} d \psi \exp \left\{ - t \frac{\sin \psi}{\sin \frac{\varphi}{2}} \right\} = 2 \int_0^{\pi/2} \frac{d \chi}{\chi} \int_{\sin^{-1} \chi}^{\pi/2} d \psi e^{-t \sin \psi/\pi}
\]

\[
= 2 \int_0^{\pi/2} d \psi \int_{\csc \psi}^{\infty} d y \frac{1}{y} e^{-t y \sin \psi} = \pi E_1 (t),
\]

where \( E_1 (t) \) denotes the first exponential integral. Thus

\[
\int_0^\pi \sigma_\varphi^2 \cos^2 \frac{1}{2} \varphi d \varphi = \pi a^2 \int_0^{\infty} E_1 (t) t R (t) d t
\]

\[
= \pi a^2 \tau_0^2 \int_0^{\infty} E_1 (\xi \tau_0) \xi R (\xi) d \xi.
\]

For \( \tau_0 \ll 1 \), we may use the expansion

\[
E_1 (x) = - \gamma - \log x + O (x) \quad (\gamma = 0.5772 \ldots),
\]

for the exponential integral \( E_1 (\xi \tau_0) \) in equation (90). In this manner we obtain

\[
\int_0^\pi \sigma_\varphi^2 \cos^2 \frac{1}{2} \varphi d \varphi \approx \pi a^2 \tau_0^2 \left[ - (\gamma + \log \tau_0) \int_0^{\infty} \xi R (\xi) d \xi - \int_0^{\infty} \xi R (\xi) \log \xi d \xi \right].
\]

The appearance of \( \log \tau_0 \) in the foregoing expression is a reflection of the singular behavior of \( \sigma_\varphi^2 \).

We shall now obtain explicit expressions for \( \sigma_\varphi^2 \) for two special forms of \( R(t) \).

i) The case \( R(t) = e^{-t/\tau_0} \).—For this case

\[
\sigma_0^2 = a^2 \int_0^{\infty} \exp \left\{ - t \left( 1 + \frac{1}{\tau_0} \right) \right\} d t = a^2 \frac{\tau_0}{1 + \tau_0},
\]

and (cf. eq. [82])

\[
\sigma_\varphi^2 = \frac{a^2}{\epsilon \sqrt{1 - \epsilon^2}} \int_{\epsilon}^{1} \frac{d y}{\sqrt{1 - \gamma^2}} \int_0^{\infty} d t \exp \left\{ - t \left[ \frac{1}{\tau_0} + \gamma \right] \right\},
\]

or

\[
\sigma_\varphi^2 = \frac{a^2 \epsilon \tau_0^2}{\sqrt{1 - \epsilon^2}} \int_\epsilon^{1} \frac{d y}{(1 - \gamma^2) \sqrt{(1 - \gamma^2)(1 - \epsilon^2)}},
\]

where it may be recalled that \( \epsilon = \sin \frac{1}{2} \varphi \).

The integral over \( y \) in equation (95) is an elementary one; and we must distinguish the three cases: \( \epsilon < \tau_0 \), \( \epsilon = \tau_0 \), and \( \epsilon > \tau_0 \). We find:

\[
\frac{\sigma_0^2 - \sigma_\varphi^2}{a^2} = \frac{1}{3} \frac{\tau_0 (1 + 2 \tau_0)}{(1 + \tau_0)^2},
\]

\( (\gamma = 1) \)

\[
= - \frac{\epsilon \gamma}{(1 - \gamma^2) \left[ \gamma + \frac{1}{\epsilon + \gamma \sqrt{1 - \gamma^2 \left( 1 - \epsilon^2 \right)} \} \right]} \log \frac{1 + \gamma \epsilon - \sqrt{\left( 1 - \gamma^2 \right) (1 - \epsilon^2)}}{\gamma + \epsilon} \quad (\gamma < 1)
\]

\[
= \frac{\epsilon \gamma}{(1 - \gamma^2) \left[ \gamma + \frac{1}{\epsilon + \gamma \sqrt{1 - \gamma^2 (1 - \epsilon^2)} \} \right]} \left\{ \frac{\pi}{2 \sin^{-1} \frac{1 + \gamma \epsilon}{\gamma + \epsilon} \} \right\} \quad (\gamma > 1),
\]

where

\[
\gamma = \frac{\epsilon \tau_0}{\tau_0} = \frac{1}{\tau_0} \sin \frac{1}{2} \varphi.
\]
From the second of equations (96) we find that, for \( \varphi \to 0 \),
\[
\frac{\sigma_0^2 - \sigma_\varphi^2}{a^2} \to -\epsilon \gamma \left[ 1 + \log \frac{1}{2} (\gamma + \epsilon) \right] \quad (\epsilon, \gamma \to 0) \tag{98}
\]
or
\[
\frac{\sigma_0^2 - \sigma_\varphi^2}{a^2} \to -\frac{\varphi^2}{4 \tau_0} \left[ 1 + \log \frac{\varphi}{4 \tau_0} \right]. \tag{99}
\]

Equation (99) should be contrasted with Ambarzumian's formula (69). However, since, for the validity of equation (99), not only \( \varphi \) but also \( \varphi/\tau_0 \) should be small, it is evident that the equation has no useful range of applicability.

ii) The case \( R(r) = e^{-r^2/r_0^2} \).—In this case
\[
\sigma_0^2 = \frac{1}{2} \sqrt{\pi} a^2 e^{r_0^2/4} \tau_0 \left[ 1 - \Phi \left( \frac{1}{2} r_0 \right) \right], \tag{100}
\]
where \( \Phi(x) \) stands for the error function,
\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{101}
\]

Next, starting from equation (77), we find that, for the case on hand, the expression for \( \sigma_\varphi \) can be reduced to the form
\[
\sigma_\varphi^2 = \frac{1}{2} \sqrt{\pi} \frac{a^2 \tau_0^2}{\sin \frac{1}{2} \varphi} e^{r_0^2/4 \sin^2 \left( \varphi/2 \right)} \int_{r_0/2 \sin \left( \varphi/2 \right)}^{\infty} dz e^{-z^2 \cos^2 \left( \varphi/2 \right)} \left[ 1 - \Phi \left( z \sin \frac{1}{2} \varphi \right) \right], \tag{102}
\]
or, alternatively,
\[
\sigma_\varphi^2 = \sqrt{\frac{\pi}{\sin \varphi}} \frac{a^2 \tau_0^2}{2} e^{r_0^2/4 \sin^2 \left( \varphi/2 \right)} \int_0^{\tau_0 \cot \left( \varphi/2 \right)/2} dx e^{-x^2} \left[ 1 - \Phi \left( x \tan \frac{1}{2} \varphi \right) \right]. \tag{103}
\]

We shall normally be interested only in values of \( \varphi \leq 10^\circ \). It appears that for such small angles we may obtain a satisfactory approximation for \( \sigma_\varphi^2 \) by using, for the error function in equation (103), the series expansion,
\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3} + \frac{x^5}{10} - \ldots \right) \quad (x \to 0). \tag{104}
\]

In this manner we find that
\[
\frac{\sigma_\varphi^2}{a^2} = \frac{\pi \tau_0}{2 \sin \varphi} e^{r_0^2/4 \sin^2 \left( \varphi/2 \right)} \left[ 1 - \Phi \left( \frac{1}{2} \tau_0 \cot \frac{1}{2} \varphi \right) \right] - \frac{1}{2} \tau_0^2 e^{r_0^2/4}. \tag{105}
\]

Since we shall also be interested in angles \( \varphi > \tau_0 \), it is clear that we cannot use any series expansion for the error function in equation (105). In other words, in this case (as in the previous case) we cannot develop any useful series expansion for \( \sigma_\varphi^2 \). And the reason for this in both cases is that, for an expansion to be valid, not only \( \varphi \) but also \( \varphi/\tau_0 \) should be small; and in practice the latter condition is not fulfilled.

Using equations (96) and (105), we have evaluated \( \sigma_\varphi^2 \) for a number of values \( \tau_0 \). The results of these calculations are illustrated in Figures 1a and 1b.

7. The effect of the diffuse galactic light on the fluctuations in brightness of the Milky Way.—The analysis in the preceding sections presupposes that the brightness of the Milky Way is due, in its entirety, to the light from the stars obscured by the distribution

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We are grateful to Miss Donna Elbert for assistance with these calculations.
of the interstellar matter. However, as was first clearly pointed out by C. T. Elvey and F. E. Roach,\(^{14}\) the observed brightness is only in part due to the light from the stars, and an appreciable fraction of it must be ascribed to the scattering of the light of the stars by the interstellar matter: this is the so-called "galactic light." While, by a suitable observational technique,\(^{15}\) the two contributory sources to the brightness of the Milky Way can be distinguished, the most extensive measures we have at the present time, namely, those due to Pannekoek,\(^{16}\) do not distinguish between them. We shall now show how we can allow for this admixture of the diffuse galactic light in the observed brightness.

\[\Sigma_{\varphi} = \frac{\sigma^2 - 1}{\sigma^2} \]

Fig. 1.—\(a\), The normalized correlation function \(\Sigma_{\varphi} = \frac{\sigma^2 - 1}{\sigma^2}\) of the brightness in two directions making an angle \(\varphi\) with each other for the case \(R(t) = e^{-r/\tau_0}\). The different curves are labeled by the values of \(\tau_0\) to which they refer. \(b\), The normalized correlation function \(\Sigma_{\varphi} = \frac{\sigma^2 - 1}{\sigma^2}\) of the brightness in two directions making an angle \(\varphi\) with each other for the case \(R(t) = e^{-r/\tau_0}\). The different curves are labeled by the values of \(\tau_0\) to which they refer. In comparing the two sets of curves in \(a\) and \(b\), we should remember that, for equal values of \(\tau_0\), the equivalent optical thickness of a cloud on the assumption that \(R(t) = e^{-r/\tau_0}\) is \(\sqrt{\pi/2} = 0.886\) times the value it has on the assumption that \(R(t) = e^{-r/\tau_0}\) (cf. eq. [28]).

The contribution to the observed intensity, \(I\), by the scattering of the light of the stars by the interstellar matter is given by

\[I_s = \int_0^\infty e^{-\tau(s)} \mathfrak{F}(s) d\tau(s),\]

where \(\mathfrak{F}(s)\) denotes the source function for the galactic light. In extending the range of integration over \(\tau(s)\) from zero to infinity, we are assuming, as in the preceding sections, that the system extends to infinity along the line of sight.

Now the source function \(\mathfrak{F}(s)\) can be expressed in the form

\[\mathfrak{F}(s) = \int I(s, n) \Phi(\Theta) d\omega/4\pi,\]

where \(I(s, n)\) is the total intensity (i.e., the intensity of the light from the stars plus the intensity of the diffuse galactic light) at \(s\) in the direction specified by a unit vector \(n\); \(\Theta\) is the angle which the line of sight makes with the direction of \(n\); and \(\Phi(\Theta)\) is the

\(^{14}\)\textit{Aph. J.}, 85, 213, 1937.

\(^{15}\)Cf. L. G. Henyey and J. L. Greenstein, \textit{Aph. J.}, 93, 70, 1941.

phase function governing the scattering process. In equation (107) the integration is
extended over all solid angles.
For a statistically uniform distribution of matter, $\mathcal{F}(s)$ can vary from point to point
only on account of the fluctuations in $I(s, n)$. But, since $I(s, n)$ is averaged over all
directions $n$ and since, as we shall see in § 8, the correlations in the intensities in neigh-
boring directions decreases extremely rapidly, it is evident that, to a high degree of ap-
proximation, we may ignore the statistical fluctuations in $\mathcal{F}(s)$ and write

$$\mathcal{F}(s) = \bar{I} \int \Phi(\theta) \frac{d\omega}{4\pi} = \omega_0 \bar{I}, \quad (108)$$

where $I$ is the total mean intensity prevailing at any point in the system and $\omega_0$ is the
albedo of the interstellar particles for single scattering.

With $\mathcal{F}(s)$ given by equation (108), equation (106) becomes

$$I_g = \omega_0 \bar{I}. \quad (109)$$

According to this equation, the contribution to the observed brightness of the Milky
Way near the galactic plane by the diffuse galactic light is a constant. This is in accord
with the indications from observations\(^\text{17}\) that there is no pronounced correlation be-
tween the intensity of the diffuse galactic light and the total brightness of the Milky
Way.

Letting $\bar{I}_s$ denote the mean intensity of the light from the stars prevailing at any point
in the system, we can rewrite equation (109) in the form

$$I_g = \omega_0 \bar{I} = \omega_0 (\bar{I}_s + \bar{I}_g), \quad (110)$$

or

$$I_g = \frac{\omega_0}{1 - \omega_0} \bar{I}_s \quad \text{and} \quad \bar{I} = \frac{\bar{I}_s}{1 - \omega_0}. \quad (111)$$

Since $I_g$ is a constant, the angular correlation, $\overline{II}$, in the total brightness in two direc-
tions making an angle $\phi$ to each other can be readily found. Thus,

$$\overline{II}_{\phi} = \frac{(\bar{I}_s + \bar{I}_g)(\bar{I}_s + \bar{I}_g)}{(\bar{I}_s + \bar{I}_g)^2} + 2 \bar{I}_s \bar{I}_g + \bar{I}_g^2; \quad (112)$$

or, substituting for $I_g$ from equation (111), we have

$$\overline{II}_{\phi} = \bar{I}_s \bar{I}_s + \frac{\omega_0^2}{(1 - \omega_0)^2} + 2 \omega_0 \bar{I}_s. \quad (113)$$

An alternative form of this equation is

$$\frac{\overline{II}_{\phi}}{I_s^2} - 1 = \frac{\overline{II}_{\phi}}{I_s^2} - \left[ \frac{\omega_0^2}{(1 - \omega_0)^2} + 2 \omega_0 \frac{1}{1 - \omega_0} \right] \frac{1}{I_s^2}. \quad (114)$$

or

$$\frac{\overline{II}_{\phi}}{I_s^2} - 1 = \frac{\overline{II}_{\phi}}{I_s^2} - \left[ \frac{1}{(1 - \omega_0)^2} \right]. \quad (115)$$

Since $\bar{I}_s = (1 - \omega_0) \bar{I}$ (cf. eq. [111]), we can also write

$$\frac{\overline{II}_{\phi}}{I_s^2} - 1 = \frac{1}{(1 - \omega_0)^2} \left( \frac{\overline{II}_{\phi}}{I_s^2} - 1 \right). \quad (116)$$

\(^\text{17}\) Cf. Elvey and Roach, op. cit., esp. Table 3, p. 238.
The quantity on the left-hand side of equation (116) can clearly be identified with $\sigma_\varphi^2$ of § 6 (cf. eq. [77]). And, if

$$s_\varphi^2 = \frac{f_\varphi}{f^2} - 1$$

represents the corresponding angular correlation in the total brightness, then it follows from equation (116) that

$$\sigma_\varphi^2 = \frac{s_\varphi^2}{(1 - \sigma_0)^2}.$$  \hspace{1cm} (118)

In particular, for $\varphi = 0$ we have (cf. eq. [116])

$$\sigma_0^2 = \frac{f_0^2}{f^2} - 1 = \frac{s_0^2}{(1 - \sigma_0)} = \frac{1}{(1 - \sigma_0)} \left( \frac{f_0^2}{f^2} - 1 \right).$$  \hspace{1cm} (119)

From equations (118) and (119) we obtain

$$\frac{\sigma_0^2 - \sigma_\varphi^2}{\sigma_0^2} = \frac{s_0^2 - s_\varphi^2}{s_0^2} = \Sigma_\varphi$$  \hspace{1cm} (120)

Equation (120) shows that, if the angular correlations in the total brightness are discussed in terms of the "normalized" quantity on the right-hand side, we can apply the theory of § 6 as though no galactic light were present.

8. An analysis of the observed fluctuations in the surface brightness of the Milky Way.— We shall now show how the theory of the preceding sections can be used to interpret the observed fluctuations in the surface brightness of the Milky Way. For this purpose we shall use the results of Pannekoek's recent survey of the southern Milky Way.\footnote{A relation essentially equivalent to this was also established by Rusakov (op. cit.) in his analysis of the angular correlations in the brightness of the Milky Way in terms of Ambarzumian's formula (69).}

It may be recalled that the observational material which formed the basis of Pannekoek's survey was a number of photographic plates taken extra-focally through a short-focus camera. The intensities measured on these plates do not, therefore, distinguish between the light of the stars and the diffuse galactic light. We must accordingly allow for this in the manner described in § 7.

Pannekoek expressed his measured brightness in the unit of one star of photographic magnitude 10 per square degree, and he presented the results of his survey in a series of beautiful charts of isophotes for the various regions. Using these charts, we can read, sufficiently accurately for our purposes, the intensities along any curve we may draw through the regions covered by the survey.

In selecting the regions for our analysis, we have tried to avoid regions which are either heavily obscured (like the region in Taurus) or which contain pronounced associations of very bright stars (such as the regions in η Carinae or near P Cygni). With these criteria in mind, the following regions were finally chosen:

Region I: $280^\circ < l < 340^\circ$ \hspace{1cm} ($|b| \leq 2^\circ$)

and

Region II: $170^\circ < l < 245^\circ$ \hspace{1cm} ($-2^\circ \leq b \leq 0^\circ$).  \hspace{1cm} (121)

The intensities in these regions along several parallel arcs of constant galactic latitude ($b$) and separated by $\frac{1}{2}^\circ$ in $b$ were read from Pannekoek's charts and plotted as curves

\footnote{A. Pannekoek and D. Koelbloed, Publ. Astr. Inst. Amsterdam, No. 9, 1949.}
against the galactic longitude \( (l) \). These curves were later approximated by step functions, in which the intensity (rounded to 5 whole units) was constant over successive intervals of \( \frac{1}{2} \)° in galactic longitude. Fourteen such graphs were made (nine for region I and five for region II); a selection from these is illustrated in Figures 2 and 3.

Now the averages to which the formulae of the preceding sections apply are strictly averages over all possible complexions of the distribution of the interstellar matter. We shall suppose that we are substantially averaging over all possible complexions when we take means along various arcs of constant galactic latitude. If the available sample is sufficiently large, this will be a valid procedure; but, unfortunately, this is not the case, and this must be kept in mind when comparing the observations with the theory.

On the assumption made in the foregoing paragraph, we can write

\[
\overline{I_\phi} = \frac{1}{l_1 - l_2 + \phi} \int_{l_1 - \phi}^{l_1} I_1 I_{1+\phi} dl ,
\]

where \( l_1 \) and \( l_2 \) are the limits in galactic longitude of the arc considered. When the intensity-curves are drawn as step functions (as in Figs. 3 and 4), the integral giving \( \overline{I_\phi} \) can be replaced by a corresponding sum. (It may be stated here that, in order to minimize
Fig. 3.—The dependence on galactic longitude of the surface brightness of the Milky Way, along arcs of constant galactic latitude \( b \) in region II \((170^\circ < 1 < 245^\circ; 0^\circ < b < -2^\circ)\). Otherwise the legend for Figure 2 applies to this figure also.

Fig. 4.—The normalized correlation function, \( \Sigma_{\varphi} = (\sigma_0^2 - \sigma_0^2)/\sigma_0^2 = (\sigma_0^2 - \sigma_0^2)/\sigma_0^2 \), of the brightness of the Milky Way, as derived from observations in regions I (dots) and II (crosses). The curves represent the predicted dependence of \( \Sigma_{\varphi} \) on \( \varphi \), for the various values of \( r_0 \); the full-line curves are derived for the correlation function \( R = e^{-r^2/r_0^2} \) (cf. Fig. 1, a), while the dashed curves are derived for \( R = e^{-r^2/r_0^2} \) (cf. Fig. 1, b).
the effects of incomplete "waves" in the fluctuations, the limits in galactic longitudes were so chosen that $I_1$, and $I_2$ were in the neighborhood of $I$. In this manner the various quantities which occur in the definition of $s_2^2$ (eq. [117]) were determined for each arc separately; they were then combined to give average values for the two regions (eq. [121]) considered. The final results of this analysis are summarized in Table 1.

It will be noticed that we have not determined the angular correlations for $\phi$'s exceeding 5°. The reason for this restriction was that it appeared that, for $\phi > 5°$, the uncertainties resulting from the small size of the available sample became very appreciable. Indeed, the derived values of $s_2^2$ are increasingly affected by sampling errors as $\phi$ increases. Nevertheless, the fact that at $\phi = 5°$ all the calculations consistently gave, for $\Sigma_\phi = (s_0^2 - s_2^2)/s_0^2$, values in the neighborhood of 10 per cent would indicate that the derived run of this quantity for $\phi \leq 5°$ is probably reliable.

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tr>
<td>ANGULAR CORRELATIONS OF THE BRIGHTNESS OBSERVED IN SELECTED REGIONS OF THE MILKY WAY</td>
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<table>
<thead>
<tr>
<th>$\phi$</th>
<th>REGION I: $200° &lt; \phi &lt; 340°$</th>
<th>REGION II: $170° &lt; \phi &lt; 245°$</th>
<th>REGION I: $200° &lt; \phi &lt; 340°$</th>
<th>REGION II: $170° &lt; \phi &lt; 245°$</th>
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<tbody>
<tr>
<td>$s_0^2$</td>
<td>$\Sigma_\phi$</td>
<td>$s_2^2$</td>
<td>$\Sigma_\phi$</td>
<td>$s_0^2$</td>
</tr>
<tr>
<td>0.0°</td>
<td>0.0381</td>
<td>0</td>
<td>0.0582</td>
<td>0</td>
</tr>
<tr>
<td>0.5°</td>
<td>0.0582</td>
<td>0.13</td>
<td>0.0423</td>
<td>0.27</td>
</tr>
<tr>
<td>1.0°</td>
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<td>0.27</td>
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<td>1.5°</td>
<td>0.0582</td>
<td>0.13</td>
<td>0.0423</td>
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<td>2.0°</td>
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<td>0.0582</td>
<td>0.13</td>
<td>0.0423</td>
<td>0.27</td>
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</table>

We should state here that the angular correlations we have derived do not agree very well with the correlations derived similarly by Rusakov for two regions in the northern Milky Way. It is possible that this discrepancy is due to the larger sampling errors in Rusakov's calculations, since he restricted himself to regions with extensions in galactic longitude even less than 25°.

As we have already remarked (cf. eq. [120]), the deduced values of

$$\Sigma_\phi = \frac{s_0^2 - s_2^2}{s_0^2}$$

(123)

can be compared directly with the predictions for $(\sigma_0^2 - \sigma_2^2)/\sigma_1^2$ based on equation (77). It is evident that, in principle, the run of $\Sigma_\phi$ should give us detailed information concerning the correlation function $R(\tau/\tau_0)$. However, in view of the observational uncertainties in the deduced values of $\Sigma_\phi$, it does not seem that at the present time anything more than the order of magnitude of the micro-scale, $\tau_0$, can be derived. For this purpose it would clearly be sufficient to compare the observed run of $\Sigma_\phi$ with the theoretical curves derived on the two special forms of $R(\tau/\tau_0)$ considered in § 6. This comparison is made in Figure 4. From an examination of this figure, it appears that

$$\tau_0 \approx 0.01$$

(124)

Returning to the absolute value of $s_0^2(=I^2/I^2 - 1)$, we may first observe that for $\tau_0 \approx 0.01$ the approximation provided by equation (65) is amply sufficient and that, therefore (cf. eq. [119]),

$$s_0^2 = (1 - \omega_0)^2 \sigma_0^2 = (1 - \omega_0)^2 a^2 \tau_0 R_0$$

(125)
Since we may expect that $R_0 \sim 1$, we can write

$$a^2 \sim \frac{s_0^2}{\tau_0 \left(1 - \omega_0\right)^2}.$$  \hspace{1cm} (126)

The albedo $\omega_0$ of the interstellar particles is not known with any degree of precision. However, for the sake of definiteness, we shall assume that

$$\omega_0 = 0.4.$$  \hspace{1cm} (127)

According to Table 1, $s_0 \approx 0.04$ in region I and $s_0 \approx 0.06$ in region II. With these values of $s_0$, $\tau_0$ and $\omega_0$ given by equations (124) and (127), we find that

$$a^2 \approx 11 \text{ (region I)} \quad \text{and} \quad a^2 \approx 16 \text{ (region II)}.$$  \hspace{1cm} (128)

Since $a^2 \approx \sqrt{\langle \delta^2 \rho \rangle} / \langle \rho \rangle$ (cf. eq. [25]), the foregoing values of $a^2$ imply that

$$\sqrt{\langle \delta^2 \rho \rangle} \approx 3.3 \langle \rho \rangle \text{ (region I)} \quad \text{and} \quad \sqrt{\langle \delta^2 \rho \rangle} \approx 4 \langle \rho \rangle \text{ (region II)}.$$  \hspace{1cm} (129)

The fact that the root mean square of the fluctuations in the density derived in this manner is of the order of three to four times the mean density itself may be taken as a quantitative confirmation of the conspicuously nonuniform distribution of the interstellar matter.