THE PROPAGATION OF SHOCK WAVES IN A STELLAR MODEL WITH CONTINUOUS DENSITY DISTRIBUTION*

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ABSTRACT

The aim of the present paper has been to investigate the properties of progressing waves which arise if a compressible-gas configuration, in which the density $\rho_0$ diminishes with increasing distance $r$ from the center, as $\rho_0 \sim r^{-3/2}$ is disturbed from its state of equilibrium by an instantaneous central explosion. It is shown that, if this explosion has been sufficiently energetic for the outgoing disturbance to possess the characteristics of a shock wave, the central part of our configuration will be effectively evacuated by the explosion up to a certain distance from the center, depending on the amount of energy liberated by the initial explosion. If the release has been instantaneous (and, in consequence, the total energy of the wave motion is independent of the time $t$), the inner boundary of our flow (inclosing the empty core) becomes, by definition, a surface of contact discontinuity.

In order to study the properties of the actual flow of gas trapped between the shock wave and the contact discontinuity, which is of the nature of a progressing wave, the fundamental system of partial differential equations of our problem has been converted to ordinary differential equations with $\xi = r^{3/4}$ as the sole independent variable and has been integrated numerically for 8 cases corresponding to different strengths (i.e., the Mach numbers) of the head wave and to the ratio of specific heats $\gamma = \frac{4}{3}$. The physical properties of our flow—such as the velocity, pressure, and density at any point of the disturbed medium—remain the same along the lines $\xi = \text{Constant}$. In particular, the radii of both boundaries limiting our expanding regime increase as $r^{3/6}$; and as for the generalized Roche model, the thickness ($\xi$-wise) of the shell between them is found to increase with the increasing Mach number of the head wave.

Table 1 contains the numerical data describing the details of the individual solutions in terms of nondimensional parameters, which can be easily converted into absolute units by a suitable choice of the initial parameters. Table 2 summarises the physical properties of the respective shock waves and indicates, in particular, the fractional amount of energy necessary to give rise to the computed phenomena.

Finally, the appendix contains a proof of the uniqueness of the solutions of the form of progressing blast waves investigated in Sections II-V. It demonstrates that, if the dependent variables $U$, $P$, and $\Omega$ of our fundamental system of partial differential equations are to be expressible in terms of an equivalent system of ordinary differential equations with $\xi(r, t)$ as the sole independent variable, $\xi(r, t)$ must necessarily be of the form $\phi(r)\psi(t)$, where $\phi(r)$ is a power of $r$ alone, while $\psi(t)$ may be either a power or an exponential of $t$. The product of the powers of both $r$ and $t$ represents, moreover, the only possible form of $\xi$ which will render the total energy of the corresponding wave motion independent of the time. If, ultimately, the expanding field of flow is to be headed by a shock front characterized by a constant Mach number (i.e., if both sides of the Rankine-Hugoniot conditions are to be functions of $\xi$ alone), the structure of the undisturbed configuration becomes uniquely specified.

I. INTRODUCTION

In a previous communication we investigated the propagation of shock waves in the envelope of a generalized Roche model, caused by a sudden expansion of its core. As is well known, a generalized Roche model (GRM) is characterized by a discontinuity in density across the interface of the core and its envelope, and for this reason it may not (except possibly under very special circumstances) come very close to stellar models of real astrophysical importance. In the present paper we shall, naturally enough, aim at removing this special feature of our previous work and shall attempt to extend it by considering the propagation of intense spherical disturbances in gaseous configurations in which the density remains a continuous function of radial distance from the center to the surface. In doing so, we shall find our problem to be considerably less tractable for an obvious reason. In a GRM practically the whole mass of the configuration was supposed

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1 Ap. J., 113, 193, 1951. This paper will hereafter be referred to as “Paper I.”
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to be confined to its core, so that the gravitational acceleration throughout the envelope varied simply as the inverse square of the distance from the center. If, however, the density is to be a continuous function of the radius \( r \), the mass \( m(r) \) interior to any arbitrary radius will likewise vary with \( r \); as a result, the field of force in which the shock waves are to propagate will no longer be of the simple inverse-square type, but its intensity will respond to any fluctuation in density caused by the passage of any disturbance. In mathematical language the mass \( m(r) \) contained within a sphere of radius \( r \) becomes an additional dependent variable of our problem, and its presence will increase the order of the system of differential equations by one.

The aim of the present paper will be to investigate, in the light of such equations, the events which will follow if the center of our compressible-gas configuration is disturbed by a radially symmetrical explosion. In doing so, we shall again consider the central explosion to be instantaneous—an assumption which necessitates the total energy of the wave motion being independent of the time—and, as in Paper I, we shall limit our present contribution to an analysis of the "progressing blast waves"—i.e., to such motions for which the velocity, pressure, and density of the disturbed field of flow can be made to depend on a sole variable \( \xi = r^2 t^2 \phi \), where \( r \) denotes the radial distance, \( t \) the time, and \( \phi \) and \( \psi \) are suitably chosen constants. The form of the fundamental equations of the problem, together with the requirement that the total energy carried off by the wave motion be independent of \( t \), will uniquely specify the values of \( \phi \) and \( \psi \); while the requirement that the expanding regime be limited on the outside by a shock wave will again impose a certain definite variation of the undisturbed density in front of the shock. Conversely, should the actual exploding configuration possess a different equilibrium structure, the fundamental system of partial differential equations describing, implicitly, the effects of explosion in terms of the independent variables \( r \) and \( t \) will not be reducible to ordinary differential equations with \( \xi \) as the sole independent variable. In reality—as represented by the phenomenon of nova outbursts—the equilibrium structure of the prenovae (which is being explored concurrently with these investigations) may not come very close to the one postulated later in this paper; at any rate, it should be understood that the reasons which have led us to adopt the latter have been prompted by mathematical convenience rather than by its physical reasonableness. We do not believe, however, that this attitude calls for any apology, for the strategy inherent in our approach is obvious. Unlike the numerical solutions of partial differential equations (which are, besides, much more difficult and laborious), the cases in which such equations can be made to depend on a single variable \( \xi \) are capable of furnishing solutions which hold good without any limit of time. In point of fact, the progressing waves—like the periodic orbits of dynamical astronomy—represent, to use the words of Henri Poincaré, "la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu’ici réputée inabordable." It is therefore only logical that an exploration of solutions of the fundamental equations of the problem should begin with waves of the progressing type; and this constitutes the task which will be dealt with in the present paper. The main conclusions arrived at in the course of this investigation are summarized in the abstract.

II. EQUATIONS OF THE PROBLEM

As in Paper I, let \( u, p, \) and \( \rho \) denote the radial velocity, pressure, and density at any point of a compressible-gas configuration at a distance \( r \) from the center and at a time \( t \). If, moreover, \( m = m(r, t) \) denotes the mass interior to \( r \), \( \gamma \) the ratio of specific heats of the constituent gas, and \( G \) the gravitation constant, the Eulerian hydrodynamical equations of motion reduce to

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{Gm}{r^2} = 0,
\]

Les Méthodes nouvelles de la mécanique céleste (Paris: Gauthier-Villars et fils, 1892), 1, 82.
where now
\[
\frac{\partial m}{\partial r} = 4\pi \rho r^2;
\]
(2)

the equation of continuity can be written as
\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) = 0,
\]
(3)

while the energy equation appropriate for polytropic gas assumes the form
\[
\frac{\partial}{\partial t} (\rho \rho^{-\gamma}) + u \frac{\partial}{\partial r} (\rho \rho^{-\gamma}) = 0.
\]
(4)

As in Paper I, the outer boundary conditions of our problem are given, by hypothesis, over a moving surface (the shock wave); but, since we do not postulate a priori the existence of any discrete core inside the configuration, the inner boundary conditions merely imply that the center is a point of no displacement. In what follows, let the subscript 0 again denote the undisturbed values of pressure, density, etc., in front of the shock wave (where they are functions of \( r \) alone), while the subscript 1 will denote the values of the respective quantities immediately behind the shock. If, moreover, \( V \) stands for the velocity of propagation of the shock front, the Rankine-Hugoniot shock-wave conditions

\[
\frac{\rho_1}{\rho_0} = \frac{\gamma - 1 + (\gamma + 1) \gamma}{\gamma + 1 + (\gamma - 1) \gamma},
\]
(5)

\[
\frac{V - u_1}{V} = \frac{\rho_0}{\rho_1},
\]
(6)

and

\[
\rho_0 V u_1 = p_1 - p_0 = (\gamma - 1) \rho_0,
\]
(7)

where we have abbreviated \( \gamma = \frac{\rho_1}{\rho_0} \) expressing the continuity of energy, mass, and momentum across the shock wave, permit us to express \( u_1, p_1, \) and \( p_1 \) in terms of undisturbed values of these quantities by means of the equations

\[
u_1 = 2 V \frac{2 \gamma \rho_0}{\gamma + 1} (\gamma + 1) \rho_0 V',
\]
(8)

\[
\frac{p_1}{\gamma + 1} = 2 \rho_0 \frac{V^2 - (\gamma - 1) \rho_0}{\gamma + 1},
\]
(9)

\[
\rho_1 = \frac{(\gamma + 1) \rho_0^2 V^2}{(\gamma - 1) \rho_0 V^2 + 2 \gamma \rho_0}.
\]
(10)

In addition, it is easy to see that
\[
m_1 = 4\pi \int_0^R \rho r^2 d r = 4\pi \int_0^R \rho_0 r^2 d r,
\]
(11)

where \( R = R(t) \) denotes the radius of the shock front. Equation (6) makes it evident that, under any circumstances, the mass particles behind the shock move with velocities inferior to that of the shock itself; hence the mass interior to the shock front at any time must be equal to that contained within a sphere of radius \( R \) in the undisturbed state; none could have escaped outside.

The system of partial differential equations (1)–(4) is one of fourth order and of second
degree, in two independent variables, with initial conditions defined over a moving boundary. As in Paper I, our augmented system of equations cannot be linearized by any transformation of the variables, and any attempt to eliminate the nonlinear terms would leave us with an approximation in which the essential features of our problem would be completely lost. On the other hand, a retention of the nonlinear terms precludes the construction of an analytical solution, so that numerical integrations appear to offer the only avenue of approach. Before embarking upon them, however, we propose again to investigate, first, the conditions for which the fundamental system (1)-(4) of partial differential equations may be reducible to one of ordinary differential equations which can be solved—numerically or otherwise—with much less difficulty.

In more specific terms we propose to seek such solutions of the basic equations as will make \( u, p, \rho, \) or \( m \) depend on \( r \) and \( t \) only through the product

\[
\xi = r^\phi p^\psi,
\]

where \( \phi \) and \( \psi \) are suitably chosen constants. Accordingly, as in Paper I, we shall assume that

\[
u = \frac{r}{l} U(\xi),
\]

\[
p = r^{\phi+2} \rho^{-\psi} P(\xi),
\]

\[
\rho = r^{\phi}\Omega(\xi),
\]

and likewise that the local velocity of sound,

\[
C^2 = \gamma \frac{p}{\rho} = \left(\frac{r}{l}\right)^2 C^2(\xi),
\]

where \( U(\xi), P(\xi), \) and \( \Omega(\xi) \) are new functions defined by the foregoing relations and the parameters \( \kappa, \lambda, \phi, \) and \( \psi \) are to be determined by physical considerations.

The first such condition is imposed by the requirement that, when the new variable \( \xi \), as defined by equation (12), is introduced, together with equations (13)-(15), into equations (1)-(4), the exponents of the remaining powers of \( r \) and \( t \) will vanish. The reader can easily verify that this will be the case, provided that

\[
\lambda + 2 - \frac{\psi}{\phi} = 0.
\]

On the other hand, the total energy \( E \) of the configuration we are considering will be given by

\[
E = 4\pi \int_a^R \frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1} - G \frac{mp}{r} r^2 dr,
\]

where \( a \) denotes a radius (so far arbitrary) at which the explosion is supposed to take place. Now if this explosion is to be instantaneous, the foregoing expression for \( E \) must obviously be independent of the time. Consistent with equation (12), we may write

\[
R(t) = \xi_0^{\phi/\psi} t^{-\psi/\phi}
\]

and

\[
a(t) = \xi_1^{\phi/\psi} t^{-\psi/\phi},
\]

\[\text{For a justification of this assertion cf. Sec. IV of this paper.}\]
where \( \xi_0 \) and \( \xi_1 \) are the values which our new independent variable \( \xi \) will assume at the shock front and the locus of explosion, respectively. Inserting equations (19) and (20) together with (13)–(15) into equation (18), we find that the kinetic and thermal energies carried by the wave motion will be independent of the time, provided that

\[
\lambda - 2 - (\kappa + 5) \frac{\psi}{\phi} = 0 ,
\]

while the gravitational energy will be time-independent if

\[
2\lambda - (2\kappa + 5) \frac{\psi}{\phi} = 0 .
\]

Equations (17) and (21)–(22) constitute the conditions which bear on the determination of \( \kappa, \lambda, \) and \( \psi/\phi \). It is instructive to compare these conditions with the corresponding equations of Paper I. In discussing the GRM, we were dealing with a nonhomogeneous system of differential equations of third order. The nonhomogeneity of the Eulerian equation of motion at once imposed the ratio \( \psi/\phi = -\frac{4}{5} \) if \( \xi \) was to replace \( r \) and \( t \) throughout as the independent variable. The required constancy of the kinetic and thermal energies with time then led to a relation between \( \kappa \) and \( \lambda \) (as represented by eq. [16] of Paper I) which automatically insured the constancy of the gravitational contribution to the total energy as well; and this left us with a (trivial) multiplicity of solutions corresponding to the same ratio of \( \psi/\phi \). In the present case, when dealing with a homogeneous system (1)–(4) of fourth order, our system does not, by itself, yield the value of \( \psi/\phi \) directly but furnishes only relation (17) between the four parameters involved. In order to specify these parameters, it is again necessary to enforce the constancy of the total energy of our wave motion. The constancy of its kinetic and thermal constituents furnishes us with equation (21), which, together with equation (17), yields

\[
\frac{\psi}{\phi} = -\frac{4}{5} \quad \text{and} \quad \lambda + 2 = -\frac{4}{5} \kappa.
\]

It is noteworthy that, as in Paper I, these are precisely the values which also satisfy equation (22) and thus automatically render constant the gravitational energy of the disturbed field of flow. As in Paper I, the foregoing equations do not specify our parameters uniquely, but provide only two relations between them. These relations are obviously satisfied by

\[
\kappa = 0 \; ; \; \lambda = -2 \; ; \; \psi = 4 \; ; \; \text{and} \; \phi = -5 ,
\]

the values which we shall hereafter adopt. It can be easily shown that no generality has been lost by this choice; for all other combinations of the four parameters consistent with equations (23) will lead to no new solutions.\(^4\)

Having done so, we may now proceed to rewrite our fundamental equations (1)–(4) in terms of our new independent variable \( \xi = r^{\psi/\phi} \). If we abbreviate, for the sake of simplicity,

\[
5U(\xi) - 4 = \Xi(\xi)
\]

and

\[
C^2(\xi) = S(\xi)
\]

\(^4\)In general, the first one of eqs. (23) permits us to assert that \( \psi = 4n \) and \( \phi = -5n \), where \( n \) may denote any number (positive or negative). It is, however, evident that no generality can be lost by setting \( n = 1 \); for any solution with \( n \neq 1 \) would merely correspond to the choice of a new variable \( \eta = \xi^n \).
and eliminate \( m \) from equation (1) by means of equation (2) by differentiation, then after some transformation we establish that our original system (1)-(4) is equivalent to the following three simultaneous ordinary differential equations of the form

\[
\frac{1}{P} \frac{dP}{d\xi} = -5 \gamma \xi \xi' + (3 \gamma + 2) \Xi + 12 (\gamma - 1)
\]

\[
\frac{1}{S} \frac{dS}{d\xi} = -5 (\gamma - 1) \xi \xi' + (3 \gamma - 1) \Xi + 12 \gamma - 14
\]

\[
\frac{d\Xi}{d\xi} = \frac{\xi \Xi (25 \gamma S - \gamma \Xi^2)}{-1} \frac{5 \xi \Xi^2 P'}{P} (2S + 5 \xi S') + 4 \pi G \gamma \Xi^2 \frac{P}{S}
\]

\[
- \Xi^2 \left[ 10 \xi S' + (15 \gamma + 4) S \right] + 5 S \left[ 5 \xi \Xi \xi'^2 - 12 (\gamma - 1) \xi \Xi' - 12 (\gamma - 1) \Xi \right] + \frac{\xi \Xi^2}{S} \left[ 5 \xi \Xi'^2 - 3 \xi \Xi' + \frac{3}{2} (\Xi + 4) (\Xi - 1) \right]
\]

where primes denote differentiation with respect to \( \xi \) and where, consistent with equations (16) and (26),

\[
S (\xi) = \gamma \frac{P (\xi)}{\Omega (\xi)}
\]

These are the equations governing the propagation of blast waves of the progressing type in the configurations which we are considering.

### III. Solution of the Equations

Having reduced the differential equations of our problem to tractable forms, it remains for us to specify the initial conditions which determine the nature of their solution. Consider, first, the boundary conditions at the shock front. On the passage through the shock, the velocity, pressure, and density are known to undergo discontinuous changes expressed by equations (8)-(10), where, consistent with the assumptions of the preceding section,

\[
V = \frac{dR}{dt} = \frac{4 R}{5 i}
\]

Moreover, if the values of \( u_i \), \( p_i \), and \( p_i \) immediately behind the shock wave, as given by equations (8)-(10), are to be consistent with equations (13)-(15) in which we have set \( \xi = \xi_0 \) and inserted from equation (24), it follows that \( p_0 \) and \( p_0 \) in equations (8)-(10) cannot be arbitrary functions of \( r \) but must be such as to render both sides of the equations dimensionally correct at all times. This is found to be true if, in front of the shock wave,

\[
p_0 = \beta r^{-5/2}
\]

which, by equation (2), leads to

\[
m_0 = 4 \pi \int_0^r \rho_0 r^2 \, dr = 8 \pi \beta \sqrt{r}
\]

and if, consistent with the equation (1) in the state of equilibrium,

\[
p_0 = \frac{8 \pi \beta^2 G}{3 r^3}
\]
where $\beta$ is an arbitrary constant. The equilibrium structure of our configuration is thus specified by a polytropic equation of state,

$$p_0 = K \rho_0^{\beta/5},$$  \hspace{1cm} (35)

where

$$K = \frac{8}{3} \pi G \beta^{3/5}. \hspace{1cm} (36)$$

The radius of our configuration is thus infinite, and so is its total mass. These facts do not, however, give rise to any complications in this connection; in point of fact, the infiniteness of the radius will absolve us from having to consider the reflection of a shock wave from a finite spherical boundary.

Let us now proceed to insert the foregoing equations (32) and (34) into equations (8)-(10). If, furthermore, we replace in the latter the values of $u_i$, $p_i$, and $\rho_i$ by expressions (13)-(15), which are supposed to hold good from the shock inward, we find that, immediately behind the shock front, when $\xi = \xi_0$,

$$U(\xi_0) = \frac{8 (1 - x)}{5 (\gamma + 1)}, \hspace{1cm} (37)$$

$$P(\xi_0) = \frac{16 \beta (\gamma - 1) \xi_0^{1/2}}{25 \gamma (\gamma + 1)} \left\{ \frac{2 \gamma}{\gamma - 1} - x \right\}, \hspace{1cm} (38)$$

$$\Omega(\xi_0) = \frac{(\gamma + 1) \beta \xi_0^{1/2}}{\gamma - 1 + 2 x}, \hspace{1cm} (39)$$

and, therefore,

$$S(\xi_0) = \frac{16 (\gamma - 1 + 2 x) [2 \gamma - (\gamma - 1) x]}{25 (\gamma + 1)^2}, \hspace{1cm} (40)$$

where we have abbreviated

$$x = \frac{25}{6} \pi \beta G \gamma \xi_0^{1/2}. \hspace{1cm} (41)$$

The fundamental system of differential equations (27)-(29) is one of fourth order; therefore, in order to be able to integrate it from the shock inward, one more initial condition at $\xi = \xi_0$ remains yet to be ascertained. This remaining boundary condition should be the value of $\xi'(\xi_0)$ or—which is the same thing—of $U'(\xi_0)$, where primes denote differentiation with respect to $\xi$. It can be evaluated from the initial differential equations (1), (3), and (4), rewritten in terms of $\xi$ in place of $r$ and $t$, in which we set $\xi = \xi_0$ and, by virtue of equations (11), (19), and (32),

$$m(R) = 8 \pi \beta \sqrt{R(t)} = 8 \pi \beta \xi_0^{-1/10} \rho^{1/5}. \hspace{1cm} (42)$$

These equations assume the explicit form

$$(4 - 5 U) \xi_0 \Omega' - 5 \xi_0 P' = U (1 - U) \Omega - 2 P - 8 \pi \beta G \xi_0^{1/5} \Omega, \hspace{1cm} (43)$$

$$5 \xi_0 \Omega U' - (4 - 5 U) \xi_0 \Omega' = (3 U - 2) \Omega, \hspace{1cm} (44)$$

$$(4 - 5 U) \xi_0 \Omega P' - \gamma (4 - 5 U) \xi_0 P \Omega' = 2 (2 - \gamma - U) P \Omega. \hspace{1cm} (45)$$

The law of density (and pressure) in the form in which we take it leads to an infinite density (and pressure) at the center. This singularity presents us, however, with a mathematical, rather than a physical, difficulty; for we may imagine that the law (32) holds only to a very small distance from the center, within which the density remains finite. This avoids the physical difficulty, without appreciably altering either the mass within any significant radius or the values of the gravity and pressure.
If, in these equations, we insert values for $U$, $P$, and $\Omega$ from equations (37)-(39), we can solve the foregoing simultaneous system for $U'$, $P'$, and $\Omega'$, any one of which is sufficient to complete the requisite number of initial conditions of our problem. The reader should notice that, unlike in Paper I, the pressure $P$ occurs in equation (29) not only through its logarithmic derivative but also explicitly. It follows, therefore, that the requisite solutions of the system (27)-(29) will also depend, apart from $\gamma$, $x$, and $\xi_0$, on the absolute value of $\beta$. Of these four, a choice of $\xi_0$ determines the position of the shock at a given time $t_0$ and depends, therefore, on our choice of the units of length and time. In what follows, we shall, for simplicity's sake, set $\xi_0 = 1$, a convention which obviously entails no loss of generality. The ratio $\gamma$ of specific heats depends, in turn, on the properties and composition of the gas constituting our configuration; so that, once a fixed value of $\gamma$ has been adopted, either one of the constants $x$ or $\beta$, related by equation (41), remains as the only arbitrary nondimensional parameter characterizing our solutions. It is easy to show that, as in Paper I, the parameter $x$ is again intimately related with the strength (i.e., the Mach number $M$) of the respective shock wave. Let us define, as usual,

$$M = \frac{V}{c_0},$$

where $c_0$ denotes the Laplacian velocity of sound in the undisturbed medium in front of the shock wave. Now, in our present case, $V$ is given by equation (31), while

$$c^2_0 = \gamma \frac{p_0}{\rho_0} = \frac{8 \pi \beta \gamma G}{3 \sqrt{r}},$$

by equations (32) and (34). Hence, it transpires immediately that

$$M^2 = \frac{1}{x},$$

which discloses that, if the expanding regime we are considering is to possess a shock wave on its head, we must have

$$0 \leq x < 1,$$

the strength of the shock being greater, the smaller the value of $x$.

Suppose, now, that a proper value of $x$ has been chosen and an integration of equations (27)-(29) started from the initial conditions as represented by equations (37), (38), (40), and (43)-(45) from $\xi_0 = 1$ in the direction of increasing $\xi$, which, for fixed $t$, corresponds to an inward march toward the center, where $\xi = \infty$. Such integrations can be continued until, for each value of $x$, a point $\xi = \xi_1$ has been reached at which $\Xi(\xi_1)$ vanishes. The place at which $\Xi(\xi_1) = 0$ again turns out to be a singular point of our solution; and our equations readily disclose what happens at this point. The mass $m(\xi)$ contained within a shell between two concentric spheres of radii $\xi_0$ and $\xi$ at time $t$ is obviously given by

$$m(\xi, t) = \frac{4 \pi}{3} \rho t^{3/2} M(\xi),$$

where

$$M(\xi) = \frac{3}{5} \int_{\xi_0}^{\xi} \gamma \frac{P}{c^2} \xi^{4/5} d\xi$$

and can be integrated simultaneously with the solution of our fundamental system (27)-(29). When we arrive at a point $\xi = \xi_1$ at which $\Xi(\xi_1) = 0$, we find that the mass $m(\xi_1)$, as defined by the above equation (50), becomes numerically equal to the mass...
$m(R)$ of the whole configuration inside the shock wave of radius $R$, which has already been expressed by equation (42). In order to describe the situation in terms of physical language, we may say that, by having produced a requisite condensation behind the postulated shock wave, we have depleted the available supply of mass in the interior of our model to such an extent that none is left near the center, all mass interior to ($\xi_i$) being swept clean and driven outward by the initial explosion at time $t = 0$.

By having assumed the initial explosion to be instantaneous, we have, moreover, compelled the sphere characterized by the radius $\xi_i$ to become a surface of contact discontinuity (i.e., one across which there is no flow of mass or energy for $t > 0$). The pressure must be continuous across this surface, and its gradient must therefore remain finite; since, moreover, there is no mass inside $\xi_i$, the pressure must be zero at either side of the interface. Accordingly, the logarithmic derivative $P'/P$ should become infinite at $\xi = 0$, as it indeed does in accordance with equation (27). Numerical integrations of the system (27)–(29) disclose, however, that the ratio $S = \gamma P/\Omega$ vanishes as $\xi \to \xi_i$; hence $P$ must approach zero faster than $\Omega$ and, if $P'$ is to be (by definition of a contact discontinuity) finite at $\xi = \xi_i$, $\Omega'(\xi_i)$ must become infinite. Therefore, we meet essentially the same situation as in the problem of a GRM of Paper I. In either case the density derivative is infinite at $\xi = \xi_i$. In the case of the Roche model, this singularity marked the boundary of the inner core whose mass was assumed to be infinite in comparison with the mass of its envelope. The actual distribution of this mass within the core is wholly irrelevant; as long as spherical symmetry is preserved, its gravitational attraction will be the same if it is distributed uniformly through the volume or confined to an infinitesimally thin hollow shell. The fact that the mass within this shell should be infinite in comparison with that of the surrounding envelope discloses that, in the case of a generalized Roche model, the singularity in the density distribution at $\xi = \xi_i$ is a nonintegrable one—in contrast to the present problem, which is characterized by an integrable density distribution for $\xi \leq \xi_i$.

The reader may notice that, as long as $\xi_i$ remains finite, there exists one possibility for the logarithmic derivatives $P'/P$ or $\Omega'/\Omega$ to assume finite values (or possibly even to vanish) at $\xi = \xi_i$. Of the density derivative this will be true if

$$\xi'(\xi) \to \frac{12}{5\gamma \xi} (\gamma - 1),$$

as $\xi \to \xi_i$ in such a way that

$$\lim_{\xi \to \xi_i} \left\{ \xi'(\xi) - \frac{12}{5\gamma \xi} (\gamma - 1) \right\} = B \xi^n(\xi),$$

where $n \geq 1$ and $B$ is an arbitrary constant. If, in particular, $n = 1$ and $B = (2 + 3\gamma)/5\xi_i$, it is evident that

$$\lim_{\xi \to \xi_i} \frac{P'}{P} = 0.$$  

The logarithmic derivative of $\Omega$ will, in turn, remain finite at $\xi = \xi_i$ if, as $\xi \to \xi_i$,

$$\xi'(\xi) \to \frac{2}{5\xi},$$

in such a way that

$$\lim_{\xi \to \xi_i} \left\{ \xi'(\xi) - \frac{2}{5\xi} \right\} = D \xi^n(\xi),$$

It will be shown below that $\xi_i = \infty$ only if $\gamma = \frac{2}{3}$.
where \( n \geq 1 \) and \( D \) is a constant; if, in particular, \( n = 1 \) and \( D = (3\gamma - 1)/5\xi_1 \), we again find that

\[
\lim_{\xi \to \xi_1} \frac{\Omega'}{\Omega} = 0. \tag{56}
\]

The reader may observe that equations (51) and (54) are compatible only if \( \gamma = 1.2 \); for any other value of \( \gamma \), only one of these relations can be satisfied.

It is easy to show by the same argument invoked in Paper I that, for \( \gamma \neq \frac{4}{3} \), the value of \( \xi \) for which \( \Xi(\xi) \) vanishes is always finite. Consider the total energy of the configuration in its undisturbed state inclosed between the surfaces of radii \( R \) and \( a \); it is given by

\[
H = 4\pi \int_a^R \left( \frac{1}{\gamma - 1} \frac{p_0}{\rho_0} - \frac{Gm_0}{r} \right) \rho_0 r^2 dr
\]

\[
= \frac{32}{3} \frac{(4 - 3\gamma)}{\gamma - 1} \pi^2 \beta^2 \log \frac{R}{a} \tag{57}
\]

by virtue of equations (32)-(34). For \( \gamma \neq \frac{4}{3} \), this latter expression evidently remains finite only if \( a > 0 \), i.e., if the density distribution (32) in the undisturbed state breaks down at a finite distance from the center. The distance at which this occurs corresponds to the singular point \( \xi_1 \) of our solution. Since this singular point again specifies the location of a contact discontinuity across which there is no flow of matter or heat, the regimes at either side of the discontinuity do not communicate and can be investigated separately. For our present purpose the actual structure of the inner core between \( 0 \leq r \leq a \) is wholly irrelevant—except that its mass and energy contents should be negligible in comparison with those outside in the concentric shell between \( a < r < R \). It is this requirement which distinguishes our present problem from that discussed in Paper I, where the converse assumption was made.

If we now revert from \( r \) to \( \xi \), equations (19) and (20) disclose that \( R/a = (\xi_1/\xi_0)^{\gamma/\gamma} \).

With the value of this ratio determined from a solution of the system (27)-(29) for any particular value of \( x \), equation (57) can be used to determine the absolute value of our arbitrary constant \( \beta \) if the total energy \( H \) of our configuration in its equilibrium state is known—or, conversely, to determine \( H \) from a given value of \( \beta \).

IV. NUMERICAL INTEGRATIONS

Equations (27)-(29) as they stand are too involved to admit of an analytical solution; so that, in order to investigate the properties of an expanding gas flow governed by such equations, recourse must be had to numerical integration. We have performed, altogether, eight such integrations, corresponding to \( \gamma = \frac{4}{3} \) (monatomic gas) and Mach numbers \( M \) ranging from \( M^2 = 5 \) to \( M^2 = \infty \). The outcome of the integrations is presented in Table 1, the successive columns of which are self-explanatory. Each solution is given from \( \xi_0 = 1 \) (corresponding to the assumed position of the shock wave) to \( \xi_1 \), where the inner contact discontinuity sets in. Table 2 summarizes the characteristic properties of the medium immediately behind the shock wave, for various values of the strength of the shock. As many decimals are retained in both tables as are considered significant. The normalized values of the successive variables permit the evaluation of the actual absolute properties of the respective expanding regime of gas flow by choosing the appropriate absolute values of \( \beta \) and \( \xi_0 \). Figures 1, 2, and 3 give the profiles of the velocity, pressure, and density, respectively, at a given time \( t = 1 \), extending from the shock wave \( (r = 1) \) inward to the core. In this manner, these figures can be regarded as instantaneous.

\(^7\) I.e. (cf. Sec. IV), if the medium in front of the shock wave is isentropic.
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**TABLE I**

Physical Properties of the Expanding Field of Flow behind Shock Waves of Different Strength

---

**Table:**

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- **Column 2:** $U$
- **Column 3:** $C$
- **Column 4:** $\beta$
- **Column 5:** $\gamma$
- **Column 6:** $M$
- **Column 7:** $P$
- **Column 8:** $Q$
- **Column 9:** $R$

**Source:** American Astronomical Society • Provided by the NASA Astrophysics Data System
TABLE I

| ϵ | U | C | Pr/β | CPr/β | M | F | ϵ | U | C | Pr/β | CPr/β | M | F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8.0 | 0.7055 | 0.0375 | 0.0325 | 0.0235 | 0.0206 | 0.0209 | 0.0219 | 0.0221 | 0.0224 | 0.0227 | 0.0230 | 0.0233 | 0.0236 | 0.0239 |
| 8.6 | 0.7121 | 0.0376 | 0.0326 | 0.0236 | 0.0207 | 0.0209 | 0.0219 | 0.0221 | 0.0224 | 0.0227 | 0.0230 | 0.0233 | 0.0236 | 0.0239 |
| 8.2 | 0.7086 | 0.0375 | 0.0325 | 0.0235 | 0.0206 | 0.0209 | 0.0219 | 0.0221 | 0.0224 | 0.0227 | 0.0230 | 0.0233 | 0.0236 | 0.0239 |
| 7.8 | 0.7051 | 0.0374 | 0.0324 | 0.0234 | 0.0205 | 0.0207 | 0.0218 | 0.0220 | 0.0223 | 0.0225 | 0.0228 | 0.0231 | 0.0234 | 0.0237 |
| 7.4 | 0.7016 | 0.0373 | 0.0323 | 0.0233 | 0.0204 | 0.0207 | 0.0218 | 0.0219 | 0.0222 | 0.0225 | 0.0228 | 0.0231 | 0.0234 | 0.0237 |
| 7.0 | 0.6981 | 0.0372 | 0.0322 | 0.0232 | 0.0203 | 0.0206 | 0.0217 | 0.0219 | 0.0221 | 0.0224 | 0.0227 | 0.0230 | 0.0233 | 0.0236 |
| 6.6 | 0.6946 | 0.0371 | 0.0321 | 0.0231 | 0.0202 | 0.0205 | 0.0216 | 0.0218 | 0.0220 | 0.0223 | 0.0226 | 0.0229 | 0.0232 | 0.0235 |
| 6.2 | 0.6910 | 0.0370 | 0.0320 | 0.0229 | 0.0201 | 0.0204 | 0.0215 | 0.0217 | 0.0219 | 0.0221 | 0.0224 | 0.0227 | 0.0230 | 0.0233 |
| 5.8 | 0.6874 | 0.0369 | 0.0319 | 0.0228 | 0.0200 | 0.0203 | 0.0214 | 0.0216 | 0.0218 | 0.0220 | 0.0223 | 0.0226 | 0.0229 | 0.0232 |
| 5.4 | 0.6838 | 0.0368 | 0.0318 | 0.0227 | 0.0199 | 0.0202 | 0.0213 | 0.0215 | 0.0217 | 0.0219 | 0.0222 | 0.0225 | 0.0228 | 0.0231 |
| 5.0 | 0.6802 | 0.0367 | 0.0317 | 0.0226 | 0.0198 | 0.0201 | 0.0212 | 0.0214 | 0.0216 | 0.0218 | 0.0221 | 0.0224 | 0.0227 | 0.0230 |
| 4.6 | 0.6766 | 0.0366 | 0.0316 | 0.0225 | 0.0197 | 0.0200 | 0.0211 | 0.0213 | 0.0215 | 0.0217 | 0.0220 | 0.0223 | 0.0226 | 0.0229 |
| 4.2 | 0.6730 | 0.0365 | 0.0315 | 0.0224 | 0.0196 | 0.0200 | 0.0211 | 0.0213 | 0.0215 | 0.0217 | 0.0220 | 0.0223 | 0.0226 | 0.0229 |
| 3.8 | 0.6694 | 0.0364 | 0.0314 | 0.0223 | 0.0195 | 0.0199 | 0.0210 | 0.0212 | 0.0214 | 0.0216 | 0.0219 | 0.0222 | 0.0225 | 0.0228 |
| 3.4 | 0.6658 | 0.0363 | 0.0313 | 0.0222 | 0.0194 | 0.0198 | 0.0209 | 0.0211 | 0.0213 | 0.0215 | 0.0218 | 0.0221 | 0.0224 | 0.0227 |
| 3.0 | 0.6622 | 0.0362 | 0.0312 | 0.0221 | 0.0193 | 0.0197 | 0.0208 | 0.0210 | 0.0212 | 0.0214 | 0.0217 | 0.0220 | 0.0223 | 0.0226 |
| 2.6 | 0.6586 | 0.0361 | 0.0311 | 0.0220 | 0.0192 | 0.0196 | 0.0207 | 0.0209 | 0.0211 | 0.0213 | 0.0216 | 0.0219 | 0.0222 | 0.0225 |
| 2.2 | 0.6550 | 0.0360 | 0.0310 | 0.0219 | 0.0191 | 0.0195 | 0.0206 | 0.0208 | 0.0210 | 0.0212 | 0.0215 | 0.0218 | 0.0221 | 0.0224 |
| 1.8 | 0.6514 | 0.0359 | 0.0309 | 0.0218 | 0.0190 | 0.0194 | 0.0205 | 0.0207 | 0.0209 | 0.0211 | 0.0214 | 0.0217 | 0.0220 | 0.0223 |
| 1.4 | 0.6478 | 0.0358 | 0.0308 | 0.0217 | 0.0189 | 0.0193 | 0.0204 | 0.0206 | 0.0208 | 0.0210 | 0.0213 | 0.0216 | 0.0219 | 0.0222 |
| 1.0 | 0.6442 | 0.0357 | 0.0307 | 0.0216 | 0.0188 | 0.0192 | 0.0203 | 0.0205 | 0.0207 | 0.0209 | 0.0212 | 0.0215 | 0.0218 | 0.0221 |

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TABLE I: Properties of the Expanding Field of Flow behind Shock Waves of Different Strength.

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representation of the behavior of the particles inside the shock wave at a given time. The interchangeability of \( r \) and \( t \) in the variable \( \xi \) also permits us to interpret these figures—with a proper reversal and change of scale—as the velocity, pressure, and density which prevail at a given fixed radius \( r = 1 \), as time increases from that of the passage of the shock to that of the arrival of the core. The arbitrary choice of the numbers \( r = 1 \) and \( t = 1 \) for which the figures are drawn is justified by the latent arbitrariness of the units in which these quantities are expressed.

A comparison of Table 2 with the corresponding table of Paper I discloses the equality of the columns \( p_1/p_0, \rho_1/\rho_0 \), and, consequently, \( T_1/T_0 \) and \( K_1/K_0 \), at corresponding \( \gamma \)'s and Mach numbers. This follows immediately from the fact that the Rankine-Hugoniot

\[
\rho_1 \rho_0 = f (M^2, \gamma), \quad \frac{p_1}{p_0} = g (M^2, \gamma).
\]

The field of flow considered in this investigation is not isentropic. As is well known, the mere existence of the shock front is bound to bring about a discontinuous increase of the entropy of gas streaming through it; and, as will be shown below, the
entropy continues to increase thereafter as $\xi \to \infty$. Let $s(\xi)$, in what follows, denote the entropy at any point within the shock wave, and $s_0$ that of the undisturbed medium in front of the shock. As is well known,

$$s(\xi) - s_0 = \frac{\mathcal{R}}{\gamma - 1} \log \frac{K}{K_0},$$

where $\mathcal{R}$ denotes the gas constant and $K_0$ and $K$ are the values of the adiabatic constants in front of and behind the shock wave. Now, in front of the shock,

$$K_0 = \rho_0 p_0^{-\gamma} = \frac{8}{3} \pi G \beta^{\frac{2}{\gamma - 2}} r_0^{\frac{\gamma - 2}{2(\gamma - 3)}}$$

by virtue of equations (32) and (34), while, behind the shock,

$$K = \rho p^{-\gamma} = \rho_0 p_0^{\frac{\gamma}{\gamma - 1}}$$

by equations (14) and (15). These equations disclose that the undisturbed medium itself will not be isentropic unless $\gamma = 1.2$; and the medium behind the shock wave will not be so, even if the entropy in front of the wave is constant. In general, we find that

$$\frac{K}{K_0} = \frac{3 \beta^{\gamma - 2} P_{\Omega}^{-\gamma}}{8 \pi G \xi^{1 - \gamma / 2}} = \frac{25}{16} \frac{\gamma}{\xi} \left( \frac{\xi_0}{\xi} \right)^{1/2} \left( \frac{P}{\beta \sqrt{\xi}} \right) \left( \frac{\Omega}{\beta \sqrt{\xi}} \right)^{\gamma};$$

and, in particular, if $K_1$ denotes the value of the adiabatic constant immediately behind the shock, an appeal to equations (38) and (39) discloses that

$$\frac{K_1}{K_0} = \frac{2 \gamma - \gamma x + x}{x (\gamma + 1)^{1+\gamma} (\gamma - 1 + 2x)^{\gamma}}.$$
which, for $x = 1$ (i.e., the Mach number $M = 1$), indeed reduces to $1$. In this case there is no entropy change across the incipient shock. The reader should note that the foregoing equation turns out to be identical with equation (53) of Paper I, which discloses that the ratio $K_1/K_0$ (and so the difference in entropy $s_1 - s_0$) across a shock wave of the same strength in both the GRM and the present model turns out to be the same. The penultimate column of Table 1 lists the auxiliary quantity $(P/\beta\sqrt{\xi})(\Omega/\beta\sqrt{\xi})^{-\gamma}$ as a function of $\xi$, which should facilitate the computation of the entropy at any point of the field of flow under investigation.

The last column of Table 1 contains the values of the integral

$$F(\xi) = \int_{\xi_0}^{\xi} \left\{ \frac{25}{24} \frac{\gamma-1}{x} \left[ \frac{U^2 \Omega}{2 \beta} + \frac{P}{\beta (\gamma - 1)} \right] - \frac{\xi^{3/5} \Omega}{5 \beta} \int_{\xi_0}^{\xi} \frac{\xi^4 \Omega d\xi}{\beta \xi^\gamma} \right\} \frac{d\xi}{\xi^2},$$

which is related with the total energy $E(\xi)$ contained, at any time, within a concentric shell extending from $\xi_0$ to $\xi$ by means of

$$E(\xi) = \frac{1}{8} \pi^2 G \beta^2 F(\xi).$$

In particular, the total energy contained in our model can be obtained from equation (64) by setting, in the latter, $\xi = \xi_0$.

A knowledge of $E(\xi_0)$ puts us, in turn, in a position to determine the amount of energy which must be released by the instantaneous initial explosion in order to produce a
shock wave of requisite strength. A sum $H$ of the thermal and gravitational energy originally stored in the envelope has already been given by equation (57), where, by virtue of equations (19) and (20), we are now entitled to set $R/a = (\xi_0/\xi_0)^{1/5}$. The difference $|E(\xi_0) - H(\xi_0)|$ therefore furnishes the absolute amount of energy whose instantaneous release was adequate to produce the computed phenomena, and the ratio $|E - H|/|H|$ expresses this amount in terms of the original contents of energy of our model.

The numerical values of the quantity $|E - H|/|H|$ corresponding to each one of our solutions can be found in the last column of Table 2, which summarizes the main physical characteristics of our solutions.

V. DISCUSSION OF THE RESULTS

In conclusion, the bearing of the results obtained in the preceding sections and in Paper I on the dynamics of the nova phenomenon may be briefly discussed. The models at the basis of our investigations—the GRM of Paper I and that of the present contribution—differ widely in their assumed internal structure. A decrease in density with increasing radial distance in the layers through which our flow propagates is not so different in either case—with $\rho$ varying as $r^{-2}$ in the envelope of a GRM, and as $r^{-2.6}$ in the model considered in the present paper—but, whereas the total mass contained in the envelope of a GRM is assumed to be negligible in comparison with the mass of its core, the model considered in this paper possessed no massive core (or, at least, the volume within which the law $\rho \sim r^{-3/2}$ could be expected to break down was considered to be small enough for the mass inclosed within it to be negligible).
This fact has one important consequence, namely, whereas in the GRM the total energy of the model is effectively stored in its core, this will not be true of our second model. Now it is a well-known fact that the energy lost by a nova in the course of its explosion represents but a minute fraction \(10^{-5} - 10^{-6}\) of the total energy stored inside the star. Since a glance at Table 2 discloses that the values of \(|E - H| / |H|\) listed in its last column for our second model are incomparably larger than those permissible by the astrophysical evidence, it follows that this model may come close to reality only in the initial phases of the explosion, when the radius \(R\) of the corresponding shock wave includes but a small fraction of the total volume of a prenova and the "bubble" created by the initial explosion at the center still continues to grow. On the other hand, the GRM may possibly offer an approximation to the actual phenomenon in advanced stages of the explosion, when the shock wave characterized by a very high Mach number has disengaged itself to a sufficient distance from the main mass of the star. The latter can then be regarded as the "core" of the GRM, and its internal structure becomes largely irrelevant as far as its attraction on the shell is concerned. The intermediate phases of a nova explosion (constituting the main part of the whole phenomenon) are, however, such that neither model considered by us so far can describe them as yet to any degree of approximation. In order to do so, a more adequate model must be sought; but this constitutes a task which must be deferred for subsequent investigation.

In conclusion, the authors wish to express their sincere appreciation to Miss Virginia K. Brenton and Mrs. Margaret D. Hill for carrying out the numerical integrations presented in this paper, and to Miss Katherine J. Campbell and Mr. F. Gilbert Davoren for editorial assistance in the preparation of the tables.

APPENDIX

The aim of the present paper has been, in brief, to investigate the properties of solutions of the fundamental equations (1)-(4) of our problem, which are of the type of progressing waves—i.e., of such solutions in which the variables \(U, P, \Omega,\) and \(S\) can be made to depend on a single independent variable \(\xi = r^\phi \psi,\) where \(\phi\) and \(\psi\) are suitably chosen numbers. We have found that the form of our equations, combined with the requirement that the total energy of the wave motion be independent of the time, has led us to a single-valued specification of \(\psi / \phi\) in the form of a rational fraction. The question may be asked, however, whether the particular solutions investigated in this paper represent the only type of progressing waves which are compatible with the basic physical requirements of our problem or whether other solutions exist which are characterized by a different functional relationship between \(\xi\) and \(r\) or \(t.\) In Paper I, where our fundamental system of equations (1)-(3) was nonhomogeneous, the question of the possible existence of other types of progressing waves besides those already investigated did not arise; for the gravity term in the Eulerian equation of motion by itself precluded the functional dependence of \(\xi\) on \(r\) and \(t\) in any other way than through a product of their powers, characterized by a fixed ratio \(\psi / \phi = -\frac{3}{2}.\) In the present case of a variable mass, where our fundamental system of equations (1)-(4) is homogeneous in the dependent variables, the possibility of other types of progressing waves besides those already investigated cannot, however, be a priori ruled out. The aim of this appendix will be to examine this specific question and to demonstrate that if the dependent variables \(U, P,\) and \(\Omega\) of our fundamental system of partial differential equations are to be expressible in terms of an equivalent system of ordinary differential equations with \(\xi(r, t)\) as the sole independent variable, \(\xi(r, t)\) must necessarily be of the form \(\phi(r)\psi(t),\) where \(\phi(r)\) is a power of \(r\) alone, and \(\psi(t)\) may be either a power or an exponential of \(t.\) A product of the powers of both \(r\) and \(t\) represents, moreover, the only admissible form of \(\xi\) which may render the total energy of the corresponding wave motion independent of the time.

In order to prove these assertions, let us assume that

\[
\xi = \xi (r, t)
\]
and seek the solution of equations (1)–(4) in terms of
\[ u = \tilde{u}(r, t) \, U(\xi), \quad p = \tilde{p}(r, t) \, P(\xi), \quad \rho = \tilde{\rho}(r, t) \, \Omega(\xi), \quad m = \tilde{m}(r, t) \, M(\xi). \]

Consider, now, one set of the seven partial derivatives involved in the system (1)–(4)—say, those of \( u \). Differentiating the first one of the foregoing equations (66) with respect to \( r \), we find that
\[ \frac{\partial u}{\partial r} = U(\xi) \frac{\partial \tilde{u}}{\partial r} + \tilde{u} U' = \tilde{u} U(\xi) \frac{\partial \xi}{\partial r} \left\{ \frac{\partial \tilde{u}/\partial r}{\tilde{u}(\partial \xi/\partial r)} + \frac{U''(\xi)}{U'(\xi)} \right\}; \]
and a similar differentiation with respect to \( t \) yields
\[ \frac{\partial u}{\partial t} = U(\xi) \frac{\partial \tilde{u}}{\partial t} + \tilde{u} U' = \tilde{u} U(\xi) \frac{\partial \xi}{\partial t} \left\{ \frac{\partial \tilde{u}/\partial t}{\tilde{u}(\partial \xi/\partial t)} + \frac{U''(\xi)}{U'(\xi)} \right\}, \]
where a prime denotes a differentiation with respect to \( \xi \).

The second terms in braces on the right-hand sides of equations (67) and (68) are, by definition, functions of \( \xi \) alone. If our solution is to represent a progressing wave (i.e., if the functions \( U, P, \Omega, \) and \( M \) in eq. [66] are to be expressible in terms of ordinary differential equations with \( \xi \) as the only independent variable), the same must necessarily be true of the whole contents of the braces in equations (67) and (68). This fact permits us to assert at once that
\[ \frac{\partial u}{\partial t} = \tilde{u} \frac{\partial \xi}{\partial t} \, F(\xi), \]
and
\[ \frac{\partial u}{\partial r} = \tilde{u} \frac{\partial \xi}{\partial r} \, G(\xi), \]
where \( F(\xi) \) and \( G(\xi) \) are functions of \( \xi \) alone.

Differentiating equation (69) with respect to \( r \) and equation (70) with respect to \( t \), we obtain
\[ \frac{\partial^2 \tilde{u}}{\partial r \partial t} = \frac{\partial \tilde{u}}{\partial r} \, \frac{\partial \xi}{\partial t} \, F(\xi) + \frac{\partial^2 \xi}{\partial r \partial t} \, F(\xi) + \tilde{u} \frac{\partial \xi}{\partial r} \frac{\partial \xi}{\partial t} \, F'(\xi) \]
\[ = \frac{\partial \tilde{u}}{\partial \xi} \frac{\partial \xi}{\partial t} \, G(\xi) + \tilde{u} \frac{\partial^2 \xi}{\partial r \partial t} \, G(\xi) + \tilde{u} \frac{\partial \xi}{\partial r} \frac{\partial \xi}{\partial t} \, G'(\xi), \]
which, by elimination of the left-hand side, yields
\[ \{ F(\xi) - G(\xi) \} \frac{\partial^2 \xi}{\partial r \partial t} + \{ F'(\xi) - G'(\xi) \} \frac{\partial \xi}{\partial r} \frac{\partial \xi}{\partial t} = 0, \]

as the partial differential equation determining \( \xi \). This equation admits immediately of the following integrals:
\[ \frac{\partial \xi}{\partial r} = \frac{f(\xi)}{F(\xi) - G(\xi)}, \]
\[ \frac{\partial \xi}{\partial t} = \frac{g(\xi)}{F(\xi) - G(\xi)}, \]
where \( f(\xi) \) and \( g(\xi) \) are, so far, arbitrary functions of their parameters.

8Primes will, as before, be used to denote differentiation whenever a function depends on a single variable only.
Consider, now, the last two terms of equation (3) of continuity. In accordance with equations (66), we find that
\[ \frac{\partial u}{\partial r} + 2 \frac{u}{r} = U(\xi) \frac{\partial \tilde{u}}{\partial r} + \tilde{u} U'(\xi) \frac{\partial \xi}{\partial r} + \frac{2 \tilde{u} U(\xi)}{r} = \frac{\tilde{u} U'(\xi)}{r} \left\{ \frac{2 U(\xi)}{U'(\xi)} + r \frac{\partial \xi}{\partial r} \right\} + U \frac{\partial \tilde{u}}{\partial r}. \] (75)

By the same argument as that used before, if \( U(\xi)/U'(\xi) \) is a function of \( \xi \) alone, so also must be \( r(\partial \xi/\partial r) \), which permits us to assert that
\[ r \frac{\partial \xi}{\partial r} = H_1(\xi) \] (76)
and, likewise, that
\[ r \frac{\partial \tilde{u}}{\partial r} = \tilde{u} H_2(\xi), \] (77)
where \( H_1(\xi) \) and \( H_2(\xi) \) are again arbitrary functions of \( \xi \). A comparison of equations (73) and (76) discloses that, of necessity,
\[ f(r) = r^{-1}, \quad H_1(\xi) = \left[ F(\xi) - G(\xi) \right]^{-1}, \] (78)
by virtue of which, equations (73) and (74) can be combined to yield
\[ r \frac{\partial \xi}{\partial r} = \frac{1}{g(t)} \frac{\partial \xi}{\partial t} = 0 \] (79)
as the necessary and sufficient condition for the function \( \xi(r, t) \) to reduce our system of partial differential equations for \( U(\xi), P(\xi), \) and \( \Omega(\xi) \) into ordinary differential equations.

The function \( g(t) \) in equation (79), moreover, is not arbitrary, and its explicit form can be determined as follows. The Lagrangian derivative of \( u \) in equation (1) can be rewritten by means of equations (67) and (68) as
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = U \frac{\partial \tilde{u}}{\partial t} + \tilde{u} U' \frac{\partial \xi}{\partial t} + \tilde{u} U'^2 \frac{\partial \tilde{u}}{\partial r} + \tilde{u}^2 U U' \frac{\partial \xi}{\partial r} \] (80)
from which it follows that
\[ \tilde{u} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \xi}{\partial t} = K_1(\xi) \] (81)
and
\[ \tilde{u} \frac{\partial \xi}{\partial r} + \frac{\partial \xi}{\partial t} = K_2(\xi), \] (82)
where \( K_1(\xi) \) and \( K_2(\xi) \) again denote arbitrary functions of \( \xi \). A comparison of equations (79) and (82) indicates that
\[ \tilde{u} = \left\{ \frac{\partial \xi}{\partial t} \right\} K_2(\xi) \] (83)

Differentiating this equation, we obtain
\[ \frac{\partial \tilde{u}}{\partial t} = g(t) K_2(\xi) + r g(t) K_2'(\xi) \frac{\partial \xi}{\partial r} \] (84)
and
\[ \frac{\partial \tilde{u}}{\partial t} = r g'(t) K_2(\xi) + r g(t) K_2'(\xi) \frac{\partial \xi}{\partial t}, \] (85)
respectively; and a division of these equations yields
\[
\frac{\partial u}{\partial t} + \frac{\partial \tilde{u}}{\partial r} = \left( \frac{g'(t)}{g(t)} \right) \left( (K_2'[\xi]) / (K_2[\xi]) \right) \left( \frac{\partial \xi}{\partial t} \right) / \left( \frac{\partial \xi}{\partial r} \right) \tag{86}
\]

On the other hand, from equations (69) and (70) it follows that
\[
\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial r} = F(\xi) \left( \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial r} \right) / G(\xi) \tag{87}
\]

by equations (73), (74), and (78). By equating the right-hand sides of equations (86) and (87), we find that the differential equation governing the function \( g(t) \) is of the form
\[
\frac{1}{g^2(t)} \frac{dg(t)}{dt} = \left\{ \frac{K_2'[^2(\xi)]}{K_2[^2(\xi)]} + F(\xi) \right\} \frac{1}{G(\xi)} \tag{88}
\]

Now the left-hand side of this equation is a function of \( t \) alone, while the right-hand side depends only on \( \xi \). Since, by equation (65), \( \xi \) is assumed to depend on both \( r \) and \( t \), it follows that the foregoing equation can be satisfied only if both its sides are zero or both equal to a constant, (say) \(-\lambda^{-1}\). If so, the equation for \( g(t) \) takes the neat form
\[
\frac{g'}{g^2} + \frac{1}{\lambda} = 0, \tag{89}
\]

which can be readily integrated.

In doing so we encounter two possibilities, depending on whether or not the ratio \( g'/g^2 \) is equal to, or different from, zero. Let us assume \( \lambda \) to be finite, in which case equation (89) integrates into
\[
g(t) = \frac{\lambda}{t + \mu}, \tag{90}
\]

where \( \mu \) denotes an arbitrary constant. The value of this constant depends on the origin from which the time is reckoned and, without loss of generality, may be set equal to zero. Let us now insert (putting \( \mu = 0 \)) the foregoing equation (90) into equation (79) and, furthermore, let \( t^{-\lambda} = \tau \). If so, equation (79), governing the function \( \xi(r, t) \), readily takes the form
\[
r \frac{\partial \xi}{\partial r} + \tau \frac{\partial \xi}{\partial t} = 0, \tag{91}
\]

which, by Euler's theorem, constitutes the necessary and sufficient condition for \( \xi(r, \tau) \) to be a homogeneous function of \( r \) and \( \tau = t^{-\lambda} \) of zero degree. In consequence, it follows that
\[
\xi(r, t) = f(r^\lambda); \tag{92}
\]

and, without loss of generality, we may put \( \xi \sim r^{\lambda} \), since any other functional relationship between \( \xi \) and \( r^\lambda \), introduced into the fundamental equations (1)–(4), would not lead to any new solutions.

In order to complete our discussion, let us consider the case which arises if both sides of equation (88) vanish. If this happens, equation
\[
\frac{1}{g^2} \frac{dg}{dt} = 0 \tag{93}
\]
integrates readily into
\[
g(t) = \text{Constant} \quad \text{(say, } \mu \neq 0). \tag{94}
\]

In this case, equation (79) ceases to express Euler's theorem on homogeneous functions, and \( \xi \) is, therefore, no longer required to be a homogeneous function of its arguments. In order to
establish the explicit form of this function, let us introduce \( f(r) = r^{-1} \) and \( g(t) = \mu \) in equations (73) and (74); integrating the former along a line \( t = \text{Constant} \), we find that

\[
\log c r = \int \{ F(\xi) - G(\xi) \} d\xi = A(\xi),
\]

where \( c \) is independent of \( r \) but may be a function of \( t \). In order to establish its form, let us differentiate equation (95) with respect to \( t \), and we obtain

\[
A'(\xi) \frac{d\xi}{dt} = \frac{c'(t)}{c(t)}.
\]

By equation (95), however, \( A'(\xi) = F(\xi) - G(\xi) \), and an appeal to equations (74) and (94) discloses that

\[
\frac{c'(t)}{c(t)} = \mu.
\]

This equation can be immediately integrated to yield

\[
c(t) = ve^{\mu t},
\]

where the integration constant \( v \) is immaterial, as it can again be absorbed in the choice of our origin of the time. Inserting equation (98) into equation (95), we establish that

\[
e^{A(\xi)} = vr e^{\mu t},
\]

which discloses that, in the present case,

\[
\xi(r, t) = f(r e^{\mu t});
\]

and, without loss of generality, we can again set \( \xi \sim r e^{\mu t} \).

The foregoing discussion demonstrates that, if our fundamental system (1)–(4) of partial differential equations is to be reducible to ordinary differential equations with \( \xi \) as the sole independent variable, the latter must be of the form

\[
\xi = rt
\]

or

\[
\xi = re^{\mu t};
\]

where the constants \( \lambda \) and \( \mu \) must be determined by supplementary physical conditions of the problem under investigation. In particular, the reader can easily verify that the alternative form of \( \xi \) as represented by equation (102) is incompatible with a requirement that the total energy of the wave motion (or even a sum of the kinetic and thermal energies) be independent of the time (which is a definition of the blast wave). As we have seen in Section II of this paper, the form of \( \xi \) as represented by equation (101) can be reconciled with the assumption of a blast wave, provided that \( \lambda = -\frac{4}{5} \). Thus, as long as the functions \( U(\xi), P(\xi), \) or \( \Omega(\xi) \) are to be expressible in terms of ordinary differential equations, the solutions of our fundamental system (1)–(4) investigated earlier in this paper represent the only possible type of progressing blast waves of our problem; and, in addition, if the expanding field of flow is to be headed by a shock front characterized by a constant Mach number throughout all time, the structure of the undisturbed configuration is uniquely specified by equations (32) and (34).