PROPAGATION OF SHOCK WAVES IN THE GENERALIZED
ROCHE MODEL*

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ABSTRACT

The aim of the present paper has been to investigate the events which take place if the core of a
generalized Roche model is made to expand as the result of an instantaneous central explosion. A sudden
expansion of the core sets off a wave of condensation traveling through the envelope; and, if the initial
explosion has been sufficiently strong, the outgoing disturbance will possess the characteristics of a shock
wave. It is shown that, if the initial explosion has been instantaneous, the velocity, pressure, and density
at any point of the disturbed medium can be made to depend on a single parameter, \( \xi = r^{-3/2} \), where \( t \)
denotes the time and \( r \) the radial distance.

The equations of the problem have been rewritten in terms of \( \xi \) as the sole independent variable and
integrated numerically for 18 cases corresponding to different Mach numbers of the shock waves and
different ratios, \( \gamma \), of specific heats of the gas constituting the envelope. The expanding regime forms a
concentric shell, limited on the outside by the shock front of radius varying as \( r^{3/2} \), while the interface of
the core confronts us with a “contact discontinuity.” For large values of the Mach number \( M \), the thick-
ness of the shell increases with increasing \( M \) and decreasing value of the ratio of specific heats; for
\( \gamma = \frac{4}{3} \) (corresponding to polyatomic gas) the radius of the core becomes zero for any strength of the
shock. It is shown that, for \( \gamma = \frac{5}{3} \) (corresponding to monatomic gas), a shock wave characterized by a
Mach number \( M = 2.877 \ldots \) is just sufficiently strong to endow the mass particles immediately behind
it with a velocity equal to one of escape from the gravitational field of the configuration, while for
\( M > 5.92 \ldots \) the whole envelope will eventually be ejected.

Table 1 contains the numerical data describing the individual solutions in terms of nondimensional
parameters which can be converted into absolute units by a suitable choice of the initial conditions; and
Table 2 summarizes the physical properties of the respective shock waves.

In a previous paper, published recently in this Journal by one of us,\(^1\) the radial
oscillations of a generalized Roche model (consisting of a compressible massive core, of
finite dimensions, surrounded by an infinitesimally thin envelope) were investigated
in some detail. Such oscillations, of the nature of standing waves characterized by a node
at the center and a loop at the surface of zero pressure, may be of astrophysical importance
in connection with the pulsation of composite stellar models. The present investigation
will, on the other hand, be concerned with the properties of running waves in stellar
interiors. More specifically, we propose to consider the propagation, through a general-
ized Roche model and certain similar configurations, of spherical disturbances which are
sufficiently energetic to give rise to expanding shock waves and whose impact may be
sufficient actually to eject matter from the surface at speeds exceeding the velocity of
escape from the gravitational field of the whole configuration. The astrophysical impli-
cations of the situation which we propose to consider lie clearly in the direction of the
nova phenomenon. We do not claim, to be sure, to deal as yet very closely in this paper
with the actual physical situation encountered in the nova outbursts. A great amount of
preparatory work must, however, be done before a solution of the actual astrophysical
problem can be even approached with any hope of success. The work summarized in the
present paper should be regarded as a first contribution to the study of the propagation
of shock waves in spherical-gas models of astrophysical interest and a first step toward
our ultimate goal.

In order to describe the physical situation analyzed in the present work, let us con-
sider, first, a classical generalized Roche model consisting of a massive core of finite

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dimensions but arbitrary structure, surrounded by an envelope of infinitesimal weight, in which the density falls off as the inverse square of the distance from the center. Suppose, now, that such a configuration becomes disturbed from its state of equilibrium by an instantaneous explosion—whatever its origin—at the center, which causes the core suddenly to expand. Its expansion will, in turn, set off in the envelope an outward-going spherical wave of condensation; and if the impulse provided by the initial explosion is sufficiently strong, the outgoing disturbance will possess the characteristics of a shock wave and will propagate in accordance with well-known laws governing the propagation of such waves. Our aim will be, in brief, to investigate the way in which the energy liberated by the explosion is converted into motion and heat through the medium of shock waves propagating in the envelope of a generalized Roche model. In doing so, however, we shall be prevented by sheer weight of mathematical difficulties from considering the most general type of the motion which may result from an instantaneous central explosion. Instead, we shall limit ourselves to investigating solutions of the type of "progressing blast waves"—in the terminology of Courant and Friedrichs—which are characterized by a shock wave on the head of the disturbed motion and which carry a sufficient impulse actually to cause an ejection of the matter from the surface. Progressing waves in compressible homogeneous media situated in the field of constant gravity have been previously studied by several writers. Our present investigation may therefore be regarded as an extension of the previous work with terrestrial applications to the more difficult astronomical case of progressing blast waves in heterogeneous models, which propagate through a fluctuating gravitational field created by the disturbed mass itself. The main conclusions arrived at in the course of this investigation have already been summarized in the abstract; so that, in what follows, our task will be to substantiate them in detail.

**EQUATIONS OF THE PROBLEM**

Let $u$, $p$, and $\rho$ denote the radial velocity, pressure, and density at any point inside our model at a distance $r$ from the center and at a time $t$. If, moreover, $m$ denotes the (constant) mass of the core, $\gamma$ the ratio of specific heats of the gas constituting it, and $G$ the gravitation constant, the Eulerian hydrodynamical equations reduce to

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} \frac{Gm}{r^2} = 0, 
$$

the equation of continuity becomes

$$
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left\{ \frac{\partial u}{\partial r} + \frac{2u}{r} \right\} = 0, 
$$

while the energy equation (appropriate for polytropic gas) assumes the form

$$
\frac{\partial}{\partial t} (p \rho^{-\gamma}) + u \frac{\partial}{\partial r} (p \rho^{-\gamma}) = 0.
$$

The reader should keep in mind that, if shock waves are to occur in the flow governed by the foregoing equations, the flow behind such a wave will no longer be isentropic, and, in consequence, there will be no closed equation of state relating $p$ and $\rho$. Such a relationship can be obtained only by a simultaneous integration of equations (1), (2),


and (3) for \( u, p, \) and \( \rho \) as functions of \( r \) and \( t \), subject to appropriate boundary conditions.

In our present problem the outer boundary conditions are given over a moving surface (the shock wave), and the inner conditions are defined on the surface of the expanding core. In what follows, let \( p_0 \) and \( \rho_0 \) denote the undisturbed values of pressure and density in front of the shock wave (where they depend only on \( r \)), and \( u_1, p_1, \) and \( \rho_1 \) be the values of the respective quantities at any point immediately after the passage of the shock. If so, the well-known Rankine-Hugoniot conditions,\(^4\) expressing the continuity of energy, mass, and momentum across the shock wave, permit us to express \( u_1, p_1, \) and \( \rho_1 \) in terms of the undisturbed values of these quantities by means of the following equations:

\[
\begin{align}
\frac{V}{\gamma + 1} - \frac{2\gamma p_0}{(\gamma + 1) \rho_0 V} & = u_1, \\
\frac{2\rho_0 V^2 - (\gamma - 1) p_0}{\gamma + 1} & = p_1, \\
\frac{(\gamma + 1) \rho_0^2 V^2}{(\gamma + 1) \rho_0 V^2 + 2\gamma p_0} & = \rho_1,
\end{align}
\]

where \( V \) denotes the velocity of propagation of the shock. On the other hand, consistent with our definition of the generalized Roche model, we should expect that the derivative \( \partial p/\partial r \) would be discontinuous at the core; but the significant feature of our problem is that neither the position of the shock nor that of the core is known to us beforehand at any moment but must be ascertained by the compatibility of prescribed conditions as the integration proceeds.

The problem of solving equations (1)-(3) subject to the conditions just enumerated is one of considerable complexity: the system of partial differential equations is one of third order and of second degree, in two independent variables, with initial conditions defined over two moving boundaries. These equations cannot be linearized by any transformation of the variables, and the neglect of nonlinear terms would eliminate the essential features of the problem. A retention of the nonlinear terms precludes, on the other hand, any possibility of the construction of an analytical solution; so that numerical integrations appear to offer the only method of approach. Numerical integration of partial differential equations in two independent variables constitutes, however, no small task. Before embarking upon it, we propose, therefore, to investigate, first, the conditions for which the fundamental system of partial differential equations may be reducible to that of ordinary differential equations, which can be solved—numerically or otherwise—with much less difficulty.

In specific terms, let us seek such solutions of the system of equations (1)-(3) in which \( u, p, \) and \( \rho \) can be made to depend on \( r \) and \( t \) only through the product \( r^\phi \psi \), where \( \phi \) and \( \psi \) are suitably chosen constants. Accordingly, we shall assume that

\[
\begin{align}
u & = \frac{r}{t} \, U (\xi), \\
p & = r^{\phi + 2\lambda + 2} P (\xi), \\
\rho & = r^{\phi + \Omega} (\xi),
\end{align}
\]

and the local velocity of sound

\[
c^2 = \frac{\rho}{\rho} \frac{p}{\rho} = \left( \frac{r}{t} \right)^2 \, C^2 (\xi),
\]

\(^4\) Cf., e.g., Courant and Friedrichs, op. cit., sec. 54.
where
\[ \xi = r^\phi \psi \] (11)

and the parameters \( \kappa, \lambda, \phi, \) and \( \psi \) are to be determined by physical considerations.

The form of equation (1) imposes the ratio \( \psi/\phi = -\frac{3}{\kappa} \) if \( \xi \) is to replace \( r \) and \( t \) throughout as the independent variable. The other parameters, \( \kappa \) and \( \lambda \), may be determined by considering the form of the total energy carried by the wave motion. If \( R(t) \) denotes the radius, at any time, of the shock front and \( a(t) \) that of the core of our generalized Roche model, it follows that the total energy \( E \) of the envelope (being the sum of its kinetic, thermal, and gravitational energies) will be given by

\[ E = 4\pi \int_a^R \left\{ \frac{1}{2} \rho u^2 + \frac{\theta}{\gamma-1} - G \frac{m}{r} \right\} r^2 dr. \] (12)

Now if the total energy of the initial explosion is to be released at an instant, the foregoing expression for \( E \) must obviously be independent of the time. Consistent with our foregoing assumptions, we have

\[ R(t) = \xi^1/\phi^{(1)} \] (13)

and
\[ a(t) = \xi^1/\phi^{(1)} , \] (14)

where \( \xi_0 \) and \( \xi_1 \) are values which the new independent variable \( \xi \) assumes at the shock and the core, respectively. Inserting the foregoing relations together with equations (7)-(9) into equation (12), we find the kinetic and thermal energies carried by the wave motion to be independent of the time, provided that

\[ \lambda - (\kappa + 5) \left( \frac{\psi}{\phi} \right) = 2, \] (15)

and it follows that \( \kappa \) and \( \lambda \) are related by

\[ \lambda = -\frac{2}{3} (\kappa + 2). \] (16)

It is interesting to note that this choice of parameters will automatically enforce the constancy of the gravitational contribution to the total energy. This constancy would impose, between \( \kappa, \lambda, \phi, \) and \( \psi, \) the relation

\[ \lambda - (\kappa + 2) \left( \frac{\psi}{\phi} \right) = 0 , \]

which, together with equation (16), again yields

\[ \frac{\psi}{\phi} = -\frac{3}{\kappa} \quad \text{and} \quad \lambda = -\frac{2}{3} (\kappa + 2). \] (17)

It should be observed that the foregoing equations do not specify our parameters uniquely but provide only two relations between them. These relations are obviously satisfied by

\[ \phi = -3 , \quad \psi = 2 , \quad \kappa = -2 , \quad \lambda = 0 , \] (18)

the values which we shall hereafter adopt. It can be easily shown that no generality has been lost by this choice; for all other combinations of the four parameters consistent with equations (17) will lead to no new solutions.

Having thus specified the values of the arbitrary parameters in equations (7)-(9), we shall rewrite equations (1)-(3) in terms of the new independent variable \( \xi = r^{-2/\phi}. \) If we abbreviate, for the sake of simplicity,

\[ 3 U(\xi) - 2 = \Xi(\xi) \] (19)
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and

\[ C^2(\xi) = S(\xi), \]  

we establish, after some transformations, that the original system of equations (1)-(3) is equivalent to the following system of three ordinary differential equations:

\[ \frac{1}{P} \frac{dP}{d\xi} = \frac{1}{3 \xi} \left\{ 3 \left( 2 \gamma - 2 \right) + 3 \gamma \Xi - 3 \gamma \xi \frac{d\Xi}{d\xi} \right\}, \]  

\[ \frac{1}{S} \frac{dS}{d\xi} = \frac{1}{3 \xi} \left\{ 2 \left( 3 \gamma - 4 \right) + (3 \gamma - 1) \Xi - 3 \left( \gamma - 1 \right) \xi \frac{d\Xi}{d\xi} \right\}, \]  

\[ \frac{1}{\Xi} \frac{d\Xi}{d\xi} = \frac{1}{3 \xi^2} \left\{ 3 + \Xi - 2 + 9 \xi Gm - \frac{27}{\gamma} \xi \frac{S}{dP} \right\}. \]  

Once these equations have been solved, the last one of our four auxiliary functions in equation (9) follows from

\[ \Omega(\xi) = \gamma \frac{P(\xi)}{S(\xi)}. \]  

SOLUTION OF THE EQUATIONS

Having reduced the differential equations of the problem to tractable form, we have yet to specify the initial conditions which determine the nature of their solution. Consider, first, the boundary conditions at the shock front. On the passage through the shock, the velocity, pressure, and density are known to undergo a discontinuous change expressed by equations (4)-(6), where, consistent with the assumptions of the preceding section,

\[ V = \frac{dR}{dt} = \frac{2 R}{3 \tau}. \]  

In front of the shock, in the undisturbed envelope of the generalized Roche model, we have, by definition,

\[ \rho_0 = \beta r^{-2}, \]  

where \( \beta \) is an arbitrary constant; and, if this envelope is to be in hydrostatic equilibrium, the integration of equation (1) discloses that the corresponding pressure must vary as

\[ p_0 = \frac{1}{3} Gm \beta r^{-3}. \]  

Insert now the foregoing relations in equations (4)-(6). If, furthermore, we replace \( u_1, p_1, \) and \( p_1 \) by expressions (7)-(9), which are expected to hold good from the shock inward, we find that immediately behind the shock front, when \( \xi = \xi_0, \)

\[ U(\xi_0) = \frac{4 \left( 1 - x \right)}{3 \left( \gamma + 1 \right)}, \]  

\[ P(\xi_0) = \frac{4 \beta \left( 2 \gamma - \gamma x + x \right)}{9 \gamma \left( \gamma + 1 \right)}, \]  

\[ \Omega(\xi_0) = \frac{\beta \left( \gamma + 1 \right)}{\gamma - 1 + 2 x}, \]  

and, therefore,

\[ S(\xi_0) = \frac{4 \left( 2 \gamma - \gamma x + x \right) \left( \gamma - 1 + 2 x \right)}{9 \left( \gamma + 1 \right)^2}, \]  

\[ \]
where

\[ x = \frac{4}{3} G m \gamma \xi_0. \]  

In order to be able to integrate equations (21)-(23) from the shock inward, we still must ascertain the values of the derivatives of \( U, P, \) and \( \Omega \) (or \( S \)) with respect to \( \xi \) immediately behind the shock wave. This can be done by means of the initial differential equations (1)-(3), which, rewritten in terms of \( U, P, \) and \( \Omega, \) assume the forms

\[
(2 - 3 U) \xi \Omega U' - 3 \xi P' = \Omega \{ U (1 - U) - G m \xi \},
\]

\[
(2 - 3 U) \xi \Omega' - 3 \xi U' = -\Omega U,
\]

\[
(2 - 3 U) \Omega \xi P' - \gamma (2 - 3 U) P \xi \Omega' = 2 \Omega P (1 - \gamma U),
\]

where primes denote differentiation with respect to \( \xi \). If, in these equations, we set \( \xi = \xi_0 \) and insert for \( U(\xi_0), P(\xi_0), \) and \( \Omega(\xi_0) \) from equations (28)-(30), we can solve the foregoing system of equations for \( U'(\xi_0), P'(\xi_0), \) and \( \Omega'(\xi_0); \) doing so, we find that, immediately behind the shock,

\[
U'(\xi_0) = -\frac{2 (5 + 6 \gamma x - 3 \gamma - 2 x)}{9 (\gamma + 1)^2 \xi_0},
\]

\[
P'(\xi_0) = -\frac{4 \beta [8 \gamma (\gamma - 1) x^2 - (21 \gamma^2 - 8 \gamma + 3) x + 2 \gamma (5 \gamma - 3)]}{27 \gamma (\gamma + 1)^2 (\gamma - 1 + 2 x) \xi_0},
\]

\[
\Omega'(\xi_0) = \frac{\beta (4 x - 4 \gamma x + \gamma - 7)}{3 (\gamma - 1 + 2 x)^2 \xi_0},
\]

and, therefore,

\[
27 (\gamma + 1)^3 \xi_0 S'(\xi_0) = 8 \{ 2 (\gamma - 1) (1 - 3 \gamma) x^2 + (15 \gamma^2 - 12 \gamma + 5) x - \gamma (6 \gamma - 10) \}. \]

It may be noticed that, since the pressure occurs in equations (21)-(23) through its logarithmic derivative only, these equations are independent of \( \beta; \) the only parameters influencing the solution are \( \gamma, x, \) and \( \xi_0. \) Of these three parameters, \( \xi_0 \) determines the position of the shock at a given time \( t_0. \) In what follows we shall arbitrarily set \( \xi_0 = 1, \) which can be done without the loss of generality by a proper choice of our units of length and time. The ratio \( \gamma \) of specific heats is characteristic of the material which we are considering; so that \( x \) remains as the only arbitrary nondimensional constant characterizing our solutions. It is easy to show that this constant is intimately related with the Mach number \( M \) (i.e., with the strength) of the shock front. As is well known,

\[
M = \frac{V}{c_0},
\]

where \( c_0 \) denotes the velocity of sound in the undisturbed medium in front of the shock wave. Now, in our present case, \( V \) is given by equation (25), while, by equations (26) and (27),

\[
c_0^2 = \gamma \frac{\rho_0}{\rho_0} = \frac{G m \gamma}{3 \tau}.
\]

Hence it transpires immediately that

\[
M^2 = \frac{1}{x},
\]
which discloses that, if the expanding regime we are considering is to be characterized by a shock wave on its head, we must have

$$0 \leq x < 1,$$

the strength of the shock being greater, the smaller the value of $x$.

Let us, furthermore, consider the relation between $x$ and the ratio of the velocity of individual particles to the velocity of escape from the gravitational field of the star. As is well known, the velocity of escape $V_{esc}$ is defined by

$$V_{esc}^2 = \frac{2Gm}{r};$$

therefore, the ratio

$$\frac{u}{V_{esc}} = y = \frac{U(\xi)}{\sqrt{(2Gm\xi)}}$$

is given by equation (7). Immediately behind the shock wave we thus have

$$y = \frac{1 - x}{\gamma + 1} \sqrt{\frac{2\gamma}{3x}},$$

relating $x$ and $y$. If any ejection of matter is to take place, we must obviously have $y > 1$. If, therefore, we are interested only in such solutions as will lead to the actual ejection of matter from the gravitational field of the star, we must choose $x$ so as to satisfy the inequality

$$0 \leq x < \frac{1}{4\gamma} \left[ 3\gamma^2 + 10\gamma + 3 - \sqrt{\left(3\gamma^2 + 10\gamma + 3\right)^2 - 16\gamma^2} \right],$$

the upper limit of which depends solely on $\gamma$. This upper limit corresponds to the velocity of ejection equal to that of the escape, while $x = 0$ would correspond to an infinite strength and Mach number of the shock and, therefore, to an infinite energy of the initial explosion.

Suppose, now, that a proper value of $x$ has been chosen and an integration of equations (21)–(23) started from the initial conditions given earlier in this section for $\xi_0 = 1$ in the direction of increasing $\xi$, which, for fixed $t$, corresponds to a march from the surface inward (the center of our configuration corresponding to $\xi = \infty$). Such integrations can proceed until, for each value of $x$, we have reached a point $\xi = \xi_1$ at which $\Xi(\xi_1)$ vanishes. The point at which $\Xi(\xi_1) = 0$ turns out to be a singular point of our solution; for the compatibility of our system of equations requires that $\Omega(\xi_1) = \infty$, though the pressure at $\xi = \xi_1$ remains continuous. When this point has been reached, we have evidently arrived at the interface of the core of our generalized Roche model. Owing to $\Xi(\xi_1) = 0$, the mass particles at $\xi = \xi_1$ are found to be moving with the velocity $V$—which means that there is no further energy transfer between the core and the envelope, in agreement with our basic premises. Thus the field of flow which we are considering is bounded by two surfaces of discontinuity; whereas, on the outer boundary (the shock wave), the pressure, density, and velocity are discontinuous with matter flowing freely across, the inner boundary is of the nature of a contact discontinuity in density but not in pressure or velocity. The flow of mass across this discontinuity is consequently zero.

It is easy to show that, for $\gamma \neq \frac{2}{3}$, the value of $\xi_1$ at which $\Xi(\xi_1)$ vanishes is always

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finite. Consider the total energy of the envelope in its undisturbed state; it is given by

\[ H = 4\pi \int_a^R \left( \frac{1}{\gamma - 1} \frac{p_0 - \frac{G m}{r}}{\rho_0 r^2} \right) r^2 dr \]

by virtue of equations (26) and (27). For \( \gamma \neq \frac{4}{3} \), this expression can remain finite only if \( a \neq 0 \). Hence, for \( \gamma > \frac{4}{3} \), the discontinuity in \( \Omega(\xi) \) is bound to occur for a finite value of \( \xi_1 \).

**NUMERICAL INTEGRATIONS**

Equations (21)–(23) as they stand are too involved to admit of the possibility of an analytical solution; so that, in order to investigate the properties of an expanding flow of gas defined by such equations, recourse must be had to numerical integration. We have performed 18 such integrations—14 for \( \gamma = \frac{5}{3} \) (monatomic gas) and 4 for \( \gamma = \frac{4}{3} \)—corresponding to a series of Mach numbers ranging from \( M^2 = 8.27924 \ldots \) (which corresponds to \( \gamma = 1 \)) to \( M^2 = \infty \). The details of such integrations are given in Table 1, the successive columns of which are self-explanatory. Each tabulation extends from \( \xi = 1 \) (assumed position of the shock wave) down to \( \xi_1 \), at which the core sets in and the density becomes discontinuous. As many decimals are retained in each tabulation as are regarded to be significant. The values so normalized permit the evaluation of the actual absolute properties of the expanding regime of gas flow corresponding to any explosion by choosing the appropriate absolute value of \( \beta \). The accompanying diagram (Fig. 1) shows a plot of \( U \) against \( C \), for a number of solutions corresponding to \( \gamma = \frac{5}{3} \) and 6 different assumed Mach numbers of expansion.

As we surmised at the beginning, the field of flow considered in this investigation is not

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**Fig. 1.—Vector field representing the solutions of our differential equations (21)–(23) for \( U = r^{-1}u \) and \( C = r^{-1}c \) of spherical progressing waves discussed in the present paper.**
<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$U$</th>
<th>$C$</th>
<th>$\beta$</th>
<th>$Q / \beta$</th>
<th>$\gamma$</th>
<th>$B / \gamma^2 - \gamma^2$</th>
<th>$F$</th>
<th>$\gamma$</th>
<th>$B / \gamma^2 - \gamma^2$</th>
<th>$F$</th>
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**TABLE I**

Physical Properties of the Expanding Field of Flow behind Shock Waves of Different Strengths.
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TABLE I

Physical Properties of the Expanding Field of Flow behind Shock Waves of Different Strengths

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Notes:
- $\gamma = \frac{5}{3}$
- $M^2 = 100$
- $\gamma = \frac{5}{3}$
- $M^2 = 300$
- $\gamma = \frac{5}{3}$
- $M^2 = 500$
- $\gamma = \frac{5}{3}$

The table continues with similar entries for different values of $\xi$, $U$, $C$, $P/\beta$, $Q/\beta$, $\gamma$, and $P_{\xi}/\gamma^{2/3}\gamma$. Each entry is followed by a footnote indicating the source of the data.
### TABLE I

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### Table I

Physical Properties of the Expanding Field of Flow behind Shock Waves of Different Strengths

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<th>$Q/\beta$</th>
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isentropic. As is well known, the difference between the entropy $s(\xi)$ at any point within the shock wave and that of the undisturbed medium in front of the shock is given by

$$s ( \xi ) - s_0 = \frac{9R}{\gamma - 1} \log \frac{K}{K_0} ,$$

(49)

where $R$ denotes the gas constant and $K_0$ and $K$ are the values of the adiabatic constants in front of and behind the shock wave. Now, in front of the shock wave,

$$K_0 = p_0 \rho_0^{-\gamma} = \frac{3}{4} G m \beta^{1-\gamma} r^{-(3-2\gamma)}$$

(50)

by virtue of equations (26) and (27); while, behind the shock,

$$K = p \rho^{-\gamma} = r^{2\gamma}\Omega^{-\gamma}$$

(51)

by virtue of equations (8) and (9). These equations disclose that the undisturbed medium itself will not be isentropic unless $\gamma = 1.5$; and that behind the shock wave will not be so even if the entropy in front of the wave is constant. In general, we find that

$$\frac{K}{K_0} = \frac{3}{4} \frac{\beta^{\gamma-1}}{G m} \frac{p_0^{-\gamma} \Omega^{-\gamma}}{\xi} = \frac{9\gamma}{4x} \left( \frac{P}{\beta \xi} \right) \left( \frac{\Omega}{\beta} \right)^{\gamma} .$$

(52)

In particular, if $K_1$ denotes the value of the adiabatic constant immediately behind the shock, an appeal to equations (29) and (30) discloses that

$$\frac{K_1}{K_0} = \frac{2\gamma - x - x}{x (\gamma - 1 + 2x) - (\gamma + 1) \gamma + 1} ,$$

(53)

which, for $x = 1$ (i.e., the Mach number $M = 1$), reduces indeed to 1; in this case, there is no entropy change across the incipient shock. If, however, $x > 1$, then $K_1 > K_0$, and the corresponding increase in entropy can be evaluated by means of equation (49). The penultimate column of Table 1 lists the auxiliary quantity $(P/\beta \xi)(\Omega/\beta)^{-\gamma}$, tabulated as a function of $\xi$, which should facilitate the computation of the entropy at any point of the field of flow under investigation.

The last column of Table 1 ultimately contains the values of the integral

$$F ( \xi ) = \int_{\xi_0}^{\xi} \left\{ \frac{U^2}{2} + \frac{1}{\gamma - 1} \frac{P}{\beta} - \frac{4x \xi \Omega}{3 \gamma \xi_0} \right\} \frac{\xi_0 d \xi}{\xi^2} ,$$

(54)

which is related to the total energy $E (\xi)$ contained at any time within a concentric shell extending from the shock inward by means of the equation

$$E ( \xi ) = \frac{4\pi \beta}{3 \xi_0} F ( \xi ) .$$

(55)

In particular, the total energy of wave motion in the envelope of our generalized Roche model can be obtained from the preceding equation (55) by setting, in the latter, $\xi = \xi_1$. A knowledge of $E (\xi_1)$ puts us, in turn, in a position to determine the amount of energy which had to be released by the instantaneous initial explosion in order to produce a shock wave of requisite strength. A sum $H$ of the thermal and gravitational energy originally stored in the envelope has already been given by equation (48), where, by virtue of equations (13) and (14), we are entitled to set $R/a = (\xi_1/\xi_0)^{1/3}$. The difference $|E (\xi_1) - H (\xi_1)|$ furnishes, therefore, the absolute amount of energy whose instantaneous release was adequate for giving rise to the computed phenomena; and the ratio $|E - H|/|H|$ expresses this amount in terms of the energy originally contained in the envelope.

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The numerical values of the quantity $|E - H|/|H|$ corresponding to each one of our solutions can be found in the last column of Table 2, which summarizes the physical characteristics of the solutions presented in Table 1. The headings of the first eight columns of Table 2 are self-explanatory. The ratios $\xi_0/\xi_1 = (a/R)^2$, compiled in the tenth column, are identical with the ratios of mean density of our configuration as a whole (limited by the radius $R$) to that of its core; while the eleventh column contains the fraction of the mass of the envelope which has been endowed with a velocity greater than that of escape from the gravitational field of the core.

| $a/R$ | $\gamma$ | $\rho_1/\rho_0$ | $\rho_1/\rho_0$ | $(l/r)u_1$ | $(l/r)c_1$ | $T_c/T_0$ | $K_1/K_0$ | $\xi_0/\xi_1$ | $|E-H|/|H|$ |
|-------|---------|-----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 8.27924 | 0.4536 | 1.000 | 10.10 | 2.936 | 0.4396 | 0.4297 | 3.44 | 1.677 | 0.0933 | 0.000 | 2.625 |
| 10.0 | 0.4623 | 1.125 | 12.25 | 3.077 | 0.4500 | 0.4307 | 3.98 | 1.882 | 0.0988 | 0.196 | 3.630 |
| 12.0 | 0.4714 | 1.225 | 14.75 | 3.200 | 0.4583 | 0.4132 | 4.61 | 2.123 | 0.1024 | 0.363 | 4.803 |
| 15.0 | 0.4715 | 1.429 | 18.50 | 3.333 | 0.4667 | 0.4055 | 5.55 | 2.486 | 0.1047 | 0.537 | 6.566 |
| 20.0 | 0.4715 | 1.679 | 24.75 | 3.479 | 0.4750 | 0.3976 | 7.11 | 3.099 | 0.1048 | 0.717 | 9.395 |
| 30.0 | 0.4654 | 2.093 | 37.25 | 3.636 | 0.4833 | 0.3896 | 10.24 | 4.332 | 0.1008 | 0.905 | 14.77 |
| 40.0 | 0.4576 | 2.438 | 49.76 | 3.721 | 0.4875 | 0.3854 | 13.37 | 5.568 | 0.0958 | 1.000 | 19.86 |
| 60.0 | 0.4432 | 3.011 | 74.74 | 3.809 | 0.4917 | 0.3813 | 19.62 | 8.042 | 0.0870 | 1.000 | 29.28 |
| 75.0 | 0.4340 | 3.377 | 93.49 | 3.846 | 0.4933 | 0.3796 | 24.31 | 9.900 | 0.0818 | 1.000 | 36.22 |
| 100.0 | 0.4213 | 3.913 | 124.76 | 3.884 | 0.4950 | 0.3779 | 32.13 | 13.001 | 0.0748 | 1.000 | 47.08 |
| 200.0 | 0.3880 | 5.562 | 249.75 | 3.942 | 0.4975 | 0.3752 | 63.36 | 25.372 | 0.0584 | 1.000 | 85.76 |
| 300.0 | 0.3679 | 6.820 | 374.74 | 3.960 | 0.4983 | 0.3744 | 94.63 | 37.811 | 0.0498 | 1.000 | 127.6 |
| 500.0 | 0.3427 | 8.821 | 624.75 | 3.976 | 0.4990 | 0.3737 | 157.13 | 62.606 | 0.0403 | 1.000 | 199.5 |
| $\infty$ | 0 | $\infty$ | $\infty$ | $\infty$ | 0 | 0.000 | 0 | 0 | 1.000 | $\infty$ |

In conclusion, the writers wish to express their indebtedness to Miss Virginia K. Brenton and Mrs. Margaret D. Hill for carrying out most of the numerical integrations presented in this paper and to Mr. Francis G. Davoren for editorial help in the preparation of the tables.

APPENDIX

In conclusion of the present investigation, we wish to point out that, if the Mach number of the shock front which we are considering were infinite (which amounts to an assumption that the pressure $p_a$ in front of the shock wave can be ignored in comparison with the pressure $p$ behind the shock), and if the density $\rho_0$ in the undisturbed medium varies as

$$\rho_0 = \beta r^v,$$

where $v$ is an arbitrary constant exponent, there exists a certain value of $\gamma$ for which the equations of motion of our problem, rewritten in terms of a single independent variable $\xi = r^{v/2}$, admit of a solution in a closed form.

In order to prove this statement, we may recall that $M = \infty$ corresponds to $x = 0$ and, therefore, to $Gm = 0$ (implying our motion to be so rapid that no finite central mass can decelerate
It significantly). If so, the Rankine-Hugoniot shock-wave conditions, simplified by ignoring \( p_0 \) in comparison with \( p_1 \), yields

\[ U(\xi_0) = -(1 - \mu^2) \frac{\psi}{\phi}, \]

\[ P(\xi_0) = (1 - \mu^2) \left( \frac{\psi}{\phi} \right)^2 \beta, \]

and

\[ \Omega(\xi_0) = \mu^{-3\beta}, \]

where we have let

\[ \mu^2 = \frac{\gamma - 1}{\gamma + 1}. \]

On the other hand, it may be readily verified that the solution of the fundamental equations in this particular case yields

\[ U(\xi) = -(1 - \mu^2) \frac{\psi}{\phi} = -\frac{2}{\gamma + 1} \frac{\psi}{\phi}, \]

\[ P(\xi) = P(\xi_0) \left( \frac{\xi}{\xi_0} \right)^{2(\phi/\psi)/\psi}, \]

\[ \Omega(\xi) = \Omega(\xi_0) \left( \frac{\xi}{\xi_0} \right)^{2(\phi/\psi)/\psi}, \]

for any value of \( \xi \). For these equations to be consistent with equations (57)–(59) immediately behind the shock wave (i.e., when \( \xi = \xi_0 \)), it is obviously necessary that

\[ \frac{\gamma - 1}{\gamma + 1} = \frac{1 - (\phi/\psi)}{3 + (\phi/\psi)}, \]

and the constancy of the energy requires, on the other hand, that

\[ \nu + 5 \frac{\psi}{\phi} + 2 = 0, \]

i.e., that

\[ \nu = -5 - 2 \frac{\phi}{\psi} = \frac{\gamma - 7}{\gamma + 1}. \]

In the particular case discussed in this paper, \( \phi/\psi = -\frac{2}{3} \) by equation (17) and \( \nu = -2 \), in accordance with equation (65). The value of \( \gamma \), corresponding to these parameters by equation (64), turns out to be \( \frac{2}{3} \) (the one appropriate for monatomic gas); and in this case the closed solution of our equations describing the properties of a gas flow behind a shock wave of an infinite strength takes the explicit form

\[ u = \frac{1}{2} \frac{r}{i}, \quad \rho = 4\beta \xi_0 \frac{r}{i^2}, \quad \beta = \frac{\beta}{3} \frac{r^3}{i^4}, \]

respectively.

It is interesting to note that another particular case of the family of solutions just deduced, corresponding to strong progressing blast waves in water, was previously discovered by Primakoff. In his case the medium in front of his shock was supposed to be homogeneous \( (\nu = 0) \), and the constancy of the energy of the whole explosion led to \( \psi/\phi = -\frac{2}{3} \). If so, the value of \( \gamma \) required to yield a solution in a closed form turned out to be \( \frac{2}{3} \) (which corresponds, very approximately, to the ratio of specific heats in water under normal conditions), exactly in agreement with our equations (64)–(66). It transpires, therefore, that Primakoff's solution happens to be a particular member of the whole family of closed solutions whose explicit form has been given above and whose existence has so far escaped notice.

\[ \text{As quoted by Courant and Friedrichs, op. cit., sec. 161.} \]