THE MONOCHROMATIC COEFFICIENTS OF DARKENING
FOR THE MAIN SEQUENCE STARS

By G. Münch and S. Chandrasekhar

In the analysis of the light curves of eclipsing variables, it is generally assumed that
the distribution of brightness over the apparent disc of a star is governed by a law of darkening of the form

\[ I_\lambda(\theta) = I_\lambda(0) \left[ 1 - \kappa_\lambda \cos \theta \right], \tag{1} \]

where \( I_\lambda(\theta) \) is the emergent intensity in the wavelength \( \lambda \) and in a direction making an
angle \( \theta \) with the outward normal and \( \kappa_\lambda \) is the coefficient of darkening which may depend on
the wavelength. Now it may be stated at once that a linear law of darkening of the form (1)
does not emerge from a theory of the continuous spectrum of a star even under the most idealized
conditions. For, under conditions of local thermodynamic equilibrium the emergent intensity is
given by

\[ I_\lambda(\theta) = \int_{0}^{\infty} B_\nu(T_\nu(\lambda)) \ e^{-\tau_\nu(\lambda)} \sec \theta \ d(\tau_\nu \sec \theta), \tag{2} \]

where \( B_\nu(T_\nu(\lambda)) \) is the Planck intensity in the wavelength \( \lambda \) and at the temperature \( T_\nu(\lambda) \)
prevailing at the optical depth \( \tau_\nu = \tau(\lambda) \) in the continuous absorption coefficient \( \kappa_\nu \)
in the wavelength.

Now if the atmosphere be assumed to be slightly non-grey, then it can be shown 3 that if \( \tau \)
is the optical thickness in a mean absorption coefficient \( \mathcal{R} \) defined in the manner

\[ \mathcal{R} = \frac{1}{F(1)^{1/3}} \int_{0}^{\infty} \kappa_\nu F_{\nu}^{(1)} \ d\nu, \tag{3} \]

where \( F_{\nu}^{(1)} \) is the monochromatic flux in a grey atmosphere and \( F \) is the net integrated flux
and if, further, \( \kappa_\nu/\mathcal{R} \) is constant with depth, then the temperature distribution is governed
by a formula of the standard type

\[ T_\nu = \frac{4}{3} T_e^4 (\tau + 2/3), \tag{4} \]

where \( T_e \) denotes the effective temperature. On these assumptions the angular distribution
of the emergent intensity in the frequency \( \nu \) can be expressed in the form

\[ I_\nu(\theta) = B_\nu(T_e) \ \mathcal{\hat{I}} \left[ \frac{h\nu}{kT_e}; \frac{\kappa_\nu}{\mathcal{R}} \right] \sec \theta, \tag{5} \]

where \( B_\nu(T_0) \) is the Planck intensity for the boundary temperature \( T_0 \) and

\[ \mathcal{\hat{I}}(\alpha;\beta) = \int_{0}^{\infty} e^{-\beta r} h_\nu(\tau) \ d(\nu r), \]

where \( h_\nu(\tau) \) is the Planck intensity at the boundary.
and
\[ b_a(\tau) = \frac{\exp \left( \left( \frac{4}{\sqrt{3}} \right)^3 \alpha \right) - 1}{\exp \left( \alpha \left[ \frac{1}{2} (\tau + q(\tau)) \right] \right) - 1} \] (6)

Now even under these severe restrictions, the emergent intensity calculated according to (5) does not lead to a linear law of darkening. The departures from linearity are most pronounced as we approach the limb; i.e., precisely in the regions which are of particular interest for the theory of eclipsing variables.

In spite of the theoretical difficulties in justifying a linear law of darkening, the immense complexity of the problem of interpreting the light curves of eclipsing variables, arising from the numerous contributory effects, it is clear that it will be advantageous to retain the linear law even if it represent only the crudest approximation from the point of view of the theory of the continuous spectrum. Now a linear law of darkening implies that the source function \( B_\lambda (T_{\text{eq}}(\lambda)) \) in (2) is linear in \( \lambda \). This clearly implies that we allow for the variation of \( B_\lambda (T_{\text{eq}}(\lambda)) \) with \( \lambda \) by a Taylor expansion about some appropriately chosen depth and retain only the first two terms. Since according to equation (4) \( T = T_\varnothing \) at \( \tau = 2/3 \), it would seem natural to make the Taylor expansion about this point. Thus:
\[ B_\lambda (T_\varnothing) = B_\lambda (T_\varnothing) + (\tau - 2/3) \int \left[ \frac{dB_\lambda (T_\varnothing)}{d\tau} \right] \tau = \tau_0, \] (7)
or
\[ B_\lambda (T_\varnothing) = B_\lambda (T_\varnothing) \left[ 1 - \frac{1}{8} \frac{\alpha}{1 - e^{\alpha}} + \frac{3}{16} \frac{\alpha}{1 - e^{\alpha}} \tau \right], \] (8)

where
\[ \alpha = \frac{hc}{kT_\varnothing}. \] (9)

Inserting this expression for \( B_\lambda (T_\varnothing) \), in the equation
\[ I_\lambda (\theta) = \int_0^\infty B_\lambda (T_\varnothing) e^{-\left(\kappa_\lambda/R\right)\tau \sec \theta} \left(\kappa_\lambda/R\right) \sec \theta \, d\tau, \] (10)
we find
\[ I_\lambda (\theta) = B_\lambda (T_\varnothing) \left[ 1 - \frac{1}{8} \frac{\alpha}{1 - e^{\alpha}} + \frac{3}{16} \frac{\alpha}{1 - e^{\alpha}} \frac{R}{R_\lambda} \cos \theta \right]. \] (11)

The coefficient of darkening is therefore given by
\[ \chi_\lambda = \left( 1 + \frac{2}{3} \left( \frac{8}{\alpha} \frac{1 - e^{\alpha}}{-1} \frac{R_\lambda}{R} \right) \right)^{-1}. \] (12)

Formula (12) can be used to determine the coefficients \( \chi_\lambda \) if \( \kappa_\lambda/R \) is known. For this purpose we have used the results of an earlier paper in which we have shown that the principal features of the continuous spectrum of stars with spectral types between A0 and G2 can be accounted for in terms of the continuous absorption coefficients of neutral hydrogen atoms and negative hydrogen ions. In that paper we have also given electron pressures in the atmospheres of the main sequence stars which will explain the observed discontinuity at the head of the Balmer series. With these electron pressures and with the values of \( \kappa_\lambda/R \) tabulated in the paper referred to, we have computed the darkening coefficients \( \chi_\lambda \) according to (12) for various wave lengths and spectral types. These values are given in the following table.

In our view, the main uncertainty in the use of the darkening coefficients \( \chi_\lambda \) given in the table arises from the linear expansion for the Planck function required for predicting a linear law of darkening. In a general way, it is clear that the linear expansion of the Planck function will lead to results which are misleading when \( \kappa_\lambda/R \) becomes appreciably less than unity; for under these circumstances the contribution to the emergent intensity from great
The Monochromatic Darkening Coefficient $\kappa_{\lambda}$ for Stars of the Main Sequence

<table>
<thead>
<tr>
<th>Sp.</th>
<th>$\theta^*$</th>
<th>$\log_{10}P_e$</th>
<th>$\lambda=3647\AA$</th>
<th>$\lambda=3647\AA$</th>
<th>$\lambda=6000\AA$</th>
<th>$\lambda=6000\AA$</th>
<th>$\lambda=6500\AA$</th>
<th>$\lambda=6500\AA$</th>
<th>$\lambda=8203\AA$</th>
<th>$\lambda=8203\AA$</th>
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<td>A1</td>
<td>0.50</td>
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<td>0.35</td>
<td>0.37</td>
<td>0.39</td>
<td>0.42</td>
<td>0.49</td>
<td>0.56</td>
<td>0.63</td>
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<td>3.3</td>
<td>0.37</td>
<td>0.38</td>
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<td>0.49</td>
<td>0.58</td>
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<td>0.50</td>
<td>0.63</td>
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<tr>
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<td>0.71</td>
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* The values of $\theta = 5040/P_e$ were obtained by interpolating in Kuiper's temperature scale.

depths will become significant and the non-constancy of $\kappa_{\lambda}/R$ will also become serious. From this we conclude that for stars not later than F2 the values of $\kappa_{\lambda}$ in the ultra violet ($\lambda<3747\AA$) and in the near infra red (6500A<$\lambda<8203\AA$) where the H absorption coefficient has a maximum, will provide a satisfactory approximation. In the region 3747A$<\lambda<5500\AA$ the values should be accepted with considerable reserve; it may be emphasized again, that the reason for this "reserve" is the inadequacy of the linear approximation. For stars of later spectral types, the same remarks apply to the use of the darkening coefficients for $\lambda<6000\AA$.

From the foregoing remarks it would seem that observations in the red are particularly valuable. There are, indeed, other considerations which suggest the greater use of this spectral region $\lambda>6000\AA$. For, in the blue and in the violet the continuous spectrum is generally distorted by the overlapping of absorption lines; and the depletion of the continuous spectrum arising from this source is of particular importance in the stars of earlier spectral types where the higher members of the Balmer series crowd. And in spectral types later than F2, the effect is pronounced in the blue and the violet regions of the spectrum. In all these cases the actual limb darkening will be less than those predicted by the values of $\kappa_{\lambda}$ which we have given.

We have not considered the O and B stars. In the atmosphere of these stars electron scattering plays the dominant role and the equation of transfer appropriate to the case of pure electron scattering has been solved exactly.

3. G. Münch, Ap.J., 102, 385, 1945 (see particularly Fig. 3).

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