ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE: XXII

S. Chandrasekhar
Yerkes Observatory
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ABSTRACT

In the present paper exact solutions are found for the problems of diffuse reflection and transmission considered in Paper XXI in a general finite approximation. The method consists in starting with the functional equations of Paper XVII governing the laws of diffuse reflection and transmission; reducing them to pairs of functional equations of the standard form,

\[ X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu' + \mu} \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] d\mu' \]  

\[ Y(\mu) = e^{-\tau_1/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] d\mu', \]

where \( \Psi(\mu) \) is an even polynomial in \( \mu \) satisfying the condition

\[ \int_0^1 \Psi(\mu) d\mu \leq \frac{1}{3}, \]

and \( \tau_1 \) is the optical thickness of the atmosphere; and, finally, relating in a unique manner the various constants occurring in the solutions with the moments of the \( X \)- and \( Y \)-functions appropriate for the problem.

There is, however, one important difference between the present theory and the corresponding theory of transfer in semi-infinite atmospheres as developed in Paper XIV. It is that, in all conservative cases of perfect scattering, the solutions of the functional equations incorporating the invariances of the problem are not unique but form a one-parametric family. For the removal of the resulting arbitrariness in the solutions, appeal must be made to the flux and the \( K \)-integrals, which conservative problems of perfect scattering always admit.

The paper is divided into five main sections. Section I is devoted to a general study of functional equations of the form (i) and (ii) and to deriving various integral properties of these functions useful in the subsequent analysis. The one-parametric nature of the solution of these equations for the case

\[ \int_0^1 \Psi(\mu) d\mu = \frac{1}{3} \]

is proved in this section; also the basic correspondence between the solutions of equations (i) and (ii) and the rational functions \( X \) and \( Y \) introduced in Paper XXI is established. The following sections deal with the problem of diffuse reflection and transmission under conditions of (i) isotropic scattering with an albedo \( q \leq 1 \); (ii) scattering in accordance with Rayleigh's phase function; (iii) scattering in accordance with the phase function \( \lambda(1 + x \cos \Theta) \); and (iv) Rayleigh scattering with proper allowance for the polarization of the radiation field.

1. Introduction.—This paper is a continuation of Paper XXI\(^1\) and completes the theory of diffuse reflection and transmission by plane-parallel atmospheres of finite optical thicknesses. By considering the functional equations for the laws of diffuse reflection and transmission derived in Paper XVII\(^2\) we shall show how the exact solutions for the various problems can be found. Now these functional equations governing the angular distributions of the reflected and the transmitted radiations are simultaneous

nonlinear nonhomogeneous systems of such high order\(^2\) that they might be considered impossible for practical solution if it were not for the guidance provided by the analysis of Paper XXI regarding the forms of the solutions to be sought. Indeed, it will appear that the solutions of the reflected and the transmitted radiations in the various cases have exactly the same forms as those found in Paper XXI, with, however, the \(X\)- and \(Y\)-functions occurring in them redefined as solutions of a simultaneous pair of functional equations of the form:

\[
X (\mu) = 1 + \mu \int_0^1 \frac{\Psi (\mu')}{\mu + \mu'} [X (\mu) X (\mu') - Y (\mu) Y (\mu')] \, d\mu' \tag{1}
\]

and

\[
Y (\mu) = e^{-\tau_1/\mu} + \mu \int_0^1 \frac{\Psi (\mu')}{\mu - \mu'} [Y (\mu) X (\mu') - X (\mu) Y (\mu')] \, d\mu' , \tag{2}
\]

where \(\tau_1\) denotes the optical thickness of the atmosphere and \(\Psi (\mu)\) is an even polynomial in \(\mu\) satisfying the condition

\[
\int_0^1 \Psi (\mu) \, d\mu \leq \frac{1}{2} . \tag{3}
\]

Equations (1) and (2) therefore play the same basic role in the theory of radiative transfer in atmospheres of finite optical thicknesses as the equation

\[
H (\mu) = 1 + \mu H (\mu) \int_0^1 \frac{\Psi (\mu')}{\mu + \mu'} H (\mu') \, d\mu' \tag{4}
\]

played in the theory of semi-infinite atmospheres.\(^4\) It is, in fact, clear that

\[
X (\mu) \to H (\mu) \quad \text{and} \quad Y (\mu) \to 0 \quad \text{as} \quad \tau_1 \to \infty . \tag{5}
\]

There is, however, one important respect in which the present theory differs from the theory of radiative transfer in semi-infinite atmospheres, namely, that, in all conservative cases of perfect scattering, the functional equations governing the angular distributions of the emergent radiations derived from the invariances discussed in Paper XVII do not suffice to characterize the physical solutions uniquely; for, as we shall see, the general solutions of the relevant equations have a single arbitrary parameter in them. Thus, for the case

\[
\int_0^1 \Psi (\mu) \, d\mu = \frac{1}{2} , \tag{6}
\]

we shall show that if \(X(\mu)\) and \(Y(\mu)\) are solutions of equations (1) and (2), then so are

\[
X (\mu) + Q\mu [X (\mu) + Y (\mu)] \tag{7}
\]

and

\[
Y (\mu) - Q\mu [X (\mu) + Y (\mu)] , \tag{8}
\]

where \(Q\) is an arbitrary constant. Similar ambiguities arise in the solutions of the more complicated systems representing general cases of perfect scattering. The physical origin of this nonuniqueness in the solution is not clear; but we shall see that in all cases the ambiguity can be removed by appealing to the "\(K\)-integral,"

\[
K = \frac{1}{2} \int_{-1}^{+1} I (\tau, \mu) \mu^2 \, d\mu = \frac{1}{4} \mu_0 F (- \mu_0 e^{-\tau_1/\mu_0} + \gamma_1 \tau + \gamma_2) , \tag{9}
\]

\(^a\) For example, in the case of Rayleigh scattering, the order of the system is eight.

which all conservative problems admit.\(^2\) (In eq. [9], \(\mu_0\) is the direction cosine of the angle of incidence of a parallel beam of radiation of net flux \(\pi F\) per unit area normal to itself, and \(\gamma_1\) and \(\gamma_2\) are two constants.)

The plan of this paper is as follows:

Section I is devoted to a general study of the functional equations (1) and (2) and to deriving certain relations useful in our subsequent analysis. The ambiguity in the solutions of equations (1) and (2) for the case (6) is proved in this section. The basic correspondence between the solutions of equations (1) and (2) and the rational functions, \(X\) and \(Y\), introduced in Paper XXI (eqs. [125] and [126]) is also established in this section. Sections II, III, IV, and V deal with the problem of diffuse reflection and transmission under conditions, respectively, of (i) isotropic scattering with an albedo \(a_0 \leq 1\); (ii) scattering in accordance with Rayleigh's phase function; (iii) scattering in accordance with the phase function \(\lambda(1 + x \cos \Theta)\); and, finally, (iv) Rayleigh scattering with proper allowance for the polarization of the radiation field.

### I. ON THE FUNCTIONAL EQUATIONS SATISFIED BY \(X\) AND \(Y\)

2. Definitions and alternative forms of the basic equations.—In dealing with the solutions of equations (1) and (2) it is convenient to introduce the following abbreviations:

\[
\begin{align*}
x_n &= \int_0^1 X(\mu) \Psi(\mu) \mu^n d\mu, \\
y_n &= \int_0^1 Y(\mu) \Psi(\mu) \mu^n d\mu, \\
a_n &= \int_0^1 X(\mu) \mu^n d\mu, & \text{and} & \beta_n &= \int_0^1 Y(\mu) \mu^n d\mu; \\
\end{align*}
\]

i.e., \(x_n\) and \(y_n\) are the moments of order \(n\) and \(X(\mu)\) and \(Y(\mu)\), weighted by the characteristic function \(\Psi(\mu)\), while \(a_n\) and \(\beta_n\) are the simple moments themselves.

Certain alternative forms of the basic equations which we shall find useful may also be noted here. Writing

\[
\frac{\mu}{\mu + \mu'} = 1 - \frac{\mu'}{\mu + \mu'}, \text{ respectively, } \frac{\mu}{\mu - \mu'} = 1 + \frac{\mu'}{\mu - \mu'},
\]

in equations (1) and (2), we readily find that

\[
\int_0^1 \frac{\mu \Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu' = 1 - [(1 - x_0) X(\mu) + y_0 Y(\mu)]
\]

and

\[
\int_0^1 \frac{\mu \Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu' = - e^{-r_1/\mu} + [y_0 X(\mu) + (1 - x_0) Y(\mu)].
\]

We also have

\[
\int_0^1 \frac{\mu \Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu' = x_1 X(\mu) - y_1 Y(\mu) - \mu + [1 - x_0] X(\mu) + y_0 Y(\mu)
\]

and

\[
\int_0^1 \frac{\mu \Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu' = y_1 X(\mu) - x_1 Y(\mu) - \mu e^{-r_1/\mu} + [y_0 X(\mu) + (1 - x_0) Y(\mu)].
\]

\(^6\)In the case of Rayleigh scattering there are two such integrals to be considered (cf. Sec. V).
The foregoing equations can be verified by writing
\[
\frac{\mu'}{\mu + \mu'} = \mu' - \frac{\mu\mu'}{\mu + \mu'},
\]
respectively,
\[
\frac{\mu'^2}{\mu - \mu'} = -\mu' + \frac{\mu\mu'}{\mu - \mu'},
\]
and using equations (13) and (14).

3. Integrodifferential equations for $X(\mu, \tau_1)$ and $Y(\mu, \tau_1).$—In equations (1) and (2), $0 < \tau_1 < \infty$ is, of course, to be regarded as some assigned constant. Nevertheless, it is sometimes convenient to emphasize explicitly the dependence of the solutions $X$ and $Y$ on $\tau_1.$ We shall then write $X(\mu, \tau_1)$ and $Y(\mu, \tau_1)$ instead of simply as $X(\mu)$ and $Y(\mu).$ And, considered as functions of $\tau_1$ also, $X$ and $Y$ satisfy certain integrodifferential equations which are of importance. We shall state them in the form of the following theorem:

Theorem 1.—If $X(\mu, \tau_1)$ and $Y(\mu, \tau_1)$ are solutions of equations (1) and (2) for a particular value of $\tau_1,$ then solutions for other values of $\tau_1$ can be obtained from the integrodifferential equations
\[
\frac{\partial X(\mu, \tau_1)}{\partial \tau_1} = Y(\mu, \tau_1) \int_0^1 \frac{d\mu'}{\mu'} \Psi(\mu') Y(\mu', \tau_1)
\]
\[
= y_{-1}(\tau_1) Y(\mu, \tau_1)
\]
and
\[
\frac{\partial Y(\mu, \tau_1)}{\partial \tau_1} + \frac{Y(\mu, \tau_1)}{\mu} = X(\mu, \tau_1) \int_0^1 \frac{d\mu'}{\mu'} \Psi(\mu') Y(\mu', \tau_1)
\]
\[
= y_{-1}(\tau_1) X(\mu, \tau_1).
\]

Proof.—According to equations (18) and (19),
\[
\mu \frac{\partial}{\partial \tau_1} \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu'
\]
\[
= \mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} \left\{ y_{-1}X(\mu) Y(\mu') + y_{-1}X(\mu') Y(\mu) 
\right. 
\]
\[
- Y(\mu) \left[ -\frac{Y(\mu')}{\mu'} + y_{-1}X(\mu') \right] 
\left. 
- Y(\mu') \left[ -\frac{Y(\mu)}{\mu} + y_{-1}X(\mu) \right] \right\} d\mu'.
\]
Hence
\[
\mu \frac{\partial}{\partial \tau_1} \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu' = y_{-1} Y(\mu).
\]

Similarly,
\[
\mu \frac{\partial}{\partial \tau_1} \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu'
\]
\[
= \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} \left[ -Y(\mu) X(\mu') + \frac{\mu}{\mu'} X(\mu) Y(\mu') \right] d\mu'.
\]
We therefore have
\[
\int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu'
\]
\[
+ \mu \frac{\partial}{\partial \tau_1} \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu' = y_{-1} X(\mu).
\]
On the other hand, if $X$ and $Y$ are solutions of equations (1) and (2), we must have

$$\frac{\partial X}{\partial \tau_1} = \mu \frac{\partial}{\partial \tau_1} \int_0^1 \frac{\Psi (\mu')}{\mu + \mu'} [X (\mu) X (\mu') - Y (\mu) Y (\mu')] \, d\mu'$$

and

$$\frac{\partial Y}{\partial \tau_1} + \frac{Y}{\mu} = \int_0^1 \frac{\Psi (\mu')}{\mu - \mu'} [Y (\mu) X (\mu') - X (\mu) Y (\mu')] \, d\mu'$$

From equations (21), (23), (24), and (25) we now conclude that, if $X(\mu, \tau_1)$ and $Y(\mu, \tau_1)$ are solutions of equations (1) and (2) for a particular value of $\tau_1$, then

$$X (\mu, \tau_1) + y_1 Y (\mu, \tau_1) \, d\tau_1$$

and

$$Y (\mu, \tau_1) + \left[ - \frac{y_1}{\mu} + y_2 X (\mu, \tau_1) \right] \, d\tau_1$$

are solutions of the same equations for an infinitesimally larger value of $\tau_1$, namely, $\tau_1 + d\tau_1$. This proves the theorem.

**Corollary.—**

$$X^2 (\mu, \tau_1) - Y^2 (\mu, \tau_1) = H^2 (\mu) - \frac{2}{\mu} \int_{\tau_1}^{\infty} Y^2 (\mu, t) \, dt.$$  

**Proof.—** Eliminating $y_{-1}$ between equations (18) and (19), we have

$$X \frac{\partial X}{\partial \tau_1} = Y \frac{\partial Y}{\partial \tau_1} + \frac{Y^2}{\mu}$$

or

$$\frac{\partial}{\partial \tau_1} (X^2 - Y^2) = \frac{2}{\mu} Y^2.$$  

Integrating equation (30) and remembering that

$$X (\mu, \tau_1) \to H (\mu) \quad \text{and} \quad Y (\mu, \tau_1) \to 0 \quad \text{as} \quad \tau_1 \to \infty$$

we obtain the result stated.

**4. Some integral properties of the functions $X$ and $Y$.—** As in the case of the $H$-functions, there are a number of integral theorems (of an essentially elementary kind) which can be proved for functions satisfying equations of the form (1) and (2). The theorems which follow are the analogues for the $X$- and $Y$-functions, of the theorems proved for the $H$-functions in Paper XIV, § 12.

**Theorem 2.—**

$$\int_0^1 X (\mu) \Psi (\mu) \, d\mu = 1 - \left[ 1 - 2 \int_0^1 \Psi (\mu) \, d\mu + \left\{ \int_0^1 Y (\mu) \Psi (\mu) \, d\mu \right\}^2 \right]^{1/2}.$$  

**Proof.—** Multiplying the equation satisfied by $X(\mu)$ by $\Psi(\mu)$ and integrating over the range of $\mu$, we have (cf. eq. [10])

$$x_0 = \int_0^1 \Psi (\mu) \, d\mu$$

$$+ \int_0^1 \int_0^1 \frac{\mu}{\mu + \mu'} \Psi (\mu) \Psi (\mu') [X (\mu) X (\mu') - Y (\mu) Y (\mu')] \, d\mu \, d\mu'.$$  

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Interchanging $\mu$ and $\mu'$ in the double integral on the right-hand side and taking the average of the two equations, we have

$$x_0 = \int_0^1 \Psi(\mu) \, d\mu + \frac{1}{2} \int_0^1 \int_0^1 \Psi(\mu) \Psi(\mu') \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] \, d\mu \, d\mu'$$

$$= \int_0^1 \Psi(\mu) \, d\mu + \frac{1}{2} \left( x_0^2 - y_0^2 \right).$$

(34)

Solving this equation for $x_0$, we have

$$x_0 = 1 \pm \left[ 1 - 2 \int_0^1 \Psi(\mu) \, d\mu + y_0^2 \right]^{1/2}.$$

(35)

The ambiguity in the sign in equation (35) can be removed by the consideration that the quantity on the right-hand side must uniformly converge to zero when $\Psi(\mu) \to 0$ uniformly in the interval $(0, 1)$. This requires us to choose the negative sign in equation (35), and the result stated follows.

**Corollary.**—In the conservative case

$$\int_0^1 \Psi(\mu) \, d\mu = \frac{1}{2},$$

(36)

we have

$$\int_0^1 [X(\mu) + Y(\mu)] \Psi(\mu) \, d\mu = 1.$$

(37)

**Theorem 3.**—

$$(1 - x_0) x_2 + y_0 y_2 + \frac{1}{2} \left( x_1^2 - y_1^2 \right) = \int_0^1 \Psi(\mu) \mu^2 \, d\mu.$$

(38)

**Proof.**—Multiplying equation (1) by $\Psi(\mu) \mu^2$ and integrating over the range of $\mu$, we have

$$x_2 = \int_0^1 \Psi(\mu) \mu^2 \, d\mu$$

$$+ \int_0^1 \int_0^1 \frac{\mu^2}{\mu + \mu'} \Psi(\mu) \Psi(\mu') \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] \, d\mu \, d\mu'$$

$$= \int_0^1 \Psi(\mu) \mu^2 \, d\mu$$

$$+ \frac{1}{2} \int_0^1 \int_0^1 (\mu^2 - \mu \mu' + \mu'^2) \Psi(\mu) \Psi(\mu') \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] \, d\mu \, d\mu'.$$

Hence

$$x_2 = \int_0^1 \Psi(\mu) \mu^2 \, d\mu + x_2 x_0 - y_2 y_0 - \frac{1}{2} \left( x_1^2 - y_1^2 \right),$$

(40)

which is equivalent to equation (38).

**Corollary.**—In the conservative case,

$$y_0 (x_2 + y_2) + \frac{1}{2} \left( x_1^2 - y_1^2 \right) = \int_0^1 \Psi(\mu) \mu^2 \, d\mu.$$

(41)

This follows from equation (38) and the corollary of theorem 2 (eq. [37]), according to which

$$x_0 + y_0 = 1.$$

(42)
Theorem 4.—When the characteristic function $\Psi(\mu)$ has the form

$$\Psi(\mu) = a + b\mu^2,$$

where $a$ and $b$ are two constants, we have the relations

$$a_0 = 1 + \frac{1}{2} \left[ a \left( a_0^2 - \beta_0^2 \right) + b \left( a_1^2 - \beta_1^2 \right) \right],$$

$$\begin{align*}
(a + b\mu^2) \int_0^1 \frac{d\mu'}{\mu + \mu'} \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] \\
= \frac{1}{\mu} \left[ X(\mu) - 1 \right] - b \left[ (a_1 - \mu a_0) X(\mu) - (\beta_1 - \mu \beta_0) Y(\mu) \right],
\end{align*}$$

and

$$\begin{align*}
(a + b\mu^2) \int_0^1 \frac{d\mu'}{\mu - \mu'} \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] \\
= \frac{1}{\mu} \left[ Y(\mu) - e^{-\gamma/\mu} \right] - b \left[ (\beta_1 + \mu \beta_0) X(\mu) - (a_1 + \mu a_0) Y(\mu) \right],
\end{align*}$$

where $a_0, \beta_0$ and $a_1, \beta_1$ are moments of order zero and one of $X(\mu)$ and $Y(\mu)$, respectively.

To prove equation (44), we simply integrate the equation satisfied by $X(\mu)$. We find

$$\begin{align*}
a_0 &= 1 + \frac{1}{2} \int_0^1 \left[ (a + b\mu^2) \mu \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] d\mu d\mu' \\
&= 1 + \frac{1}{2} \int_0^1 \left[ (a + b\mu) \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] d\mu d\mu' \\
&= 1 + \frac{1}{2} \left[ a \left( a_0^2 - \beta_0^2 \right) + b \left( a_1^2 - \beta_1^2 \right) \right].
\end{align*}$$

The relation (46) can be proved in the following manner:

$$\begin{align*}
a \int_0^1 \frac{d\mu'}{\mu - \mu'} \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] \\
= \int_0^1 a + b\mu^2 \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] d\mu' \\
+ b \int_0^1 \left( \mu + \mu' - \frac{\mu^2}{\mu - \mu'} \right) \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] d\mu' \\
= \frac{1}{\mu} \left[ Y(\mu) - e^{-\gamma/\mu} \right] + b \left[ (a_1 + \mu a_0) Y(\mu) - (\beta_1 + \mu \beta_0) X(\mu) \right] \\
- \frac{1}{\mu} \left[ Y(\mu) - e^{-\gamma/\mu} \right] - b \left[ (a_1 + \mu a_0) X(\mu) - (\beta_1 + \mu \beta_0) Y(\mu) \right] \\
= \frac{1}{\mu} \left[ Y(\mu) - e^{-\gamma/\mu} \right] - b \left[ (a_1 + \mu a_0) X(\mu) - (\beta_1 + \mu \beta_0) Y(\mu) \right].
\end{align*}$$

Hence the result. Equation (45) follows quite similarly.

5. The nonuniqueness of the solution in the conservative case. The standard solution.—

We shall now prove the following theorem:

Theorem 5.—In the conservative case,

$$\int_0^1 \Psi(\mu) d\mu = \frac{3}{2},$$

The condition $\int_0^1 \Psi(\mu) d\mu \leq \frac{1}{2}$ requires that $a + \frac{3}{2}b \leq \frac{1}{2}$. 

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the solutions of equations (1) and (2) are not unique; more particularly, if \( X(\mu) \) and \( Y(\mu) \) are solutions, then so are

\[
X(\mu) + Q\mu [X(\mu) + Y(\mu)]
\]

and

\[
Y(\mu) - Q\mu [X(\mu) + Y(\mu)],
\]

where \( Q \) is an arbitrary constant.

Proof.—Writing

\[
F(\mu) = X(\mu) + Q\mu [X(\mu) + Y(\mu)]
\]

and

\[
G(\mu) = Y(\mu) - Q\mu [X(\mu) + Y(\mu)],
\]

we verify that

\[
F(\mu) F(\mu') - G(\mu) G(\mu') = X(\mu) X(\mu') - Y(\mu) Y(\mu') + Q(\mu + \mu') [X(\mu) + Y(\mu)] [X(\mu') + Y(\mu')]
\]

and

\[
G(\mu) F(\mu') - F(\mu) G(\mu') = Y(\mu) X(\mu') - X(\mu) Y(\mu') - Q(\mu - \mu') [X(\mu) + Y(\mu)] [X(\mu') + Y(\mu')].
\]

Hence

\[
\mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [F(\mu) F(\mu') - G(\mu) G(\mu')] d\mu' = \mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu' + Q\mu [X(\mu) + Y(\mu)] \int_0^1 [X(\mu') + Y(\mu')] \Psi(\mu') d\mu'.
\]

Using equation (1) and the corollary of theorem 2 (eq. [37]), we have

\[
\mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [F(\mu) F(\mu') - G(\mu) G(\mu')] d\mu' = X(\mu) - 1 + Q\mu [X(\mu) + Y(\mu)]
\]

\[
= F(\mu) - 1.
\]

Similarly,

\[
\mu \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [G(\mu) F(\mu') - F(\mu) G(\mu')] d\mu' = Y(\mu) - e^{-\tau/\mu} - Q\mu [X(\mu) + Y(\mu)]
\]

\[
= G(\mu) - e^{-\tau/\mu}.
\]

Hence \( F(\mu) \) and \( G(\mu) \) satisfy the same equations as \( X(\mu) \) and \( Y(\mu) \), and the theorem follows.

Corollary.—The solutions derivable from a given one according to equations (52) and (53) form a one-parametric family which can be generated by any of its members.

Proof.—Let

\[
F_1(\mu) = F(\mu) + Q_1\mu [F(\mu) + G(\mu)]
\]

and

\[
G_1(\mu) = G(\mu) - Q_1\mu [F(\mu) + G(\mu)],
\]

where \( Q_1 \) is an arbitrary constant.
where \( Q_1 \) is an arbitrary constant. According to theorem 5, \( F_1 \) and \( G_1 \) are also solutions of equations (1) and (2). On the other hand, since (cf. eqs. [52] and [53])

\[
F(\mu) + G(\mu) = X(\mu) + Y(\mu) ,
\]

we can express \( F_1 \) and \( G_1 \) alternatively in the forms

\[
F_1(\mu) = X(\mu) + (Q + Q_1) \mu [X(\mu) + Y(\mu)]
\]

and

\[
G_1(\mu) = Y(\mu) - (Q + Q_1) \mu [X(\mu) + Y(\mu)] .
\]

In other words, \( F_1(\mu) \) and \( G_1(\mu) \) can also be derived directly from \( X(\mu) \) and \( Y(\mu) \).

It would appear that, in a given conservative case, all the solutions are included in one and only one family. In nonconservative cases, on the other hand, it would seem that the solutions are unique.

In view of the ambiguity in the solutions of equations (1) and (2) in conservative cases, it would be convenient to select, in each case, a particular member of the one-parametric family of solutions as a \textit{standard solution}.

\textbf{Definition.}—In a conservative case we shall define the solutions which have the property

\[
X_0 = \int_0^1 X(\mu) \Psi(\mu) \, d\mu = 1
\]

and

\[
y_0 = \int_0^1 Y(\mu) \Psi(\mu) \, d\mu = 0
\]

as the standard solutions of equations (1) and (2).

Such solutions can always be found; for, if a particular \( X \) and \( Y \) do not satisfy equations (64) and (65), we can always find a \( Q \) such that the solutions derived from \( X \) and \( Y \) in the manner of equations (52) and (53) have the required property. Standard solutions defined in this manner have several interesting properties. We shall state them in the form of the following theorems:

\textbf{Theorem 6.}—The standard solutions are invariant to increments of \( \tau_1 \) according to the integrodifferential equations of theorem 1.

Multiplying equations (18) and (19) by \( \Psi(\mu) \) and integrating over the range of \( \mu \), we have

\[
\frac{dx_0}{d\tau_1} = y_0 y_{-1} = 0
\]

and

\[
\frac{dy_0}{d\tau_1} = - (1 - x_0) y_{-1} = 0 .
\]

\textbf{Theorem 7.}—Let \( X(\mu, \tau_1) \) and \( Y(\mu, \tau_1) \) denote the standard solutions of equations (1) and (2) in a conservative case for a particular value of \( \tau_1 \). Consider the solutions

\[
F(\mu, \tau_1) = X(\mu, \tau_1) + Q \mu \left[ X(\mu, \tau_1) + Y(\mu, \tau_1) \right]
\]

and

\[
G(\mu, \tau_1) = Y(\mu, \tau_1) - Q \mu \left[ X(\mu, \tau_1) + Y(\mu, \tau_1) \right]
\]

of equations (1) and (2) derived from \( X \) and \( Y \) and continue them for other values of \( \tau_1 \) according to the equations of theorem 1. These solutions for other values of \( \tau_1 \) can, in turn, be derived from the standard solutions appropriate for these values of \( \tau_1 \) with vary-

\[\text{Since } x_0 + y_0 = 1, \text{ eq. (64) implies eq. (65) and vice versa.}\]
ing values of \( Q \). The quantity \( Q \), considered as a function of \( \tau_1 \) in this manner, satisfies the differential equation

\[
\frac{d}{d\tau_1} \left( \frac{1}{Q} \right) - \frac{2y-1}{Q} = -1. \tag{70}
\]

Proof.—According to equations (18) and (19),

\[
\frac{\partial F}{\partial \tau_1} = G \int_0^1 \frac{d\mu'}{\mu'} \Psi (\mu') G (\mu'), \tag{71}
\]

and

\[
\frac{\partial G}{\partial \tau_1} + \frac{G}{\mu} = F \int_0^1 \frac{d\mu'}{\mu'} \Psi (\mu') G (\mu'). \tag{72}
\]

Now (cf. eq. [69])

\[
\int_0^1 \frac{d\mu'}{\mu'} \Psi (\mu') G (\mu') = \int_0^1 \frac{d\mu'}{\mu'} \Psi (\mu') Y (\mu') - Q \int_0^1 [X (\mu') + Y (\mu')] \Psi (\mu') d\mu'. \tag{73}
\]

Hence

\[
\frac{\partial F}{\partial \tau_1} = (y-1 - Q) G, \tag{74}
\]

and

\[
\frac{\partial G}{\partial \tau_1} + \frac{G}{\mu} = (y-1 - Q) F. \tag{75}
\]

On the other hand, since \( X \) and \( Y \) remain standard solutions when continued for other values of \( \tau_1 \), we must have

\[
\frac{\partial F}{\partial \tau_1} = \frac{\partial X}{\partial \tau_1} + Q\mu \left( \frac{\partial X}{\partial \tau_1} + \frac{\partial Y}{\partial \tau_1} \right) + \mu \left( X + Y \right) \frac{dQ}{d\tau_1}
\]

\[
= y-1 Y + Q\mu \left( y-1 Y - \frac{Y}{\mu} + y-1 X \right) + \mu \left( X + Y \right) \frac{dQ}{d\tau_1}
\]

\[
= (y-1 - Q) Y + \mu \left( X + Y \right) \left( y-1 Q + \frac{dQ}{d\tau_1} \right). \tag{76}
\]

We can re-write the foregoing equation in the form

\[
\frac{\partial F}{\partial \tau_1} = (y-1 - Q) \left[ Y - Q\mu \left( X + Y \right) \right] + \mu \left( X + Y \right) \left[ Q \left( y-1 - Q \right) + y-1 Q + \frac{dQ}{d\tau_1} \right], \tag{77}
\]

or

\[
\frac{\partial F}{\partial \tau_1} = (y-1 - Q) G + \mu \left( X + Y \right) \left( 2y-1 Q - Q^2 + \frac{dQ}{d\tau_1} \right). \tag{78}
\]

Comparing equations (74) and (78), we must have

\[
\frac{dQ}{d\tau_1} + 2y-1 Q - Q^2 = 0. \tag{79}
\]

A similar consideration of the equation for \( \partial G/\partial \tau_1 \) leads to the same equation for \( Q \).
Equation (79) can be re-written in the form
\[
\frac{1}{Q^2} \frac{dQ}{d\tau_1} + \frac{2y-1}{Q} = 1,
\] (80)
which is equivalent to equation (70).

The various relations (eqs. [13]-[16] and [41]) derived in the preceding sections for solutions of equations (1) and (2) in general take particularly simple forms for standard solutions of conservative cases. We shall collect these relations in the form of the following theorem:

**Theorem 8.**—For the standard solutions in a conservative case we have the relations
\[
x_0 = 1, \quad y_0 = 0,
\] (81)
\[
x_1^2 - y_1^2 = 2 \int_0^1 \Psi (\mu) \mu^2 d\mu,
\] (82)
\[
\int_0^1 \frac{\mu' \Psi (\mu')}{\mu + \mu'} \left[ X (\mu) X (\mu') - Y (\mu) Y (\mu') \right] d\mu' = 1,
\] (83)
\[
\int_0^1 \frac{\mu' \Psi (\mu')}{\mu - \mu'} \left[ Y (\mu) X (\mu') - X (\mu) Y (\mu') \right] d\mu' = \frac{e^{-r_1}/\mu}{1 + \frac{e^{-r_1}/\mu}{1 - \frac{e^{-r_1}/\mu}{1 - k^2 \mu^2}}},
\] (84)
and
\[
\int_0^1 \frac{\mu' \Psi (\mu')}{\mu - \mu'} \left[ X (\mu) X (\mu') - Y (\mu) Y (\mu') \right] d\mu' = x_1 X (\mu) - y_1 Y (\mu) - \mu.
\] (85)

6. The correspondence between the solutions of equations (1) and (2) and the functions \(X\) and \(Y\) introduced into the solution of the equations of transfer in a finite approximation.—

In solving the equations of transfer appropriately for the problem of diffuse reflection and transmission in Paper XXI, we found that we had to introduce certain functions, \(X\) and \(Y\), involving the nonvanishing roots of a characteristic equation of the form
\[
1 = 2 \sum_{i=1}^n a_i \Psi (\mu_i) \left\{ 1 - \frac{1}{1 - k^2 \mu_i^2} \right\},
\] (87)
where, as usual, the \(\mu_i\)'s are the zeros of \(P_{2n}(\mu)\) and the \(a_i\)'s are the corresponding Gaussian weights. In terms of these functions \(X\) and \(Y\) it was possible to express the solutions of the emergent radiations in closed forms in all cases considered. In analogy with the theory of the \(H\)-functions (Paper XIV, § 11), we may therefore expect that the functions \(X\) and \(Y\) appearing in the solutions in a finite approximation are rational approximations to the solutions of equations (1) and (2) when they are replaced by their “finite forms,” namely,
\[
X (\mu) = 1 + \mu \sum_{i=1}^n \frac{a_i \Psi (\mu_i)}{\mu_i + \mu_i} \left[ X (\mu) X (\mu_i) - Y (\mu) Y (\mu_i) \right]
\] (88)
and
\[
Y (\mu) = e^{-r_1}/\mu + \mu \sum_{i=1}^n \frac{a_i \Psi (\mu_i)}{\mu_i - \mu_i} \left[ Y (\mu) X (\mu_i) - X (\mu) Y (\mu_i) \right].
\] (89)
We shall now examine in what sense the functions $X$ and $Y$ introduced in Paper XXI, equations (125) and (126), are related to equations (88) and (89).

The definitions of the functions $X$ and $Y$ in Paper XXI (eqs. [125] and [126]) suggest that, in seeking solutions of the equations (88) and (89), we try the forms

$$X (\mu) = F (\mu) - e^{-r/\mu} G (-\mu) \tag{90}$$

and

$$Y (\mu) = e^{-r/\mu} F (-\mu) - G (\mu), \tag{91}$$

where $F(\mu)$ and $G(\mu)$ are certain rational functions in $\mu$, satisfying the conditions

$$F (-\mu_j) = G (-\mu_j) = 0 \quad (j = 1, \ldots, n). \tag{92}$$

For the forms (90) and (91)

$$X (\mu) = e^{-r/\mu} Y (-\mu) \quad \text{and} \quad Y (\mu) = e^{-r/\mu} X (-\mu). \tag{93}$$

For $X$ and $Y$ related in this manner, it may be directly verified that equations (88) and (89) are equivalent to each other and that therefore it would suffice to consider only one of them.\(^8\)

Now substituting for $X$ and $Y$ according to equations (90) and (91) in equations (88) and (89) and remembering the further conditions (eq. [92]) imposed on $F$ and $G$, we find, after some minor reductions, that

$$F (\mu) - e^{-r/\mu} G (-\mu) = 1 + \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} \left[ F (\mu) F (\mu_j) - G (\mu) G (\mu_j) \right]$$

$$+ \mu e^{-r/\mu} \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} \left[ G (-\mu) F (\mu_j) - F (-\mu) G (\mu_j) \right]. \tag{94}$$

Equating the terms with and without the exponential factor in this equation, we obtain

$$F (\mu) = 1 + \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} \left[ F (\mu) F (\mu_j) - G (\mu) G (\mu_j) \right] \tag{95}$$

and

$$G (-\mu) = \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} \left[ G (-\mu) F (\mu_j) - F (-\mu) G (\mu_j) \right]. \tag{96}$$

We can re-write these equations alternatively in the forms

$$F (\mu) \left[ 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} F (\mu_j) \right] + G (\mu) \left[ \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} G (\mu_j) \right] = 1 \tag{97}$$

and

$$G (-\mu) \left[ 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} F (\mu_j) \right] + F (-\mu) \left[ \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} G (\mu_j) \right] = 0. \tag{98}$$

Solving for the quantities in brackets in equations (97) and (98), we find

$$F (-\mu) = \left[ F (\mu) F (-\mu) - G (\mu) G (-\mu) \right] \left[ 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi (\mu_j)}{\mu + \mu_j} F (\mu_j) \right] \tag{99}$$

\(^8\) It is of interest to note in this connection that the substitution (93) makes the functional equations (1) and (2) also equivalent to each other.

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and

\[ G(-\mu) = [F(-\mu) F(-\mu) - G(\mu) G(-\mu)] \left[ -\mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} G(\mu_j) \right]. \] (100)

So far we have pursued only the consequences of assumptions (90), (91), and (92) regarding the form of the solutions of equations (88) and (89) adopted. We shall now write down explicitly the formulae for \( F(\mu) \) and \( G(\mu) \) suggested by the expressions for \( X(\mu) \) and \( Y(\mu) \) given in Paper XXI (eqs. [125] and [126]) and see how well they satisfy equations (99) and (100).

From a comparison of equations (90) and (91) and the equations (125) and (126) of Paper XXI, we conclude that

\[ F(\mu) = \frac{(-1)^n P(-\mu)}{W(\mu)} \frac{C_0(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} \prod_{i=1}^{n} (\mu + \mu_i) \prod_{a} (1 + k_a \mu) \frac{C_0(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} \] (101)

and

\[ G(\mu) = \frac{(-1)^n P(-\mu)}{W(\mu)} \frac{C_1(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} \prod_{i=1}^{n} (\mu + \mu_i) \prod_{a} (1 + k_a \mu) \frac{C_1(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}}, \] (102)

where \( C_0(\mu) \) and \( C_1(\mu) \) are certain polynomials in \( \mu \), of degree \( n \) in nonconservative cases and \( n - 1 \) in conservative cases, satisfying the conditions (cf. Paper XXI, eqs. [50], [108], and [109])

\[ C_0\left(1/k_a\right) = \lambda_a C_1\left(-1/k_a\right) \] (103)

and

\[ \lambda_a = e^{\kappa a^2} a \frac{P\left(-1/k_a\right)}{P\left(+1/k_a\right)}. \] (104)

According to the theorems proved in Paper XXI, § 4, the relations (103) and (104) are sufficient to determine \( C_0(\mu) \) and \( C_1(\mu) \) uniquely, apart from two arbitrary constants of proportionality in \( C_0(\mu) + C_1(\mu) \) and \( C_0(\mu) - C_1(\mu) \). For the particular "normalization" adopted in Paper XXI (eqs. [100] and [101])

\[ C_0(\mu) \to \prod_{a>0} (1 + k_a \mu) = R(-\mu) \quad \text{as} \quad \tau_1 \to \infty, \] (105)

and

\[ C_1(\mu) \to 0 \quad \text{as} \quad \tau_1 \to \infty. \] (106)

\[ ^9 \] In equations (103) and (104) (as in eqs. [101] and [102]), \( a \) runs through positive and negative indices corresponding to all the nonvanishing roots \( k_a(a = \pm 1, \ldots, \pm n \text{ or } \pm n + 1 \text{ and } k_a = -k_{-a}) \) of the characteristic equation.
These further conditions, which we shall now also require, suffice to characterize the functions $C_0(\mu)$ and $C_1(\mu)$ without any arbitrariness.

Remembering that, in the approximation in which we are at present working,

\[ H(\mu) = \frac{1}{\frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{a>0} (1 + \kappa_a\mu)}} \tag{107} \]

we can re-write equations (101) and (102) in the forms

\[
F(\mu) = \frac{H(\mu)}{R(\mu)} \frac{C_0(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} \quad \text{and} \quad G(\mu) = \frac{H(\mu)}{R(\mu)} \frac{C_1(-\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}}. \tag{108}
\]

For $F(\mu)$ and $G(\mu)$, defined, in this manner (cf. Paper XXI, eq. [105]),

\[
F(\mu) F(-\mu) = G(\mu) G(-\mu)
\]

\[
= \frac{H(\mu) H(-\mu)}{[C_0^2(0) - C_1^2(0)] W(\mu)} [C_0(\mu) C_0(-\mu) - C_1(\mu) C_1(-\mu)] = H(\mu) H(-\mu). \tag{109}
\]

Using this result and also equations (108) in equations (99) and (100), we find that our problem is reduced to examining the validity of the equations

\[
\frac{C_0(\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} = H(\mu) R(-\mu) \left[ 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} F(\mu_j) \right] \tag{110}
\]

and

\[
\frac{C_1(\mu)}{[C_0^2(0) - C_1^2(0)]^{1/2}} = H(\mu) R(-\mu) \left[ -\mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} G(\mu_j) \right], \tag{111}
\]

or, equivalently,

\[
C_0(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} [C_0^2(0) - C_1^2(0)]^{1/2} P(-\mu) \left[ 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} F(\mu_j) \right] \tag{112}
\]

and

\[
C_1(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} [C_0^2(0) - C_1^2(0)]^{1/2} P(-\mu) \left[ -\mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} G(\mu_j) \right]. \tag{113}
\]

The validity (or otherwise) of equations (112) and (113) will depend essentially on whether the quantities on the right-hand sides of these equations are related in the manner required by equations (103). To examine this we have to evaluate the summations which occur in equations (112) and (113).

To carry out the summations in equations (112) and (113), we have first to break $F(\mu)$ and $G(\mu)$ into partial fractions. This requires us to treat the conservative and the nonconservative cases separately.

Considering, first, the nonconservative case, we have $2n$ distinct roots for the char-
acteristic equation (87), which occur in pairs \((k_a = -k_{-a}, a = 1, \ldots, n)\), and \(F(\mu)\) and \(G(\mu)\) can be expressed in the forms

\[
F(\mu) = \sum_{a=-n}^{+n} \frac{L_a}{1 + k_a \mu} + \frac{1}{k_1^2 \ldots k_n^2 \mu_1 \ldots \mu_n} \frac{c_0^{(n)}}{[C_0^2(0) - C_1^2(0)]^{1/2}}
\]

(114)

and

\[
G(\mu) = \sum_{a=-n}^{+n} \frac{L_a e^{-k_a \tau_1}}{1 - k_a \mu} + \frac{1}{k_1^2 \ldots k_n^2 \mu_1 \ldots \mu_n} \frac{c_1^{(n)}}{[C_0^2(0) - C_1^2(0)]^{1/2}},
\]

(115)

where the \(2n\) constants \(L_a (a = \pm 1, \ldots, \pm n)\) are to be determined from the conditions (cf. eq. [92])

\[
F(-\mu_j) = G(-\mu_j) = 0 \quad (j = 1, \ldots, n),
\]

(116)

and \(c_0^{(n)}\) and \(c_1^{(n)}\) are the coefficients of the highest power, \(\mu^n\), in \(C_0(\mu)\) and \(C_1(\mu)\).

To verify that \(F(\mu)\) and \(G(\mu)\), defined in the manner of the foregoing equations, agree with our earlier definitions (eqs. [108]), we first observe that conditions (116) enable us to express \(F(\mu)\) and \(G(\mu)\) in the forms

\[
F(\mu) = \frac{(-1)^n P(-\mu)}{\mu_1 \ldots \mu_n} f(-\mu)
\]

(117)

and

\[
G(\mu) = \frac{(-1)^n P(-\mu)}{\mu_1 \ldots \mu_n} g(-\mu),
\]

(118)

where \(f(\mu)\) and \(g(\mu)\) are polynomials of degree \(n\) in \(\mu\); and that, further,

\[
f\left(1/k_a\right) = \lambda_a g\left(-1/k_a\right) \quad (a = \pm 1, \ldots, \pm n),
\]

(119)

where \(\lambda_a\) has the same meaning as in equation (104).\(^{11}\) These latter conditions arise from a comparison of the values

\[
L_a = \frac{(-1)^n P\left(1/k_a\right)}{\mu_1 \ldots \mu_n} f\left(1/k_a\right) \quad (a = \pm 1, \ldots, \pm n)
\]

(120)

and

\[
L_a e^{-k_a \tau_1} = \frac{(-1)^n P\left(-1/k_a\right)}{\mu_1 \ldots \mu_n} g\left(-1/k_a\right) \quad (a = \pm 1, \ldots, \pm n),
\]

(121)

which follow from equations (114), (115), (117), and (118). In accordance with the theorems of Paper XXI, § 4, we therefore conclude that \(f(\mu)\) and \(g(\mu)\) must be expressible in the forms

\[
f(\mu) = q_0 C_0(\mu) + q_1 C_1(\mu) \quad \text{and} \quad g(\mu) = q_0 C_1(\mu) + q_1 C_0(\mu),
\]

(122)

where \(q_0\) and \(q_1\) are constants. And, finally, from a comparison of the coefficients of the highest power of \(\mu\) in \(f\) and \(g\) as deducible from equations (114) and (115) and equations (117) and (118), respectively, we readily verify that

\[
q_0 = \frac{1}{[C_0^2(0) - C_1^2(0)]^{1/2}} \quad \text{and} \quad q_1 = 0,
\]

(123)

as required.

\(^{10}\) As in Paper XXI (cf. p. 153, n. 6), in all summations and products over \(a\) there is no term with \(a = 0\).

\(^{11}\) Negative values of \(a\) are permitted (cf. n. 9).
**RADIATIVE EQUILIBRIUM**

For convenience we shall re-write equations (114) and (115) in the forms

\[
F(\mu) = \sum_{a=-n}^{+n} \frac{L_a}{1 + k_a \mu} + a
\]

and

\[
G(\mu) = \sum_{a=-n}^{+n} \frac{L_a e^{-k_a \mu}}{1 - k_a \mu} + b,
\]

where

\[
a = \frac{1}{k_1^2 \cdots k_n^2 \mu_1 \cdots \mu_n} \frac{(n)}{C_0} \left( C_0^2(0) - C_1^2(0) \right)^{1/2}
\]

and

\[
b = \frac{1}{k_1^2 \cdots k_n^2 \mu_1 \cdots \mu_n} \frac{(n)}{C_1} \left( C_0^2(0) - C_1^2(0) \right)^{1/2}.
\]

Moreover (cf. eqs. [120] and [121]),

\[
L_a = \frac{(-1)^n P(+1/k_a)}{\mu_1 \cdots \mu_n W_a(1/k_a)} \frac{C_0(1/k_a)}{[C_0^2(0) - C_1^2(0)]^{1/2}},
\]

and

\[
L_a e^{-k_a \mu} = \frac{(-1)^n P(-1/k_a)}{\mu_1 \cdots \mu_n W_a(1/k_a)} \frac{C_1(-1/k_a)}{[C_0^2(0) - C_1^2(0)]^{1/2}}.
\]

Returning, now, to the evaluation of the summations on the right-hand sides of equations (112) and (113), we consider, first,

\[
\Sigma_1(\mu) = 1 - \mu \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} F(\mu_j).
\]

Since \(F(-\mu_j) = 0\) (j = 1, \ldots, n) we can, without altering anything, extend the summation also over negative values of j. We thus have

\[
\Sigma_1(\mu) = 1 - \mu \sum_{j=-n}^{+n} \frac{a_j \Psi(\mu_j)}{\mu + \mu_j} \left( \sum_{a=-n}^{+n} \frac{L_a}{1 + k_a \mu_j} + a \right),
\]

where we have further substituted for \(F(\mu_j)\) according to equation (124). Remembering that, according to the characteristic equation defining the roots \(k_\alpha\) and \(k_\beta\) (cf. eq. [87]),

\[
1 = \sum_{j=-n}^{+n} \frac{a_j \Psi(\mu_j)}{1 + k_\alpha \mu_j} = \sum_{j=-n}^{+n} \frac{a_j \Psi(\mu_j)}{1 + k_\beta \mu_j}
\]

for all \(\alpha\)’s and \(\beta\)’s (\(= \pm 1, \ldots, \pm n\)), we have for \(\mu = 1/k_\alpha\)

\[
\Sigma_1(1/k_\alpha) = 1 - \sum_{j=-n}^{+n} \frac{a_j \Psi(\mu_j)}{1 + k_\alpha \mu_j} \left( \sum_{a=-n}^{+n} \frac{L_a}{1 + k_\beta \mu_j} + a \right)
\]

\[
= 1 - \left( 1 - a - \sum_{j=-n}^{+n} \sum_{\beta=-n}^{+n} \frac{a_j \Psi(\mu_j)}{(1 + k_\alpha \mu_j)(1 + k_\beta \mu_j)} \right).
\]
Alternatively, inverting the order of the summation, we can also write
\[
\Sigma_1 (1/k_a) = 1 - a - \sum_{\beta \neq a}^{+n} L_\beta - k_a - k_\beta \sum_{j=-n}^{+n} a_j \Psi (\mu_j) \left( \frac{k_a}{1 + k_a \mu_j} - \frac{k_\beta}{1 + k_\beta \mu_j} \right)
\]
(133)

\[
\frac{-L_a \sum_{j=-n}^{+n} a_j \Psi (\mu_j)}{(1 + k_a \mu_j)^2}.
\]

Hence
\[
\Sigma_1 (1/k_a) = 1 - a - \sum_{\beta = -n}^{+n} L_\beta + L_a \left[ 1 - \sum_{j=-n}^{+n} \frac{a_j \Psi (\mu_j)}{(1 + k_a \mu_j)^2} \right].
\]
(134)

But
\[
1 - \sum_{j=-n}^{+n} \frac{a_j \Psi (\mu_j)}{(1 + k_a \mu_j)^2} = \sum_{j=-n}^{+n} \frac{a_j \Psi (\mu_j)}{1 + k_a \mu_j} \left( 1 - \frac{1}{1 + k_a \mu_j} \right)
\]
(135)
\[
= \frac{\sum_{j=-n}^{+n} a_j \mu_j \Psi (\mu_j)}{1 + k_a \mu_j}.
\]

We therefore have (cf. eq. [124])
\[
\Sigma_1 (1/k_a) = 1 - F (0) + L_a k_a \sum_{j=-n}^{+n} a_j \mu_j \Psi (\mu_j) (1 + k_a \mu_j)^2 \quad (a = \pm 1, \ldots, \pm n)。
\]
(136)

The summation
\[
\Sigma_2 (\mu) = -\mu \sum_{j=1}^{n} a_j \Psi (\mu_j) G (\mu_j)
\]
(137)
\[
= -\mu \sum_{j=-n}^{+n} a_j \Psi (\mu_j) G (\mu_j)
\]
can be similarly reduced. For \( \mu = -1/k_a \) we find
\[
\Sigma_2 (-1/k_a) = -G (0) + L_a k_a e^{-k_a \tau} \sum_{j=-n}^{+n} a_j \mu_j \Psi (\mu_j) (1 + k_a \mu_j)^2 \quad (a = \pm 1, \ldots, \pm n)。
\]
(138)

An expression for the quantity
\[
\sum_{j=-n}^{+n} a_j \mu_j \Psi (\mu_j) (1 + k_a \mu_j)^2
\]
(139)
which occurs in both equations (136) and (138), can be found by differentiating the identity\textsuperscript{12}
\[
1 - \sum_{j=-n}^{+n} a_j \Psi (\mu_j) = \frac{1}{1 + \mu_j / z} = \frac{1}{H (z) H (-z)}
\]
(140)
\[
= \frac{\mu_1^2 \cdots \mu_n^2}{P (z) P (-z)} \prod_{a=-n}^{+n} (1 + k_a z),
\]
\textsuperscript{12} Cf. eq. (285) in the author's Gibbs Lecture (reference given in n. 4).
with respect to $z$ and setting $z = 1/k_a$. In this manner we find

$$k_a \sum_{j=1}^{n} a_j \mu_j \Psi_j (\mu_j) = \mu_1 \mu_2 \ldots \mu_n \frac{W_a(1/k_a)}{P(1/k_a)P(-1/k_a)}.$$  \hspace{1cm} (141)

Using this result in equations (136) and (138) and substituting also for $L_a$ and $L_a e^{-k_a x_1}$ according to equations (127) and (128), we have

$$\Sigma_1 (1/k_a) = 1 - F(0) + (-1)^n \mu_1 \mu_2 \ldots \mu_n \frac{C_0(1/k_a)}{[C_0^2(0) - C_1^2(0)]^{1/2}P(-1/k_a)}$$  \hspace{1cm} (a = \pm 1, \ldots, \pm n) \hspace{1cm} (142)

and

$$\Sigma_2 (-1/k_a) = -G(0) + (-1)^n \mu_1 \mu_2 \ldots \mu_n \frac{C_1(-1/k_a)}{[C_0^2(0) - C_1^2(0)]^{1/2}P(+1/k_a)}$$  \hspace{1cm} (a = \pm 1, \ldots, \pm n). \hspace{1cm} (143)

The right-hand sides of equations (112) and (113) for $\mu = +1/k_a$, respectively $-1/k_a$, therefore become

$$\frac{(-1)^n}{\mu_1 \ldots \mu_n} [C_0^2(0) - C_1^2(0)]^{1/2}P(-1/k_a)[1 - F(0)] + C_0(1/k_a)$$  \hspace{1cm} (144)

and

$$\frac{(-1)^{n+1}}{\mu_1 \ldots \mu_n} [C_0^2(0) - C_1^2(0)]^{1/2}P(+1/k_a)G(0) + C_1(-1/k_a).$$  \hspace{1cm} (145)

Now the validity of equations (112) and (113) requires that expressions (144) and (145) be exactly $C_0(1/k_a)$ and $C_1(-1/k_a)$. This will be the case only if $F(0) = 1$ and $G(0) = 0$. But, according to equations (108),

$$F(0) = \frac{C_0(0)}{[C_0^2(0) - C_1^2(0)]^{1/2}} \quad \text{and} \quad G(0) = \frac{C_1(0)}{[C_0^2(0) - C_1^2(0)]^{1/2}};$$  \hspace{1cm} (146)

and it is not true that $C_1(0) = 0$ identically, in all approximations, and for all values of $\tau_1$. However, according to equations (105) and (106), the conditions $F(0) = 1$ and $G(0) = 0$ will be met with increasing accuracy as $\tau_1 \to \infty$. Also, actual numerical calculations have shown that the errors with which the conditions $F(0) = 1$ and $G(0) = 0$ are met in the third or the fourth approximations (in our method of solution) are not large even for values of $\tau_1$ of the order of 0.25 or less.

Turning, next, to the consideration of conservative cases, we have only $2n - 2$ non-vanishing roots for the characteristic equation. The expressions corresponding to (114) and (115) for $F(\mu)$ and $G(\mu)$ in partial fractions are, in consequence,

$$F(\mu) = \sum_{a=-n+1}^{-1} \frac{L_a}{1 + k_a \mu} + L_0 - \frac{1}{k_1^2 \ldots k_{n-1}^2 \mu_1 \ldots \mu_n} \left[ \frac{c_0^{(n-1)}}{[C_0^2(0) - C_1^2(0)]^{1/2}} \mu \right]$$  \hspace{1cm} (147)

and

$$G(\mu) = \sum_{a=-n+1}^{-1} \frac{L_a e^{-k_a x_1}}{1 - k_a \mu} + \xi_0 - \frac{1}{k_1^2 \ldots k_{n-1}^2 \mu_1 \ldots \mu_n} \left[ \frac{c_1^{(n-1)}}{[C_0^2(0) - C_1^2(0)]^{1/2}} \mu \right],$$  \hspace{1cm} (148)

where the $2n$ constants, $L_{\pm a}(a = 1, \ldots, n - 1), L_0$, and $\xi_0$ are again to be determined by conditions of the form (116) and $c_0^{(n-1)}$ and $c_1^{(n-1)}$ are, respectively, the coefficients...
of the highest power, \( \mu^{n-1} \), in \( C_0(\mu) \) and \( C_1(\mu) \), defined in terms of the reduced number of characteristic roots appropriate for the conservative case.\(^{13}\) With expressions (147) and (148) for \( F(\mu) \) and \( G(\mu) \), the rest of the analysis proceeds exactly as in the nonconservative case. The only difference is that use must also be made of the equation

\[
\sum_{j=-n}^{+n} \frac{a_j \Psi(\mu_j)}{1 + k_\alpha \mu_j} = 0 \quad (a = \pm 1, \ldots, \pm n + 1), \tag{149}
\]

which is valid only in conservative cases (cf. Paper XIV, eq. [159]).

One special characteristic of the solution (148) should be noted. We have

\[
\sum_{j=-n}^{+n} a_j \Psi(\mu_j) Y(\mu_j) = \sum_{j=-n}^{+n} a_j \Psi(\mu_j) G(\mu_j) = \sum_{j=-n}^{+n} a_j \Psi(\mu_j) G(\mu_j)
\]

\[
= \sum_{a=-n+1}^{-1} L_a e^{-k_a r_1} \left( \sum_{j=-n}^{+n} a_j \Psi(\mu_j) \right) + \sum_{j=-n}^{+n} a_j \Psi(\mu_j).
\]

Since (conservative case!)

\[
\sum_{j=-n}^{+n} a_j \Psi(\mu_j) = 1,
\]

we have

\[
\sum_{j=-n}^{+n} a_j \Psi(\mu_j) Y(\mu_j) = \sum_{a=-n+1}^{+n} L_a e^{-k_a r_1} + \sum_{j=-n}^{+n} a_j \Psi(\mu_j).
\]

We have already seen that the nonvanishing of \( G(0) \) is a measure of the inaccuracy of our scheme of approximation. We therefore conclude that, for the exact solutions in the limit of infinite approximation,

\[
\int_0^1 Y(\mu) \Psi(\mu) \, d\mu = 0, \tag{153}
\]

and that the functions \( X \) and \( Y \), defined in terms of the reduced number of the characteristic roots in conservative cases, must be associated with the standard solutions of the functional equations (1) and (2) as defined in § 5 (eqs. [64] and [65]).

We now summarize the conclusions that we have reached in the form of the following theorem:

**Theorem 9.**—The functions \( X(\mu) \) and \( Y(\mu) \) defined in terms of the nonvanishing roots of the characteristic equation

\[
1 = 2 \sum_{j=1}^{n+1} a_j \Psi(\mu_j) \frac{1}{1 - k^2 \mu_j^2} \tag{154}
\]

in the manner

\[
X(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} \frac{1}{[C_0^2(0) - C_1^2(0)]^{1/2}} \frac{1}{W(\mu)}
\]

[\( P(\mu) \) \( C_0(-\mu) - e^{-r/P(\mu) C_1(\mu)} \)]

\(^{13}\) See particularly the remarks in Paper XXI, n. 12.
and
\[ Y(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n \left[ C_0(0) - C_1(0) \right]^{1/2}} \frac{1}{W(\mu)} \]

where
\[ C_0(\mu) = \sum_{2^{n-1} \text{ terms}} \varepsilon_i^{(0)} \frac{\prod_{i=1}^{l} \prod_{m=1}^{n-l} (k_{r_i} + k_{s_m})}{\prod_{i=1}^{l} \prod_{m=1}^{n-l} (k_{r_i} - k_{s_m})} \frac{1}{1 + k_{r_i} \mu} \left( 1 - k_{s_m} \mu \right) \]

and
\[ C_1(\mu) = (-1)^{n-1} \sum_{2^{n-1} \text{ terms}} \varepsilon_i^{(1)} \frac{\prod_{i=1}^{l} \prod_{m=1}^{n-l} (k_{r_i} + k_{s_m})}{\prod_{i=1}^{l} \prod_{m=1}^{n-l} (k_{r_i} - k_{s_m})} \times \frac{1}{1 + k_{r_i} \mu} \left( 1 - k_{s_m} \mu \right) \]

where
\[ \varepsilon_i^{(0)} = 1 \text{ for integers of the form } n - 4l \]
\[ = -1 \text{ for integers of the form } n - 4l - 2 \]
\[ = 0 \text{ otherwise} \]

and
\[ \varepsilon_i^{(1)} = 1 \text{ for integers of the form } n - 4l - 1 \]
\[ = -1 \text{ for integers of the form } n - 4l - 3 \]
\[ = 0 \text{ otherwise} \]

and
\[ \lambda_n = \frac{P(-1/k_n)}{P(+1/k_n)} \]

are, in the limit of infinite approximation, to be associated with the solutions of the functional equations
\[ X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu'')}{\mu'' + \mu} \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right] d\mu' \]

and
\[ Y(\mu) = e^{-r_0/\mu} + \mu \int_0^1 \frac{\Psi(\mu'')}{\mu'' - \mu} \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right] d\mu' \]

In conservative cases, \( X(\mu) \) and \( Y(\mu) \) (defined in terms of the reduced number of nonvanishing characteristic roots) are in the limit of infinite approximation to be associated with the standard solutions of equations (162) and (163), having the property
\[ \int_0^1 X(\mu) \Psi(\mu) d\mu = 1 \quad \text{and} \quad \int_0^1 Y(\mu) \Psi(\mu) d\mu = 0. \]
II. ISOTROPIC SCATTERING WITH AN ALBEDO $\sigma_0 \leq 1$

7. Equations of the problem.—For the problem of diffuse reflection and transmission by an atmosphere scattering radiation isotropically with an albedo $\sigma_0 \leq 1$, the basic equations are (cf. Paper XVII, eqs. [115]–[120])

$$I(0, \mu) = \frac{\sigma_0}{4\mu} FS(\mu, \mu_0); \quad I(\tau, -\mu) = \frac{\sigma_0}{4\mu} FT(\mu, \mu_0), \quad (165)$$

$$\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right) S(\mu, \mu_0) = X(\mu) X(\mu_0) - Y(\mu) Y(\mu_0), \quad (166)$$

$$\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) T(\mu, \mu_0) = Y(\mu) X(\mu_0) - X(\mu) Y(\mu_0), \quad (167)$$

$$\frac{\partial S}{\partial \tau_1} = Y(\mu) Y(\mu_0), \quad (168)$$

and

$$\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \frac{\partial T}{\partial \tau_1} = \frac{1}{\mu_0} X(\mu) Y(\mu_0) - \frac{1}{\mu} Y(\mu) X(\mu_0). \quad (169)$$

Further, the definitions of $X(\mu)$ and $Y(\mu)$ in terms of $S(\mu, \mu_0)$ and $T(\mu, \mu_0)$ are

$$X(\mu) = 1 + \frac{1}{2\mu_0} \int_{-1}^{1} S(\mu, \mu') \frac{d\mu'}{\mu'} \quad (170)$$

and

$$Y(\mu) = e^{-\tau_s/\mu} + \frac{1}{2\mu_0} \int_{0}^{1} T(\mu, \mu') \frac{d\mu'}{\mu'}. \quad (171)$$

In virtue of equations (166), (167), (170), and (171), we have the equations

$$X(\mu) = 1 + \frac{1}{2\mu_0} \int_{0}^{1} \frac{d\mu'}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] \quad (172)$$

and

$$Y(\mu) = e^{-\tau_s/\mu} + \frac{1}{2\mu_0} \int_{0}^{1} \frac{d\mu'}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')]. \quad (173)$$

Thus $X$ and $Y$ satisfy functional equations of the form considered in Section I, with the characteristic function

$$\Psi(\mu) = \frac{1}{2}\sigma_0 = \text{constant}. \quad (174)$$

In considering the foregoing equations, it is of interest to establish, first, that equations (168) and (169) are really equivalent to the integrodifferential equations of theorem 1 (§ 3).

Thus, differentiating equation (170) with respect to $\tau_1$ and using equation (168), we have

$$\frac{\partial X}{\partial \tau_1} = \frac{1}{2}\sigma_0 Y(\mu) \int_{0}^{1} \frac{d\mu'}{\mu'} Y(\mu'). \quad (175)$$

Next, differentiating equation (171), we have

$$\frac{\partial Y}{\partial \tau_1} = -\frac{1}{\mu} e^{-\tau_s/\mu} + \frac{1}{2\mu_0} \int_{0}^{1} \frac{d\mu'}{\mu - \mu'} \left[\frac{\mu}{\mu'} X(\mu) Y(\mu') - Y(\mu) X(\mu')\right]. \quad (176)$$
and, combining this with equation (173), we obtain
\[
\frac{\partial Y}{\partial \tau} + \frac{Y}{\mu} = \frac{1}{2} \sigma_0 X(\mu) \int_0^1 \frac{d\mu'}{\mu^2} Y(\mu') .
\] (177)

It is seen that equations (175) and (177) are in agreement with equations (18) and (19) of theorem 1.

Finally, we may note that, according to equations (165)–(167) we can express the reflected and the transmitted intensities in the forms
\[
I(0, \mu) = \frac{1}{2} \sigma_0 F \left[ \frac{\mu_0}{\mu + \mu_0} \left( X(\mu) X(\mu_0) - Y(\mu) Y(\mu_0) \right) \right]
\] (178)

and
\[
I(\tau, -\mu) = \frac{1}{2} \sigma_0 F \left[ \frac{\mu_0}{\mu - \mu_0} \left( Y(\mu) X(\mu_0) - X(\mu) Y(\mu_0) \right) \right].
\] (179)

8. The case $\omega_0 < 1$.—Comparing the expressions for the emergent intensities given in the preceding section (eqs. [178] and [179]) with those given in Paper XXI (Sec. I, eqs. [127] and [128]), we observe that we have here a confirmation and an illustration of the correspondence enunciated in theorem 9 between the functions $X$ and $Y$ occurring in the solutions of the equations of transfer in a finite approximation, and the functions defined in terms of the functional equations in the exact theory.

9. The ambiguity in the solution of the functional equations in the case $\omega_0 = 1$ and its resolution by an appeal to the K-integral.—When $\omega_0 = 1$, the equations (172) and (173) belong to the conservative class discussed in § 5, and, according to theorem 5, the solutions of these equations, in this case, are not unique, the general solutions being, in fact, expressible in the forms
\[
X(\mu) = Q\mu \left[ X(\mu) + Y(\mu) \right]
\] (180)

and
\[
Y(\mu) = Q\mu \left[ X(\mu) + Y(\mu) \right],
\] (181)

where $Q$ is an arbitrary constant and $X(\mu)$ and $Y(\mu)$ are the standard solutions, having for the characteristic function $\frac{1}{2}$ the property
\[
a_0 = \int_0^1 X(\mu) \, d\mu = 2 \quad \text{and} \quad \beta_0 = \int_0^1 Y(\mu) \, d\mu = 0.
\] (182)

With solutions (180) and (181) of the equations
\[
X(\mu) = 1 + \frac{1}{2} \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} \left[ X(\mu) X(\mu') - Y(\mu) Y(\mu') \right]
\] (183)

and
\[
Y(\mu) = e^{-\tau_1/\mu} + \frac{1}{2} \mu \int_0^1 \frac{d\mu'}{\mu - \mu'} \left[ Y(\mu) X(\mu') - X(\mu) Y(\mu') \right],
\] (184)

the expressions (178) and (179) for the emergent intensities take the forms
\[
I(0, \mu) = \frac{1}{2} \sigma_0 F \left[ \frac{1}{\mu_0 + \mu} \left[ X(\mu_0) X(\mu) - Y(\mu_0) Y(\mu) \right] \right.
\]
\[
+ Q \left[ X(\mu_0) + Y(\mu_0) \right] \left[ X(\mu) + Y(\mu) \right] \left] \right]
\] (185)
and
\[ I(\tau, -\mu) = \frac{1}{4} \mu_0 F \left\{ \frac{1}{\mu_0 - \mu} \left[ Y(\mu_0) X(\mu) - X(\mu_0) Y(\mu) \right] - Q \left[ X(\mu_0) + Y(\mu_0) \right] \left[ X(\mu) + Y(\mu) \right] \right\}. \]

Solutions (185) and (186) for the emergent intensities involve the arbitrary constant \( Q \), and there is nothing in the framework of the equations of § 7, for the case \( \omega_0 = 1 \), which will remove this arbitrariness. We therefore conclude that the various invariances considered in Paper XVII are not sufficient to determine the physical solutions uniquely in conservative cases. We shall encounter further examples of this in Sections III and V. But it should be noted in the present context that a comparison of solutions (185) and (186) with those obtained in Paper XXI (Sec. II, eqs. [167]–[172]) provides a confirmation and an illustration of what is stated in theorem 9, namely, that the \( X \)- and \( Y \)-functions defined in terms of the reduced number of nonvanishing characteristic roots in conservative cases and which occur in the solutions of the equations of transfer in a finite approximation, are, in the framework of the exact theory, to be associated with the standard solutions of the corresponding functional equations.

We now turn to the matter of the arbitrariness in solutions (185) and (186) and the manner in which it is to be resolved.

The equation of transfer appropriate to the problem on hand is
\[ \frac{d I}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-\infty}^{+\infty} I(\tau', \mu') d\mu' - \frac{1}{4} F e^{-\tau} \mu. \]

From this equation, two integrals which the problem admits can be derived. They are
\[ F(\tau) = 2 \int_{-\infty}^{+\infty} I(\tau, \mu) \mu \, d\mu = \mu_0 F \left( e^{-\tau/\mu_0} + \gamma_1 \right) \]
and
\[ K(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} I(\tau, \mu) \mu^2 \, d\mu = \frac{1}{4} \mu_0 F \left( -\mu_0 e^{-\tau/\mu_0} + \gamma_1 + \gamma_2 \right), \]
where \( \gamma_1 \) and \( \gamma_2 \) are two constants. The first of these represents the flux integral. We shall refer to the second as the "K-integral."

Applying the integrals (188) and (189) at \( \tau = 0 \) and \( \tau = \tau_1 \), we have
\[ F(0) = 2 \int_{0}^{1} I(0, \mu) \mu \, d\mu = \mu_0 F \left( 1 + \gamma_1 \right), \]
\[ F(\tau_1) = -2 \int_{0}^{1} I(\tau_1, -\mu) \mu \, d\mu = \mu_0 F \left( e^{-\tau_1/\mu_0} + \gamma_1 \right), \]
\[ K(0) = \frac{1}{2} \int_{0}^{1} I(0, \mu) \mu^2 \, d\mu = \frac{1}{4} \mu_0 F \left( -\mu_0 + \gamma_2 \right), \]
and
\[ K(\tau_1) = \frac{1}{2} \int_{0}^{1} I(\tau_1, -\mu) \mu^2 \, d\mu = \frac{1}{4} \mu_0 F \left( -\mu_0 e^{-\tau_1/\mu_0} + \gamma_1 + \gamma_2 \right). \]

On the other hand, we can also evaluate \( F(0) \), \( F(\tau_1) \), \( K(0) \), and \( K(\tau_1) \) according to the solutions (185) and (186) for \( I(0, \mu) \) and \( I(\tau_1, -\mu) \). In this manner we shall obtain four relations between the three constants \( \gamma_1 \), \( \gamma_2 \), and \( Q \). However, it will appear that two of these four relations are equivalent and that, in fact, they just suffice to determine all the constants uniquely.
The integrals defining $F(0)$, $F(t_1)$, $K(0)$, and $K(t_1)$ in terms of $I(0, \mu)$ and $I(t_1, -\mu)$, given by equations (185) and (186), can all be evaluated by using the various relations valid for standard solutions and collected under theorem 8 (eqs. [81]–[86]). We find

\begin{equation}
F(0) = \mu_0 F \left\{ 1 + \frac{1}{2} Q (a_1 + \beta_1) \left[ X (\mu_0) + Y (\mu_0) \right] \right\},
\end{equation}
\begin{equation}
F(t_1) = \mu_0 F \left\{ e^{-r_1/\mu_0} + \frac{3}{2} Q (a_1 + \beta_1) \left[ X (\mu_0) + Y (\mu_0) \right] \right\},
\end{equation}
\begin{equation}
K(0) = \frac{1}{2} \mu_0 F \left\{ -\mu_0 + \frac{1}{2} a_1 X (\mu_0) - \frac{1}{2} \beta_1 Y (\mu_0) + \frac{3}{2} Q (a_2 + \beta_2) \left[ X (\mu_0) + Y (\mu_0) \right] \right\},
\end{equation}
\begin{equation}
K(t_1) = \frac{1}{4} \mu_0 F \left\{ -\mu_0 e^{-r_1/\mu_0} + \frac{3}{2} \beta_1 X (\mu_0) - \frac{1}{2} a_1 Y (\mu_0)
- \frac{3}{2} Q (a_2 + \beta_2) \left[ X (\mu_0) + Y (\mu_0) \right] \right\},
\end{equation}
where $a_n$ and $\beta_n$ are the moments of order $n$ of $X(\mu)$ and $Y(\mu)$, respectively (cf. eq. 11).

It is now seen that equations (190) and (194) and (191) and (195), in agreement with each other, determine

\begin{equation}
\gamma_1 = \frac{1}{2} Q (a_1 + \beta_1) \left[ X (\mu_0) + Y (\mu_0) \right].
\end{equation}

From equations (192) and (196) we next find that

\begin{equation}
\gamma_2 = \frac{3}{2} a_1 X (\mu_0) - \frac{1}{2} \beta_1 Y (\mu_0) + \frac{3}{2} Q (a_2 + \beta_2) \left[ X (\mu_0) + Y (\mu_0) \right].
\end{equation}

Finally, from equations (193) and (197) we obtain

\begin{equation}
\gamma_1 \tau_1 + \gamma_2 = \frac{1}{2} \beta_1 X (\mu_0) - \frac{1}{2} a_1 Y (\mu_0) - \frac{3}{2} Q (a_2 + \beta_2) \left[ X (\mu_0) + Y (\mu_0) \right].
\end{equation}

Now, substituting for $\gamma_1$ and $\gamma_2$ in equation (200), according to equations (198) and (199), we find

\begin{equation}
\frac{1}{2} Q (a_1 + \beta_1) \tau_1 = -\frac{1}{2} (a_1 - \beta_1) - Q (a_2 + \beta_2).
\end{equation}

Hence,

\begin{equation}
Q = -\frac{a_1 - \beta_1}{(a_1 + \beta_1) \tau_1 + 2 (a_2 + \beta_2)}.
\end{equation}

With this determination of $Q$ in terms of the optical thickness, $\tau_1$, of the atmosphere and the moments of the standard solutions of equations (183) and (184), we have removed the arbitrariness left by the functional equations in the solutions for the emergent intensities. It is in some ways remarkable that an explicit appeal to the $K$-integral is necessary to resolve the arbitrariness left by the functional equations. We shall see later that similar appeals to the $K$- integrals are necessary in the two other cases of perfect scattering that we shall consider (namely, Rayleigh scattering and scattering in accordance with Rayleigh's phase function) to resolve the ambiguities in the solutions of the functional equations incorporating the various invariances of the problem.

10. The verification that $Q$ satisfies the differential equation of theorem 7.—It is apparent that the quantity $Q$ as introduced in § 9 must satisfy the differential equation of theorem 7. In our present context we can write this equation (eq. [70]) in the form

\begin{equation}
\frac{d}{dt} \left( \frac{1}{Q} \right) - \frac{1}{Q} \beta_{-1} = -1,
\end{equation}

since

\begin{equation}
y_{-1} = \frac{1}{2} \int_0^t \frac{d\mu}{\mu} Y (\mu') = \frac{1}{2} \beta_{-1}.
\end{equation}

We shall now show that $Q$ as defined by equation (202) satisfies equation (203).
Making use of the relation (cf. theorem 8, eq. [82])

\[ a_1^2 - \beta_1^2 = 4 \int_0^1 \mu^2 d\mu = \frac{4}{3}, \]  

we first re-write equation (202) in the form

\[ \frac{1}{Q} = -\frac{3}{4} \left[ (a_1 + \beta_1)^2 \tau_1 + 2 (a_2 + \beta_2) (a_1 + \beta_1) \right]. \]  

From this equation we then obtain

\[
\frac{d}{d\tau_1} \left( \frac{1}{Q} \right) = -\frac{3}{4} \left[ (a_1 + \beta_1) \left\{ (a_1 + \beta_1) + 2 \tau_1 \frac{d}{d\tau_1} (a_1 + \beta_1) \right\} + 2 (a_2 + \beta_2) \frac{d}{d\tau_1} (a_1 + \beta_1) + 2 (a_1 + \beta_1) \frac{d}{d\tau_1} (a_2 + \beta_2) \right].
\]  

To simplify equation (207) further, we observe that, according to equations (175) and (177), we now have

\[ \frac{d}{d\tau_1} (X + Y) = \frac{1}{2} \beta_1 (X + Y) - \frac{Y}{\mu}. \]  

Multiplying this equation by \( \mu^n \) and integrating over the range of \( \mu \), we obtain

\[ \frac{d}{d\tau_1} (a_n + \beta_n) = \frac{1}{2} \beta_1 (a_n + \beta_n) - \beta_{n-1}. \]  

In particular,

\[ \frac{d}{d\tau_1} (a_1 + \beta_1) = \frac{1}{2} \beta_1 (a_1 + \beta_1) \]  

(since, according to eq. [182], \( \beta_0 = 0 \)), and

\[ \frac{d}{d\tau_1} (a_2 + \beta_2) = \frac{1}{2} \beta_1 (a_2 + \beta_2) - \beta_1. \]  

Using the foregoing relations in equation (207), we find, after some minor reductions, that

\[ \frac{d}{d\tau_1} \left( \frac{1}{Q} \right) = -\frac{3}{4} \left[ \beta_1 \left\{ (a_1 + \beta_1)^2 \tau_1 + 2 (a_2 + \beta_2) (a_1 + \beta_1) \right\} + (a_1 + \beta_1)^2 - 2 \beta_1 (a_1 + \beta_1) \right]. \]  

Hence (cf. eqs. [205] and [206])

\[ \frac{d}{d\tau_1} \left( \frac{1}{Q} \right) = \frac{\beta_1 - \langle a_1^2 - \beta_1^2 \rangle}{Q} = \frac{\beta_1}{Q} - 1. \]  

This completes the verification.

\( (To \ be \ continued) \)