REPORTS ON THE PROGRESS OF ASTRONOMY

stellar dynamics

Since the early investigations of Eddington, Jeans and Schwarzschild, two problems have occupied a central position in stellar dynamics. The first of these relates to stellar encounters, its importance for stellar dynamics and the manner in which it is best incorporated into the framework of a general theory. The second relates to the consequences for the dynamical theory which the known kinematics of stellar motions implies. Through recent investigations, the solutions to these two problems have reached a comparatively advanced stage and it would seem proper to review in general terms the results to which they have led.

I. Stellar Encounters

Considering first the problem relating to stellar encounters, it is simplest to picture them in terms of the simple two-body problem. On this idealization, during an encounter, each star describes a hyperbola relative to the other, and the result of an encounter is that each star suffers certain increments, \( \Delta u_{\parallel} \) and \( \Delta u_{\perp} \), in the velocity in directions which are parallel and perpendicular respectively to the initial direction of motion. While the exact amounts of these increments \( \Delta u_{\parallel} \) and \( \Delta u_{\perp} \) which a star experiences will differ in individual cases, it is evident, in view of the long-range character of the inverse square force, that for the encounters occurring most frequently, these will be extremely small compared to the initial velocities of the stars. The directions and magnitudes of the motions of stars can accordingly be influenced by stellar encounters only as the result of the cumulative effect of a large number of encounters. It is, therefore, seen that in this respect the inverse square encounters which are relevant in stellar dynamics, differ in a fundamental way from those with which we are familiar in gas kinetic theory. For, in ordinary gas theory, single encounters in general lead to radical changes in the directions and magnitudes of the motions of individual molecules. Indeed the analogy which has sometimes been made between gas dynamics and stellar dynamics does not seem very appropriate: it would appear far closer to draw a comparison between the motions of stars as influenced by stellar encounters and Brownian motion. For, as is well known, colloidal particles describing Brownian movement owe their motions to collisions with the molecules of the surrounding fluid though individual collisions can hardly make "an impression" on the particle: again it is the cumulative effect of a large number of collisions that is responsible for the observed motions. This analogy which exists between Brownian motion and the motions induced by stellar encounters is far-reaching and is capable of a wide generalization. We shall accordingly pursue this a little further.

In describing Brownian motion we distinguish between random fluctuations and the systematic decelerations consequent to the operation of Stokes's law. This distinction, which is generally made, is brought out most clearly in Langevin's form for the equation of motion:

\[
\frac{du}{dt} = -\eta u + A(t)
\]

of a Brownian particle where \( u \) denotes the velocity of the particle. According to this equation, the influence of the surrounding medium can be split up into two parts: first, a systematic part \( -\eta u \) representing a dynamical friction experienced by the particle, and, second, a fluctuating part described by \( A(t) \). In the classical theories of Brownian
motion the frictional term \(-\eta u\) is assumed to be governed by Stokes's law, while the fluctuating part \(A(t)\) is assumed to vary extremely rapidly compared with the variations of \(u\), and in such a way that the increment in velocity experienced by the particle due to this term and during two successive intervals \((t, t+\Delta t)\) and \((t+\Delta t, t+2\Delta t)\) are not correlated if \(\Delta t\) is large compared with the average time between successive collisions. The manner in which the two terms in equation (1) operate is understood most clearly if we consider the increment in the velocity \(\Delta u\) which the particle experiences in an interval \(\Delta t\) satisfying the conditions stated. Writing

\[
\Delta u = -\eta u \Delta t + \delta u(\Delta t)
\]

(2)

the term \(\delta u(\Delta t)\) is of the nature of an "error" with a Gaussian distribution. Moreover,

\[
\delta u = 0; \quad |\delta u|^2 = 2q\Delta t,
\]

(3)

where \(q\) may be called the diffusion coefficient in the velocity space. In order that particles describing Brownian motion in accordance with the foregoing laws restore and maintain a Maxwellian distribution of velocities among themselves, it is necessary that \(q\) and \(\eta\) are related according to

\[
q = \frac{1}{2} |u|^2 \eta,
\]

(4)

where \(\frac{1}{2} |u|^2 \) (\(= 3kT/m\)) denotes the mean square velocity of the particles. And finally, it can be shown that the function \(W(u, t)\) governing the probability of occurrence of the velocity \(u\) at time \(t\) for a Brownian particle must satisfy the Fokker-Planck equation

\[
\frac{\partial W}{\partial t} = \text{div}_u (q \text{grad}_u W + \eta Wu).
\]

(5)

According to this equation we may visualize the motions of the representative points in the velocity space as a process of diffusion in which the rate of flow across an element of surface \(d\sigma\) is given by

\[
-(q \text{grad}_u W + \eta Wu) \cdot \mathbf{1}_d d\sigma,
\]

(6)

where \(\mathbf{1}_d\) is a unit vector normal to the element of surface considered.

Returning to the stellar case, we shall now show how the effect of stellar encounters can be treated in a manner exactly analogous to the theory of Brownian motion described above.

A straightforward analysis based on the classical two-body problem leads to the following expressions for the increments in velocity \(\Delta u_{||}\) and \(\Delta u_\perp\) which a star of mass \(m\) and velocity \(u\) suffers as the result of an encounter with another star of mass \(m_1\) and velocity \(v_1\):

\[
\Delta u_{||} = -\frac{2m_1}{m + m_1}[(u-v_1 \cos \theta) \cos \psi + v_1 \sin \theta \cos \Theta \sin \psi] \cos \psi
\]

(7)

and

\[
\Delta u_\perp = \frac{2m_1}{m + m_1}[u^2 + v_1^2 - 2uv_1 \cos \theta - \{(u-v_1 \cos \theta) \cos \psi + v_1 \sin \theta \cos \Theta \sin \psi]^2]^{1/2} \cos \psi.
\]

(8)

In equations (7) and (8) \(\theta\) denotes the angle between the two vectors \(u\) and \(v_1\), \(\Theta\) the inclination of the orbital plane to the plane containing \(u\) and \(v_1\),

\[
\cos \psi = \left(1 + D^2(u^2 + v_1^2 - 2uv_1 \cos \theta)^2/G^2(m_1 + m)^2\right)^{-1/2},
\]

(9)

\(D\) the impact parameter and \(G\) the constant of gravitation.

Now consider an interval of time \(\Delta t\) long compared with the time required for two average stars to separate by a distance equal to the average distance between the stars but short compared with the intervals during which the velocity of a star may be expected
to change appreciably. (In practice this implies that $\Delta t \sim 10^6$ years.) During such an interval of time the increments in velocities $\Delta u_\parallel$ and $\Delta u_\perp$ which a star with an initial velocity $u$ may be expected to suffer in directions parallel and perpendicular respectively to its initial direction of motion, can be obtained by simply averaging the expressions for $\Delta u_\parallel$ and $\Delta u_\perp$ over the relevant ranges of the various parameters describing an encounter. With certain approximations appropriate for the problem it is found that

$$\Sigma \Delta u_\parallel = -4 \pi m_1(m_1 + m)NG^2 \frac{1}{u} \left( \log \left( \frac{D_0 |u|^2}{G(m_1 + m)} \right) \right) |u|^1 f(v_1)dv_1 \tag{10}$$

and

$$\Sigma \Delta u_\perp = 0, \tag{11}$$

where $D_0$ denotes the average distance between the stars, $N$ the number of stars per unit volume, $|u|^2$ the mean square velocities of the "field stars", and $f(v_1)$ the distribution function governing the probability of occurrence of a star with velocity $|v_1| = v_1$. According to equation (10), we may say that the star experiences dynamical friction with a coefficient of dynamical friction $\eta$ given by

$$\eta = 4 \pi N m_1(m_1 + m) \frac{G^2}{u^3} \left( \log \left( \frac{D_0 |u|^2}{G(m_1 + m)} \right) \right) |u|^1 f(v_1)dv_1. \tag{12}$$

It may be noted that in contrast to the classical case, $\eta$ is now a function of $|u|$ such that

$$\eta \rightarrow \eta_0 = \text{constant} \quad \text{as} \quad |u| \rightarrow 0,$$

while

$$\eta \rightarrow \text{constant} |u|^{-3} \quad \text{as} \quad |u| \rightarrow \infty. \tag{13}$$

This decrease in the frictional coefficient as $|u|$ increases is of course to be expected on physical grounds.

Again, from equation (7) we similarly find after averaging over the various quantities that

$$\Sigma \Delta u_\parallel = \frac{8}{3} \pi N m_1 G^2 \frac{1}{u} \left( \log \left( \frac{D_0 |u|^2}{G(m_1 + m)} \right) \right) |u|^2 \Delta t f(v_1)dv_1, \tag{15}$$

which represents, in analogy, with equation (3) a diffusion in the velocity space. The completeness of the analogy of our present problem with Brownian motion is seen even more clearly when we note that, according to equations (12) and (15) (cf. eq. (4)),

$$\frac{\Sigma \Delta u_\parallel}{\eta \Delta t} = \frac{2}{3} \frac{m_1}{m_1 + m} |u|^2. \tag{16}$$

It would therefore appear that the most convenient way in which we may study the influence of stellar encounters is to picture it as causing a Brownian motion. Quantitatively the changes in velocities which a star experiences on this account will be governed by the Fokker-Planck equation (5) with $\eta$, now given by equation (12). When there is an external field of force ($K$) present, the phenomenon in the six-dimensional phase space of a single star will similarly be governed by the equation

$$\frac{\partial W}{\partial t} + u \cdot \nabla W + K \cdot \nabla u \cdot W = \nabla_a (g \nabla_a W + \eta W u). \tag{17}$$

We shall now illustrate the usefulness of incorporating stellar encounters in a general dynamical theory in the manner we have described, by considering the problem of the rate of escape of stars from galactic clusters like the Pleiades.

As was first clearly recognized by Ambarzumian and Spitzer, an important factor in the evolution of the galactic and the globular clusters is their gradual impoverishment
due to the escape of stars. We shall now show how we can evaluate this rate of escape of stars quite rigorously.

To be specific, we shall suppose that in order that a star may escape from a cluster it is only necessary that it acquire a velocity greater than or equal to a certain critical velocity which we shall denote by $v_\infty$. On this assumption the probability that a star will have acquired the necessary velocity of escape during a certain time $t$ can be evaluated quite simply from the probability $p(v_\infty, t)$ that a star having an initial velocity $|u| = v_0$ at time $t = 0$ will acquire for the first time the velocity $|u| = v_\infty$ during $t$ and $t + dt$. For, on integrating $p(v_\infty, t)$ over $t$ from 0 to $t$, we shall obtain the probability $Q(v_\infty, t)$ that the star will have acquired the velocity $v_\infty$ during the entire interval from 0 to $t$. And finally, averaging $Q(v_\infty, t)$ over the relevant range of the initial velocities, we shall obtain the expectation $\bar{Q}(t)$ that a star will have acquired the velocity $v_\infty$ during the time $t$.

The advantage of formulating the problem in the manner described above is that the function $p(v_\infty, t)$ can be determined in terms of the solution of equation (5) which satisfies certain appropriate boundary conditions. For, remembering the interpretation of this equation as representing a process of diffusion taking place in the velocity space, it is evident that $p(v_\infty, t)$ will be given by

$$p(v_\infty, t) = -\frac{4\pi \eta}{|u|^2} \left[ \frac{\partial W(|u|, t)}{\partial |u|} \right]_{|u| = v_\infty},$$

(18)

where $W(|u|, t)$ denotes a spherically symmetric solution of equation (5) (with $\eta$ given by equation (12)), which satisfies the boundary conditions

$$W(|u|, t) = 0 \text{ for } |u| = v_\infty \text{ for all } t > 0$$

(19)

and

$$W(|u|, t) \to \frac{1}{4\pi v_0^3} \delta(|u| - v_0) \text{ as } t \to 0,$$

(20)

where $\delta$ stands for the $\delta$-function of Dirac. Passing over the method by which the foregoing boundary value problem has been solved, we may quote the final result. It is found that to a sufficient approximation we can write

$$Q(t) = 1 - e^{-0.0075/t_0},$$

(21)

where

$$\frac{1}{t_0} = \frac{1}{|u|} \to \eta = 8\pi N m^2 G^2 \left( \log \left[ \frac{D_0}{2 G m} \right] \right) \left( \frac{3}{2} |u| \right)^{3/2} \frac{4}{3\pi^{1/2}}.$$

(22)

According to this definition of $t_0$, it is of the order of the reciprocal of the coefficient of dynamical friction for values of velocities of the order of the root mean square velocity in the system. We may therefore take $t_0$ as a measure of the time required to restore a Maxwellian distribution starting from an arbitrary initial distribution of velocities. In other words, $t_0$ is a measure of the time of relaxation of the system.

Returning to equation (21), we conclude that the mean life of a cluster may be estimated to be $133 t_0$. For the Pleiades, $\sqrt{|u|^2}$ is estimated at 0.43 km./sec. Using this value of $\sqrt{|u|^2}$ and the known dimensions and population of this cluster, the time of relaxation $t_0$ is found to be $2 \times 10^7$ years. Its mean life is therefore $3 \times 10^9$ years. This estimate for the mean life of the Pleiades is typical of galactic clusters, and we can accordingly find in these estimates an argument in favour of the currently adopted time scale of the order of $3 \times 10^9$ years.

We have discussed this treatment of the escape of stars from clusters at this length only to illustrate the advantage of formulating the effect of stellar encounters on the motions of stars in the manner of the theory of Brownian motion and using the generalized Fokker-Planck equation (17) as the fundamental equation of stellar dynamics.
II. DYNAMICS OF STELLAR SYSTEMS WITH DIFFERENTIAL MOTIONS AND STAR STREAMING

Turning our attention next to the second of the two major problems stated at the outset, it is first necessary to formulate the laws of stellar kinematics in a sufficiently general way. It may be said that the two principal characteristics of stellar motions in a system like our galaxy are (1) the existence of a field of differential motions and (2) the phenomenon of star streaming which the distribution of the peculiar velocities exhibit. We owe to Milne the first clear formulation of the first of these two concepts: essentially it implies that we are able to define in an unambiguous way a local standard of rest for each small region of the galaxy and that the different local standards of rest are in relative motion. In other words, we are able to express the function \( \Psi(x, y, z; U, V, W; t) \) governing the distribution of the velocities \( (U, V, W) \) in the various parts of the galaxy \( (x, y, z) \) in the form

\[
\Psi = \Psi(x, y, z; U - U_0, V - V_0, W - W_0; t),
\]

where \( U_0, V_0, \) and \( W_0 \) (defining the field of differential motions) are functions of position and time only. (Sometimes, as when dealing with the system of the high velocity stars, it may be necessary to generalize the foregoing and write

\[
\Psi = \sum_i \Psi_i(x, y, z; U - U_0^{(i)}, V - V_0^{(i)}, W - W_0^{(i)}; t),
\]

where the summation over the index \( i \) corresponds to the fact that under these circumstances we are to regard the stellar system as (formally) consisting of several distinct sub-systems.) On the other hand the existence of star streaming implies that the distribution of the peculiar velocities \( (u = U - U_0, v = V - V_0; \ w = W - W_0) \) is governed by Schwarzschild's ellipsoidal law:

\[
\Psi(u, v, w) = \frac{N}{\pi^{3/2}} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^{1/2} e^{-(au^2 + bv^2 + cw^2 + 2fuv + 2gwu + 2hvw)},
\]

where the coefficients of the velocity ellipsoid \( a, b, c, f, g \) and \( h \), and the number of stars per unit volume, \( N \), at \( (x, y, z) \), may all be expected to be functions of position and time. Combining equations (23) and (25) we have

\[
\Psi = e^{-(Q + \sigma)},
\]

where \( Q \) is a general homogeneous quadratic form in the residual velocities and

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\]
With the statement of stellar kinematics expressed in a distribution function of the form (26), our problem now is, what does this imply for dynamics? To answer this question we have simply to examine the conditions under which the equation of continuity (eq. (17)) will admit a solution of the form (26), or more generally a solution of the form

$$\Psi = \Psi(Q + \sigma),$$

(28)

where $\Psi$ is an arbitrary function of the argument specified. One circumstance simplifies this problem considerably. And that is, under the present conditions in the galaxy we may ignore the collision terms in this equation. The reason why we can do this is the following: as we have seen in the preceding section, the relative importance of the effect of stellar encounters in influencing the motions of stars can be inferred from a knowledge of the time of relaxation $t_0$. For, if we are primarily interested in time-intervals which are short compared to $t_0$, it is evident that the influence of stellar encounters can be ignored. And it is readily verified that under the conditions, for example in the general neighbourhood of the Sun, $t_0 \sim 10^{14}$ years, and the time-intervals we are interested in are far shorter than this. Accordingly, we may reformulate our fundamental problem in the following terms:

**Under what conditions will the equation of continuity**

$$\frac{D\Psi}{Dt} = \frac{\partial\Psi}{\partial t} + U \frac{\partial\Psi}{\partial x} + V \frac{\partial\Psi}{\partial y} + W \frac{\partial\Psi}{\partial z} - \frac{\partial V}{\partial x} \frac{\partial\Psi}{\partial x} - \frac{\partial V}{\partial y} \frac{\partial\Psi}{\partial y} - \frac{\partial V}{\partial z} \frac{\partial\Psi}{\partial z} = 0,$$

(29)

**regarded as a partial differential for $\Psi$, admit a solution of the form (28)?** (In equation (29) we have used $V = V(x, y, z; t)$ to denote the gravitational potential.)

For the assumed form of $\Psi$ the equation of continuity reduces to

$$\frac{D}{Dt}(Q + \sigma) = 0.$$

(30)

In this form we recognize that our problem is essentially one of determining the field of force in which a particle describing an orbit may possess an integral which is a general quadratic form in the velocities. Up to a point the progress toward the solution of this problem is simple and straightforward. We substitute for $Q + \sigma$ in equation (30) and find that it breaks up into a system of twenty partial differential equations. Of these, a first group of ten equations involves only the coefficients of the velocity ellipsoid, while a second group of six equations involves the differential motions as well. These sixteen equations can be solved explicitly and suffice to determine the space dependence of the coefficients of the velocity ellipsoid and the components of the differential motion. However, their dependence on time is left unspecified, as the general solutions introduce a large number of arbitrary functions of time. (Actually, the general solution of these equations introduces twenty arbitrary functions of time and six constants of integration.) But it should not be assumed that the solution to the physical problem involves this degree of arbitrariness. Severe restrictions are imposed by the last four equations, which for their validity require the satisfaction of certain integrability conditions. These integrability conditions appear as a set of six simultaneous partial differential equations for $V(x, y, z; t)$, and the problem reduces to discussing the compatibility of these equations. In its most general form this discussion has not so far been completed. But a number of special solutions have been found, and we shall briefly indicate the nature of some of the more important of these.

Considering as our first example stellar systems in steady states, it is found that three of the six integrability conditions which we mentioned reduce to a single linear homogeneous partial differential equation for $V(x, y, z)$. This equation for $V$ can be
solved quite generally, and shows that with a suitable choice of the origin and orientation of our co-ordinate axes, \( \mathbf{V} \) must have the form

\[
V(x, y, z) = V(\sigma; z + \alpha \theta),
\]

where \( \sigma \) denotes the distance from the Z-axis, \( \theta \) the azimuthal angle measured in the \( (x, y) \)-plane and \( \alpha \) a constant. In other words, \( \mathbf{V} \) must in general be characterized by helical symmetry. However, for finite stellar systems it is evident that we must require that \( \alpha = 0 \); for, the single valuedness of \( \mathbf{V} \) will otherwise require that \( \mathbf{V} \) is periodic in \( z \) with a period \( 2\pi \alpha \). We have accordingly the following fundamental theorem:

For stellar systems with differential motions which are in steady states and are of finite spatial extent the gravitational potential \( \mathbf{V} \) must necessarily be characterized by axial symmetry.

The remaining integrability conditions may now be discussed, and it appears that for gravitational potentials which have an axial symmetry but are otherwise unrestricted, the most general integral which is quadratic in the velocities must be a combination of the energy and the angular momentum integrals:

\[
I_1 = \Pi^2 + \Theta^2 + Z^2 + 2V; \quad I_2 = \sigma \Theta,
\]

where \( \Pi, \Theta \) and \( Z \) are the linear velocities in the radial, transverse and the Z-directions respectively. Thus \( Q + \sigma \) must be reducible to the form:

\[
Q + \sigma = \alpha I_1 + \beta I_2^2 + \gamma I_2,
\]

where \( \alpha, \beta, \gamma \) are constants. Solutions of this general form have been used particularly by Oort and Lindblad.

A second example we shall consider relates to the two-dimensional problem treating stellar systems with circular symmetry and in non-steady states. It can be shown that in this case the most general non-trivial form of \( \mathbf{V}(\sigma, t) \) which is compatible with the integrability conditions of the problem is

\[
V(\sigma, t) = -\frac{\phi}{2\sigma} \sigma^2 + \frac{1}{\phi^2} V_1 \left( \frac{\sigma}{\phi} \right),
\]

where \( V_1 \) is an arbitrary function of the argument specified and \( \phi \) is a certain function of time only which appears in the solutions for the coefficients of the velocity ellipse and the components of the differential motions. Thus:

\[
a = \phi^2 + \kappa_2 \sigma^2; \quad b = \phi^2 + \kappa_2 \sigma^2; \quad h = -\kappa_2 \sigma \theta
\]

and

\[
U_0 = \frac{\phi}{\phi^2} + \frac{\beta y}{\phi^2 + \kappa_2 \sigma^2}; \quad V_0 = \frac{\phi}{\phi^2} - \frac{\beta x}{\phi^2 + \kappa_2 \sigma^2},
\]

where \( \kappa_2, \beta \) are constants. Moreover,

\[
\sigma = -\frac{\beta^2 \sigma^2}{\phi^2 + \kappa_2 \sigma^2} + 2V_1 \left( \frac{\sigma}{\phi} \right) + \text{constant}.
\]

With the solutions for \( a, b, h, U_0 \) and \( V_0 \) given by equations (35) and (36), we find that the velocity ellipse can be reduced to the form

\[
Q = \phi^2 (\Pi - \Pi_0)^2 + (\phi^2 + \kappa_2 \sigma^2)(\Theta - \Theta_0 )^2,
\]

where

\[
\Pi_0 = \frac{\phi}{\phi} \sigma \quad \text{and} \quad \Theta_0 = -\frac{\beta \sigma}{\phi^2 + \kappa_2 \sigma^2}.
\]

While the conditions of the problem thus far introduced do not restrict \( \phi \) and allow it to be an arbitrary function of time, it would appear that other considerations will in fact
limit $\phi$ to a particular form. Thus, if we should suppose that the form of the solution (34) is valid also for $\sigma \to \infty$, it is evident that the boundedness of $V$ will demand that

$$-\frac{\dot{\phi}}{2\phi} = \text{constant} \frac{1}{\phi^4} \quad (40)$$

For the term $-(\phi/2\phi)\sigma^2$ must be cancelled by a term in $\sigma^2$ in $V_1$, and this can occur only with a coefficient proportional to $\phi^{-4}$ (cf. eq. (34)). Equation (40) is readily integrated and leads for $\phi$ to a solution of the form

$$\phi = \phi_0 [1 + \alpha (t - t_0)^2]^{1/2}, \quad (41)$$

where $\phi_0$, $\alpha$ and $t_0$ are constants.

The third and final example we shall refer to and for which the general solution has been found relates to stellar systems in non-steady states and with a spherical distribution of the residual velocities. For this case the distribution function has the form

$$\Psi = \Psi [a(U - U_0)^2 + (V - V_0)^2 + (W - W_0)^2 + \sigma], \quad (42)$$

where $\Psi$ is an arbitrary function of the argument specified and $a$, $U_0$, $V_0$, $W_0$ and $\sigma$ are all functions of position and time, arbitrary in the first instance. It can be shown that this form of $\Psi$ implies that $V$ satisfies a certain linear non-homogeneous partial differential equation in $x$, $y$, $z$ and $t$. This equation for $V$ has been solved, and while its discussion is too long to undertake within the limits of this report, it may be stated that the importance of this solution for general dynamical theory is that there is reason to believe that the form for $V$ which is thus disclosed is likely to be in fact the most general form which is compatible with our fundamental kinematical postulates. We may further add that the solution for $V$ and particularly $\sigma$ are of sufficient generality as to be suggestive for the interpretation of spiral structure in extra-galactic nebulae.

Bibliography.—The history and the background for the problem considered in this section, with full references, will be found in


Later references are:


In the two preceding sections we have summarized the present state of the two classical problems of the subject. We shall now consider some other problems of more recent origin.

III. THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS

In section I we idealized stellar encounters in terms of the two-body problem of classical dynamics. This idealization, while it is the simplest and the most convenient one to make, has its shortcomings. For example, in evaluating quantities like $\Sigma\Delta u_1$ and $\Sigma\Delta u_1^2$, we find that the integrals over the impact parameter $D$ diverge logarithmically if they are extended from 0 to $\infty$. In practice this divergence is avoided by limiting the range of integration from 0 to a certain maximum value $D_0$ which is assumed to be of the order of the average distance between the stars. This procedure is justified on the grounds that for values of the impact parameter which are of the order of, or greater than, the average distance between the stars, the two-body approximation must fail and the concept of individual two-body encounters must be replaced by that of a multiple encounter in which several stars partake. This argument is, of course, sound,
but it would be preferable if the arbitrariness in the cut-off value of $D$ could be avoided and a general theory developed in which the influence of all the neighbouring stars on a given one will be taken into account. It would appear that the elements for such a theory of multiple encounters will be provided by an analysis of the fluctuating gravitational field to which a star is subjected on account of the changing complexion of the local stellar distribution. While a rigorous analysis of such fluctuating fields is likely to be very difficult, a model for analysis which suggests itself and which would seem appropriate for the conditions under which the two-body approximation fails, is the following:

Let the fluctuations in the stellar distribution which occur be subject only to the restriction of a constant average number of stars per unit volume. Further, let the stars be assumed to describe linear trajectories independently of each other and with a Gaussian distribution of the velocities in a suitably chosen frame of reference. We ask: What is the statistics of the fluctuating gravitational field in such a system? Or, more specifically: What is the probability distribution of the force per unit mass, $F$, acting on a star? What is the rate of change to be expected in $F$ acting on a star moving with a velocity $v$? What is the correlation in the forces acting on a star at two different instants of time? And finally, what is the correlation in the forces acting simultaneously at two points in the system separated by a finite distance? We shall now briefly indicate how these questions have been answered.

The force per unit mass acting on a star at a particular instant of time (say $t=0$) is given by

$$F = G \sum_i \frac{M_i}{|r_1 - r_i|} \mathbf{r}_i,$$

where $M_i$ denotes the mass of a typical “field star” and $r_i$ its instantaneous position vector relative to the star under consideration. Further, in equation (43) the summation is extended over all the neighbouring stars. Our first question concerns the probability

$$W(F) dF_x dF_y dF_z = W(F) dF,$$

that $F$ occurs in the range $F, F + dF$. By the application of a general method, due to Markoff, this probability distribution can be found. Thus measuring $F$ in the units

$$Q_H = (\frac{4}{3})^{2/3} 2\pi G (M^{3/2})^{2/3} N^{2/3} = 2.603 G (M^{3/2})^{2/3} N^{2/3},$$

where $N$ denotes the number of stars per unit volume, and writing

$$F = Q_H \beta$$

and $|F| = Q_H \beta$,

it can be shown that

$$W(\beta) = \frac{1}{2\pi^2 \beta^3} \int_0^\infty e^{-\left(\frac{1}{2}\beta \sigma \eta^2\right) x} \sin x \, dx.$$ 

The integral on the right-hand side of this equation has been evaluated numerically for various values of $\beta$, thus specifying the distribution function for $F$.

To answer the remaining questions concerning the statistics of the gravitational field we have to consider the joint distributions of $F$ and one or other of the following quantities:

$$\frac{dF}{dt} = G \sum_i M_i \left\{ \frac{\mathbf{V}_i}{|r_i|^3} - \frac{3}{|r_i|^5} \frac{\mathbf{r}_i (\mathbf{r}_i \cdot \mathbf{V}_i)}{|r_i|^3} \right\},$$

$$F(t) = G \sum_i M_i \frac{\mathbf{r}_i + \mathbf{V}_i t}{|r_i + \mathbf{V}_i t|^3},$$

and

$$F_1 = G \sum_i M_i \frac{\mathbf{r}_i - \mathbf{r}_1}{|r_i - \mathbf{r}_1|^3}.$$
In equations (48) and (49) $V_\i$ denotes the velocity of the field star relative to the one under consideration. It is possible to discuss these joint probability distributions and obtain the various first moments of $\mathbf{F}, \mathbf{F}(t)$ and $\mathbf{F}_\i$ for given $\mathbf{F}$ and $\mathbf{v}$. However, as the analysis is too involved to be summarized here we shall content ourselves with the consideration of one particular problem which appears to have some general interest. The problem we have in mind relates to the stability of wide binary stars.

As is well known, binary stars have a tendency towards disruption caused by the tidal effects of the nearby stars. Now this tendency toward disruption can be estimated as follows. For a given separation between the stars there exists a definite distribution function $W(F_0, F_\i)$ governing the probability that forces of intensities $F_0$ and $F_\i$, respectively, will act simultaneously on the two components of the binary. In other words, a differential acceleration governed by a definite probability law will operate on the system, which will tend to accelerate one star relative to the other. More particularly

$$\Delta F_0 = (F_0 - F_\i) \cdot \mathbf{l}_F,$$

where $\mathbf{l}_F$ is a unit vector parallel to the direction of $F_0$, represents the amount by which the star "o" will be systematically accelerated relative to the star "i". It is clear that it is this systematic acceleration of one component relative to the other which will be chiefly responsible for such changes in the orbital elements as may be effected. The average net increase in the velocity of the star "o" relative to the star "i" during a time $\Delta t$, long compared to the periods of the elementary fluctuations in $\mathbf{F}$, will therefore be given by

$$\Delta v_{0,1} = \Delta F_0 / \Delta t.$$  \hspace{1cm} (52)

The theory of the correlation of the forces acting simultaneously at two points separated by a finite distance, which we have outlined, enables the explicit evaluation of $\Delta F_0$. The formulae simplify considerably when we consider separations which are less than a fifth of the average distance between the stars. Under these conditions equation (52) reduces to

$$\Delta v_{0,1} = 4\pi GMNa / a,$$  \hspace{1cm} (53)

where $a$ denotes the separation. Since the mean square velocity of a star in its relative orbit is given by

$$V^2 = G(M_1 + M_2) / a,$$  \hspace{1cm} (54)

where $M_1$ and $M_2$ denote the masses of the two components and $a$ the semi-major axis of the orbit, we may define the time of dissolution, $\tau$, of a binary by the equation

$$\Delta v_{0,1} = (V^2)^{1/2},$$  \hspace{1cm} (55)

or using equation (53) we have

$$\tau \approx \frac{(M_1 + M_2)^{1/2}}{4\pi GMNa^{3/2}}.$$  \hspace{1cm} (56)

From this formula it appears that the time of dissolution of binaries with semi-major axes between $10^3$ to $10^4$ astronomical units lies in the range $7 \times 10^{10}$ to $2 \times 10^{9}$ years. And since, according to Ambarzumian, we can infer from the distribution of the semi-major axes of binaries that conditions of equipartition have not been attained in the galaxy, we may conclude that the time scale cannot exceed $3 \times 10^9$ years by any very large factor.

Bibliography:

IV. Lindblad's Theory of Spiral Structure in Nebulæ

As is well known, Lindblad's theory of spiral structure in nebulae derives from his theorem that circular orbits at the edges of highly flattened systems will be dynamically unstable. Thus, as he showed several years ago, circular orbits at the periphery of a homogeneous oblate spheroid will be unstable if the eccentricity of the spheroid is greater than 0.834 and that, moreover, stars describing such orbits will, on slight perturbations, depart from the edges along trajectories having a spiral character. There can be hardly any doubt that stars at the edges of more highly elliptical systems must be describing orbits near instability and that this instability must be of considerable significance for the future development of such systems. But it appears a difficult matter to trace in detail the various consequences which the instability we have described may lead to and, in particular, to decide as to whether the points at which material will depart along the spiral orbits will be at rest in space or at rest in a frame of reference rotating with the system. In this connection, Lindblad appeals to Bryan's theorem that Maclaurin spheroids first become unstable at an eccentricity $e = 0.953$ for sectorial harmonic waves which have two opposite maxima. But it is hard to visualize how theorems valid for rotating homogeneous incompressible fluids can be applied to stellar systems with times of relaxation which may be as high as $10^{14}$ years. However, Lindblad presents a variety of arguments to support his view that the points of ejection must be fixed in space and that in consequence the winding of the spiral arms must be such that they are described outward in the direction of rotation. In particular, Lindblad believes that his photometric studies of NGC 7331 in different colours, as well as the polarization measurements of Öhman in the Andromeda nebula, confirm his conclusions regarding the sense of rotation. But Hubble and Mayall, discussing similar (and sometimes the same) material, come to exactly the opposite conclusions. The matter clearly requires further investigations both on the theoretical and on the observational sides.

Bibliography.—A general account of Lindblad's theory, together with a fairly complete list of his writings up to 1941, will be found in


Later references are:


S. Chandrasekhar.