ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. VII

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ABSTRACT

In this paper the problem of the radiative equilibrium of a stellar atmosphere in local thermodynamic equilibrium is reconsidered with a view to examining in detail the effects of a continuous absorption coefficient, \( \kappa_\nu \), dependent on frequency. It is shown that when the material departs from grayness the corrections which have to be made to the temperature distribution are of two kinds: first, because the integrated Planck intensity \( B \) is not, in general, equal to the average integrated intensity \( J = \frac{\int J v I v dv}{\int I v dv} \) and, second, because the energy density of the radiation does not have the gray atmospheric value. To evaluate the corrections arising from these two sources, a systematic method of approximation has been developed in which the first approximation is assumed to be given by the solution for a gray atmosphere with a \( \kappa_\nu \) equal to a certain appropriately chosen mean absorption coefficient \( \bar{\kappa} \).

It is found that the best way of choosing \( \bar{\kappa} \) is to define it as a straight mean of \( \kappa_\nu \), weighted according to the net monochromatic flux \( F_\nu^{(1)}(\tau) \) of radiation of frequency \( \nu \) in a gray atmosphere:

\[
\bar{\kappa} = \int_0^\infty \kappa_\nu \frac{F_\nu^{(1)}}{F} \, d\nu
\]

Moreover, in a second approximation,

\[
B^{(2)} = J^{(2)} + \frac{1}{2} \int_0^\infty \left( \frac{\kappa_\nu}{\bar{\kappa}} - 1 \right) \frac{dF_\nu^{(1)}}{d\tau} \, d\nu,
\]

where \( J^{(2)} \) is the solution appropriate for \( J \) in this approximation. The solution for \( J^{(2)} \) has, in turn, been obtained in an \( n \)th approximation.

To facilitate the use of the solutions in the higher approximations obtained in this paper, the monochromatic fluxes \( F_\nu^{(1)}(\tau) \) and their derivatives \( dF_\nu^{(1)}/d\tau \) have been evaluated for a range of values of \( \tau \) and \( \varepsilon = h\nu/kT_e \) (\( T_e \) denoting the effective temperature).

1. Introduction.—The solution to the problem of radiative transfer in an atmosphere in local thermodynamic equilibrium is fundamental to all investigations which are in any way related to the continuous spectrum of the stars. More particularly, the basic problem is one of solving the equation of transfer

\[
\mu \frac{dI_\nu}{d\tau} = -\kappa_\nu I_\nu + \kappa_\nu B_\nu,
\]

under conditions of a constant net flux of radiation

\[
\pi F = \pi \int_0^\infty F_\nu dv = 2\pi \int_0^\infty \int_{-1}^{+1} I_\nu d\mu \, dv,
\]

in all the frequencies and where \( B_\nu(T) \) is the Planck function for the temperature \( T \) at \( x \). The complete solution to this problem depends only on the distribution of temperature in the atmosphere; for, once this is known, the source function \( B_\nu(T) \) for the radiation of frequency \( \nu \) is known at all points in the atmosphere and the determination of the intensity \( I_\nu \) at any point and in any given direction is immediate (see eq. [25] below). However, when the continuous absorption coefficient \( \kappa_\nu \) is allowed to be an arbitrary function of \( \nu \) and \( x \), the determination of the temperature distribution in the atmosphere is by no means a simple matter. Indeed, there exists at the present time no satisfactory attempt to solve the basic problem with any degree of generality. And what is already known in this connection\(^1\) can be summarized quite simply.

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From the equation of transfer (1) we readily obtain

$$\frac{dK_\nu}{pd\nu} = -\frac{1}{\kappa} \kappa F_\nu,$$

(2)

where, as usual, we have written

$$K_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu^2 d\mu.$$

(3)

Defining the mean absorption coefficient \( \bar{\kappa} \) by means of the relation

$$\frac{1}{\bar{\kappa}} \int_0^{\infty} \frac{dK_\nu}{pd\nu} d\nu = \int_0^{\infty} \frac{1}{\kappa_\nu} \frac{dK_\nu}{pd\nu} d\nu,$$

(4)

we can re-write equation (2) as an equation for integrated radiation in the form

$$\frac{dK}{d\tau} = \frac{1}{4} F,$$

(5)

where \( \tau \) denotes the optical depth in the mean absorption coefficient \( \bar{\kappa} \), measured from the boundary of the atmosphere normally inward. Remembering the constancy of the net integrated flux, we can integrate equation (5) to give

$$K = \frac{1}{4} F \tau + \text{constant}.$$

(6)

Using the conventional Milne-Eddington type of approximations, we readily derive from equation (6) the formula

$$J = \int_0^{\infty} J_\nu d\nu = \frac{1}{2} \int_0^{\infty} \int_{-1}^{+1} I_\nu \mu^2 d\nu = \frac{1}{2} F (1 + \frac{3}{2} \tau).$$

(7)

But we cannot, in general, pass directly from this equation for \( J \) to an equation for the integrated Planck intensity \( B \) (= \( \sigma T^4/\pi \)). For, according to the equation of transfer (1), the constancy of the net integrated flux implies only that

$$\int_0^{\infty} \kappa_\nu J_\nu d\nu = \int_0^{\infty} \kappa_\nu B_\nu d\nu;$$

(8)

and we cannot infer (except when the material is "gray") that \( J = B \). In other words, the formula for the temperature distribution which is in common use, namely,

$$T^4 = \frac{1}{2} T^4_\nu (1 + \frac{3}{2} \tau).$$

(9)

where \( \tau \) is the optical depth measured in the "Rosseland mean" absorption coefficient,

\(^2\)In practice this is defined by

$$\frac{1}{\bar{\kappa}} \int_0^{\infty} \frac{d\nu}{d\mu} d\nu = \int_0^{\infty} \frac{1}{\kappa_\nu} \frac{d\nu}{d\mu} d\nu.$$

But this is not the same as the mean absorption coefficient defined by equation (4). It would therefore seem that the use of the conventional Rosseland mean absorption coefficient in the theory of stellar atmospheres is of questionable value. On the other hand, an alternative form of equation (4), namely,

$$\bar{\kappa} = \frac{1}{F} \int_0^{\infty} \kappa_\nu F_\nu d\nu,$$

would appear to provide a more satisfactory way of defining \( \kappa \), as it has the further advantages of being a straight mean. Further remarks relating to this matter are made in § 5.
cannot, strictly speaking, be regarded as the solution of the fundamental equations in any well-defined scheme of approximation. Thus, while the use of equation (9) cannot be strictly defended, there does seem to be some evidence for the belief that the temperature distribution derived on the gray-body assumption provides a "first approximation" to the true distribution in some sense or another. For example, it is well known from the studies of Milne and Lindblad that the law of darkening in the different wave lengths over the solar disk and the intensity distribution in the continuous spectrum of the sun agree quite well with what can be predicted on the gray-body assumption. But attempts to improve on this agreement by allowing for a variation of $\kappa_\nu$ with $\nu$ on a "second approximation" generally indicate that the departures from grayness of the material is quite large. It would, accordingly, appear that the solution for the temperature distribution on the gray-body assumption does provide a good first approximation even for substantial variations of $\kappa_\nu$ with $\nu$. But, as to in what sense it is a first approximation can be understood only when we develop a systematic well-defined scheme of approximation and explicitly work out a second approximation. It is the object of this paper to provide such a scheme and to estimate the errors which are involved in the use of the solution (9).

2. An iteration method for solving the equation of transfer for an atmosphere in local thermodynamic equilibrium and with a continuous absorption coefficient depending on the wave length.—According to our remarks in the preceding section, it would appear that the temperature distribution in the atmosphere derived on the assumption of grayness of the material provides a first approximation. Assuming that this is the case, let $\kappa$ be a certain mean absorption coefficient (undefined for the present) such that with $\kappa_\nu = \kappa = \text{constant}$ we obtain a first approximation. Let $\tau$ denote the optical depth measured in terms of this absorption coefficient. Further, let

$$\delta_\nu = \frac{\kappa_\nu}{\kappa} - 1,$$

so that $\delta_\nu$ is a measure of the departure from grayness of the material. With these definitions the equation of transfer (1) takes the form

$$\mu \frac{d I_\nu}{d \tau} = I_\nu - B_\nu + \delta_\nu (I_\nu - B_\nu).$$

We suppose that this equation can be solved in two steps. First, we find the solution of the equation

$$\mu \frac{d I_{\nu}^{(1)}}{d \tau} = I_{\nu}^{(1)} - B_{\nu}^{(1)} \quad [B_{\nu}^{(1)} = B_{\nu} (T^{(1)})].$$

appropriate to the problem on hand, and use this solution in the term which occurs as the factor of $\delta_\nu$ in equation (11). Thus, the second approximation will be given by the solution of

$$\mu \frac{d I_{\nu}^{(2)}}{d \tau} = I_{\nu}^{(2)} - B_{\nu}^{(2)} + \delta_\nu \frac{d I_{\nu}^{(1)}}{d \tau}.$$

Formally, there is, of course, no difficulty in extending this method of iteration to obtain solutions to as high an approximation as may be needed. Thus, in the $n$th approximation we shall have

$$\mu \frac{d I_{\nu}^{(n)}}{d \tau} = I_{\nu}^{(n)} - B_{\nu}^{(n)} + \delta_\nu [I_{\nu}^{(n-1)} - B_{\nu}^{(n-1)}].$$

This method of iteration will obviously converge if $\delta_0$ is sufficiently small. But the point to which attention may be drawn at this stage is that in practice the success of this method is not impaired even when $\delta_0$ takes moderately large values of the order of 2 or 3 (see § 5 below).

In this paper we shall consider only the first two approximations.

In their integrated forms, equations (12) and (13) are

$$\mu \frac{dI^{(1)}}{d\tau} = I^{(1)} - B^{(1)}$$

and

$$\mu \frac{dI^{(2)}}{d\tau} = I^{(2)} - B^{(2)} + \mu \int_0^\infty \delta_\nu \frac{dI^{(1)}}{d\tau} d\nu.$$  \hspace{1cm} (16)

The foregoing equations have, of course, to be solved under the conditions of a constant net integrated flux. In the first two approximations this condition requires, respectively, that

$$B^{(1)} = J^{(1)}$$

and

$$B^{(2)} = J^{(2)} + \frac{1}{2} \int_0^\infty \int_{-1}^{+1} \delta_\nu \frac{dI^{(1)}}{d\tau} \mu d\mu d\nu.$$  \hspace{1cm} (17)

Equation (18) can be re-written in the form

$$B^{(2)} = J^{(2)} + \frac{1}{4} \int_0^\infty \delta_\nu \frac{dF^{(1)}}{d\tau} d\nu.$$  \hspace{1cm} (18)

In other words, in the second approximation the integrated Planck intensity $B$ differs from $J$ by an amount which depends on the nonconstancy of the monochromatic fluxes $F_\nu$. From general considerations we may expect (and this is confirmed by the calculations to be presented in § 5) that, unless $\delta_\nu$ varies too widely over the relevant ranges of $\nu$, the departures from constancy of the monochromatic fluxes $F_\nu$ will be of the second order of smallness. It is precisely on this account that the temperature distribution derived on the assumption of grayness of the material is as satisfactory as it has been found to be in practice.

We now proceed to the solutions of equations (15) and (16).

3. The solution in the $(1, n)$ approximation.—Equations (15) and (17) together reduce to an equation of transfer of a standard type which has been treated in sufficient detail in earlier papers of this series.\(^4\) Adopting in particular the scheme of approximation developed in paper II we can, in the $n$th approximation, write (II, eqs. [19] and [26])

$$I^{(1)} = \frac{3}{2} F \left\{ \sum_{a=1}^{n-1} \frac{L^{(1)}_a}{1 + \mu_a k_a} e^{-k_a \tau} + \mu + Q^{(1)} + \tau \right\} (i = \pm 1, \ldots, \pm n).$$  \hspace{1cm} (20)

where the $n$ constants $L^{(1)}_a$, ($a = 1, \ldots, n - 1$), and $Q^{(1)}$ are determined by the equations (II, eq. [21])

$$\sum_{a=1}^{n-1} \frac{L^{(1)}_a}{1 - \mu_a k_a} + Q^{(1)} - \mu_i = 0 \quad (i = 1, \ldots, n).$$  \hspace{1cm} (21)

\(^4\) Ap. J., 99, 180, 1944, and 100, 76, 1944. These papers will be referred to as "I" and "II," respectively.
and the \(k_a\)'s are the \(n - 1\) distinct nonzero positive roots of the characteristic equation (II, eq. [10])

\[
2 = \sum_{i=1}^{n-1} \frac{a_i}{1 + \mu_i k_a}.
\]  

(22) 

On this approximation the solution for \(J^{(1)}\) is (II, eq. [29])

\[
J^{(1)} = B^{(1)} = \frac{1}{4} F (\tau + q [\tau]) = \frac{1}{4} F \left(\tau + Q^{(1)} + \sum_{a=1}^{n-1} T_a^{(1)} e^{-k_a \tau}\right).
\]  

(23) 

The corresponding formula giving the temperature distribution is

\[
(T^{(1)})^4 = \frac{8}{n} T_\epsilon^4 \left(\tau + Q^{(1)} + \sum_{a=1}^{n-1} T_a^{(1)} e^{-k_a \tau}\right).
\]  

(24) 

Corresponding to the distribution of temperature (24), there is a determinate distribution of the intensities at various frequencies and at various levels. Thus, the intensity of the radiation of frequency \(\nu\) at an optical depth \(\tau\) and in a direction making an angle \(\theta\) with the positive normal is given by

\[
I^{(1)}_\nu(\tau, \theta) = \int_{\tau}^{\infty} e^{-(t-\tau)} \sec \theta B_v(T_\nu^{(1)}) \sec \theta dt \left(0 < \theta < \pi / 2\right)
\]

\[
- \int_{\tau}^{\infty} e^{-(t-\tau)} \sec \theta B_v(T_\nu^{(1)}) \sec \theta dt \left(\pi / 2 < \theta < \pi\right).
\]  

(25) 

In terms of this solution for \(I^{(1)}_\nu(\tau, \theta)\) we can readily derive explicit formulae for quantities such as \(J, F,\) etc. In fact, it can be shown quite generally that

\[
\int_{-1}^{+1} I^{(1)}_\nu(\tau, \mu) \mu^j d\mu = \int_{-1}^{+1} B_v(T_\nu^{(1)}) E_{j+1}(t - \tau) dt
\]

\[
+ (-1)^j \int_{-1}^{+1} B_v(T_\nu^{(1)}) E_{j+1}(\tau - t) dt.
\]  

(26) 

where \(E_{j+1}(x)\) stands for the exponential integral

\[
E_{j+1}(x) = \int_{1}^{\infty} \frac{e^{-xw}}{w^{j+1}} dw.
\]  

(27) 

In our further work we shall find that we also need expressions for the quantities

\[
\int_{-1}^{+1} \frac{dI^{(1)}_\nu}{d\tau} \mu^i d\mu.
\]  

(28) 

From the equation of transfer (15) we find that

\[
\int_{-1}^{+1} \frac{dI^{(1)}_\nu}{d\tau} \mu^i d\mu = \int_{1}^{+1} I^{(1)}_\nu(\tau, \mu) \mu^i d\mu - \frac{2}{j} \epsilon_{j, \text{odd}} B_v(T_\nu^{(1)}),
\]  

(29) 

where

\[
\epsilon_{j, \text{odd}} = 1 \quad \text{if } j \text{ is odd}
\]

\[
0 \quad \text{otherwise.}
\]  

(30) 

* In the summation on the right-hand side of this equation there is no term with \(i = 0\).
Using equation (26), we can re-write equation (29) as

\[
\int_{-1}^{+1} \frac{dI_{\tau}^{(1)}}{d\tau} \mu d\mu = \int_{\tau}^{\infty} B_{\tau} (T_{\tau}^{(1)}) E_{j} (t - \tau) dt
\]

\[
+ ( - 1 ) i^{-1} \int_{0}^{\infty} B_{\tau} (T_{\tau}^{(1)}) E_{j} (\tau - t) dt - \frac{2}{j} \epsilon_{ij} \text{odd} B_{\tau} (T_{\tau}^{(1)}) .
\]

(Eq. 31)

Equations (24), (25), (26), and (31) may be said to represent the solution to our problem in the \((1, n)\) approximation referring to the fact that the equations of the first approximation have in turn been solved in an \(n\)th approximation.

4. The solution of the equations in the \((2, n)\) approximation.—Turning next to the solution of the equations (16) and (18), we have to consider the equation

\[
\mu \frac{dI_{\tau}^{(2)}}{d\tau} = I_{\tau}^{(2)} - \frac{1}{2} \int_{-1}^{+1} I_{\tau}^{(2)} \mu d\mu + \mu \int_{\tau}^{\infty} \delta_{\tau} \frac{dI_{\mu}^{(1)}}{d\tau} d\mu - \frac{1}{2} \int_{0}^{\infty} \int_{-1}^{+1} \delta_{\mu} \frac{dI_{\tau}^{(1)}}{d\tau} d\mu d\nu .
\]

(Eq. 32)

In solving this integrodifferential equation we shall follow the methods developed in the earlier papers of this series and replace the integrals which occur on the right-hand side of this equation by sums according to Gauss's formula for numerical quadratures. Thus, in the \(n\)th approximation, the equivalent systems of linear equations are

\[
\mu \frac{dI_{\tau}^{(2)}}{d\tau} = I_{\tau}^{(2)} - \frac{1}{2} \sum_{i \neq j} \mu_{i} \int_{\tau}^{\infty} \delta_{\tau} \frac{dI_{\mu}^{(1)}}{d\tau} d\mu
\]

\[
- \frac{1}{2} \int_{0}^{\infty} \delta_{\mu} \sum_{i \neq j} \mu_{i} \frac{dI_{\tau}^{(1)}}{d\tau} d\mu (i = \pm 1, \ldots, \pm n) ,
\]

(Eq. 33)

where we have used \(I_{\tau}^{(2)}\) and \(I_{\mu}^{(1)}\) to denote, respectively, \(I_{\tau}^{(2)}(\tau, \mu)\) and \(I_{\mu}^{(1)}(\tau, \mu)\) and the rest of the symbols have the same meanings as in paper II.

The system of equations represented by (33) is most conveniently solved by the method of the variation of the parameters. Thus, since the homogeneous part of the system of equations (33) is of the same form as that considered in II, equation (6), we seek a solution of our present system of the form (cf. II, eqs. [18] and [26])

\[
I_{\tau}^{(2)} = \frac{2}{(n - 1) \mu_{\tau}^{(2)}(\tau)} \sum_{a=1}^{n-1} \frac{e^{-k_{a} \tau}}{1 + \mu_{i} \kappa_{a}} L_{a}^{(2)} (\tau)
\]

\[
+ \frac{1}{(n - 1) \mu_{\tau}^{(2)}(\tau)} \sum_{a=1}^{n-1} \frac{e^{+k_{a} \tau}}{1 - \mu_{i} \kappa_{a}} L_{a}^{(2)} (\tau) + \mu_{i} + Q_{a}^{(2)} (\tau) + \tau (i = \pm 1, \ldots, + n) ,
\]

(Eq. 34)

where, as we have indicated, \(I_{\tau}^{(2)}, L_{a}^{(2)} (a = 1, \ldots, n - 1),\) and \(Q^{(2)}\) are all to be considered as functions of \(\tau\). It will be noticed that in writing the solution in this form we have treated as variable only \((2n - 1)\) of the \(2n\) constants of integration which the general solution of the homogeneous system associated with equation (33) involves. This is, however, permissible in view of the fact that equation (33) admits the flux integral

\[
\Sigma \mu_{i} i_{\tau}^{(2)} = \text{constant} ,
\]

(Eq. 35)

and we can arrange that this constant of integration has the same value as in the \((1, n)\) approximation corresponding to the circumstance of a given constant net integrated flux.
Substituting for $I^{(2)}_{1}$ from equation (34) in equation (33), we obtain the variational equation

$$
\frac{1}{2} F \mu_i \left\{ \sum_{s=1}^{n-1} \frac{e^{-k_s r}}{1 + \mu_s k_s} \frac{dL^{(2)}_{s}}{d\tau} + \sum_{s=1}^{n-1} \frac{e^{+k_s r}}{1 - \mu_s k_s} \frac{dL^{(2)}_{s}}{d\tau} \right\} = \mu_i \int_{0}^{\infty} \delta_{\nu} \frac{dI_{r_{1}}^{(1)}}{d\tau} dv - \frac{1}{2} \int_{0}^{\infty} \delta_{\nu} \Sigma a_i \mu_i \frac{dI_{r_{2}}^{(1)}}{d\tau} dv \quad (i = \pm 1, \ldots, \pm n). \tag{36}
$$

Of the $2n$ equations represented in the foregoing equation, only $2n - 1$ are linearly independent, since the equation derived from (36) by multiplying with $a_i$ and summing over all $i$'s is identically satisfied (cf. II, eq. [25]). The rank of the systems (36) is, accordingly, $(2n - 1)$—in agreement with the fact that we have only $(2n - 1)$ functions $L^{(2)}_{a_{1}}, L^{(2)}_{a_{2}}, (a = 1, \ldots, n - 1)$, and $Q^{(2)}$ to determine.

The order of the system of equations (36) can be further reduced to $(2n - 2)$ by a proper averaging of the continuous absorption coefficient $k_{\nu}$ to yield a $\bar{k}$ to be used in the first gray-body approximation. Thus, multiplying equation (36) by $a_i \mu_i$ and summing over all $i$'s, we obtain

$$
\frac{1}{2} F \frac{dQ^{(2)}}{d\tau} = \int_{0}^{\infty} \delta_{\nu} \Sigma a_i \mu_i \frac{2 dI_{r_{1}}^{(1)}}{d\tau} dv, \tag{37}
$$

use having been made of the relations (II, eq. [24] and eq. [58] below)

$$
\sum_{i} a_i \mu_i^2 = \frac{2}{3} \quad \text{and} \quad \sum_{i} \frac{a_i \mu_i^2}{1 + \mu_i k_a} = 0. \tag{38}
$$

Accordingly, if we arrange that

$$
\int_{0}^{\infty} \delta_{\nu} \Sigma a_i \mu_i \frac{2 dI_{r_{1}}^{(1)}}{d\tau} dv = 2 \int_{0}^{\infty} \delta_{\nu} \frac{dK^{(1)}_{\nu}}{d\tau} dv = 0, \tag{39}
$$

we shall have the integral

$$
Q^{(2)} = \text{constant} = Q^{(1)} + \Delta Q \quad \text{(say).} \tag{40}
$$

But equation (39) implies that the mean absorption coefficient $\bar{k}$ has to be defined according to

$$
\bar{k} \int_{0}^{\infty} \frac{dK^{(1)}_{\nu}}{d\tau} dv = \int_{0}^{\infty} \kappa_{\nu} \frac{dK^{(1)}_{\nu}}{d\tau} dv, \tag{41}
$$

or, alternatively (cf. eqs. [2] and [5]),

$$
\bar{k} = \frac{1}{F} \int_{0}^{\infty} \kappa_{\nu} F^{(1)}_{\nu} dv; \tag{42}
$$

for with this choice of $\bar{k}$ the departures of $\kappa_{\nu}$ from $\bar{k}$, when similarly averaged, will be zero:

$$
\int_{0}^{\infty} \delta_{\nu} \frac{dK^{(1)}_{\nu}}{d\tau} dv = \frac{1}{4} \int_{0}^{\infty} \delta_{\nu} F^{(1)}_{\nu} dv = 0. \tag{43}
$$

Attention may be drawn here to a consequence of the foregoing method of averaging $\kappa_{\nu}$ for the important special case when $\delta_{\nu}$ is independent of $\tau$, i.e., when $\kappa_{\nu}/\bar{k}$ is independent of depth. In this case we can re-write equation (19) in the form

$$
B^{(2)} = I^{(2)} + \frac{1}{4} \frac{d}{d\tau} \int_{0}^{\infty} \delta_{\nu} F^{(1)}_{\nu} dv, \tag{44}
$$
and equation (43) now implies that
\[ B^{(2)} = J^{(2)} \quad (\kappa_r / \bar{\kappa} \text{ independent of } \tau). \] (45)

With \( \bar{\kappa} \) defined as in equation (42), the variational equations become
\[
\frac{3}{2} F \mu_i \left\{ \sum_{a=1}^{n-1} \frac{e^{-k_a \tau}}{1 + \mu_ik_a} \frac{dL_{(a)}^{(2)}}{d\tau} + \sum_{a=1}^{n-1} \frac{e^{+k_a \tau}}{1 - \mu_ik_a} \frac{dL_{(a)}^{(2)}}{d\tau} \right\}
= \mu_i \int_0^\infty \delta_\nu \frac{dI_{r,i}^{(1)}}{d\tau} \, dv - \frac{1}{2} \int_0^\infty \delta_\nu \sum_a \mu_j \frac{dI_{r,i}^{(1)}}{d\tau} \, dv \quad (i = \pm 1, \ldots, \pm n). \] (46)

Though \( 2n \) equations are represented in the foregoing equation, only \( 2n - 2 \) of these are linearly independent, corresponding to the \( (2n - 2) \) functions \( L_{a}^{(2)} \) and \( L_{-a}^{(2)} \), \( (a = 1, \ldots, n - 1) \), which are to be determined.

In view of that fact that the rank of the system (46) is less than the number of equations, it appears that the most symmetrical way of treating the variational equations is the following:

Multiply equation (46) by \( a_i \mu_i^{m-1} \), \( (m = 1, \ldots, 2n) \), and sum over all \( i \)'s. We obtain
\[
\frac{3}{2} F \left\{ \sum_{a=1}^{n-1} D_{m,a} \frac{e^{-k_a \tau}}{1 + \mu_ik_a} \frac{dL_{(a)}^{(2)}}{d\tau} + (-1)^m \sum_{a=1}^{n-1} D_{m,a} \frac{e^{+k_a \tau}}{1 - \mu_ik_a} \frac{dL_{(a)}^{(2)}}{d\tau} \right\}
= \delta_m - \frac{\epsilon_m, \text{ odd}}{m} \bar{\kappa}_m \quad (m = 1, \ldots, 2n), \] (47)

where we have written
\[ D_{m,a} = \sum_i a_i \mu_i^m \frac{1}{1 + \mu_ik_a} = (-1)^m \sum_i a_i \mu_i^m \frac{1}{1 - \mu_ik_a} \] (48)

and
\[ \delta_m = \int_0^\infty \delta_\nu \sum_i a_i \mu_i^m \frac{dI_{r,i}^{(1)}}{d\tau} \, dv \] (49)

and where \( \epsilon_m, \text{ odd} \) has the same meaning as in equation (30). It may be noted that in deriving equation (47) use has been made of the relation
\[ \sum_i a_i \mu_i^m = \frac{2}{m} \epsilon_m, \text{ odd} \quad (m = 1, \ldots, 4n). \] (50)

There is a simple recursion formula which \( D_{m,a} \) satisfies and which enables a direct evaluation of this quantity. We have
\[ D_{m,a} = \frac{1}{k_a} \sum_i a_i \mu_i^m \left( 1 - \frac{1}{1 + \mu_ik_a} \right), \] (51)

or, using equation (50),
\[ D_{m,a} = \frac{1}{k_a} \left( \frac{2}{m} \epsilon_m, \text{ odd} - D_{m-1,a} \right) \quad (m = 1, \ldots, 4n), \] (52)

which is the required recursion formula. For odd, respectively even, values of \( m \), the formula takes the forms
\[ D_{2j-1,a} = \frac{1}{k_a} \left( \frac{2}{2j - 1} - D_{2j-2,a} \right) \] (53)
Combining the relations (53) and (54), we have

\[
D_{2j-1, a} = \frac{1}{k_a} \left( \frac{2}{2j-1} + \frac{1}{k_a} D_{2j-3, a} \right)
\]  \hspace{1cm} (55)

and

\[
D_{2j, a} = -\frac{1}{k_a^2} \left( \frac{2}{2j-1} - D_{2j-2, a} \right).
\]  \hspace{1cm} (56)

On the other hand, in our present notation the equation for the characteristic roots \( k_a \) can be written as (cf. eq. [22])

\[
D_{0, a} = 2 \quad (a = 1, \ldots, n-1).
\]  \hspace{1cm} (57)

From the recursion formula (52) we now conclude that

\[
D_{2, a} = -\frac{1}{k_a} D_{1, a} = -\frac{1}{k_a^2} (2 - D_0, a) = 0
\]  \hspace{1cm} (58)

and

\[
D_{3, a} = \frac{2}{3k_a} \quad \text{and} \quad D_{4, a} = -\frac{2}{3k_a^2}.
\]  \hspace{1cm} (59)

The formula (55) will now enable us to determine \( D_{m, a} \) successively for all odd values of \( m \) greater than 3; similarly, the expressions for the even values of \( m \) follow from equation (56). In this manner we find that

\[
D_{2j-1, a} = -\frac{2}{(2j-1)k_a} + \frac{2}{(2j-3)k_a^3} + \ldots + \frac{2}{3k_a^{2j-3}} \quad (j = 2, \ldots, n).
\]  \hspace{1cm} (60)

and

\[
D_{2j, a} = -\frac{2}{(2j-1)k_a^2} - \frac{2}{(2j-3)k_a^4} - \ldots - \frac{2}{3k_a^{2j-2}} \quad (j = 2, \ldots, n).
\]  \hspace{1cm} (61)

In terms of these \( D_{2j, a} \)'s it is possible to write down an alternative form of the characteristic equation which, in contrast to equation (22), does not require an explicit knowledge of the Gaussian weights and divisions. For, if \( p_{2j, i} \) \((j = 0, \ldots, n)\), are the coefficients of the polynomial \( P_{2n}(\mu) \), so that

\[
P_{2n}(\mu) = \sum_{j=0}^{n} p_{2j}\mu^j,
\]

it is evident that

\[
\sum_{j=0}^{n} p_{2j}D_{2j, a} = \sum_{j=0}^{n} \frac{a_i}{1 + \mu_i k_a} \left( \sum_{j=0}^{n} p_{2j} \mu_i^j \right) = 0 \quad (a = 1, \ldots, n-1),
\]

since the \( \mu_i \)'s are, by definition, the zeros of \( P_{2n}(\mu) \). Hence, with the definitions

\[
-D_{2j} = \frac{2}{(2j-1)k_a^2} + \frac{2}{(2j-3)k_a^4} + \ldots + \frac{2}{3k_a^{2j-2}} \quad (j = 2, \ldots, n)
\]

and

\[
D_0 = 2 \quad \text{and} \quad D_2 = 0,
\]

the equation for the characteristic roots takes the form

\[
\sum_{j=0}^{n} p_{2j}D_{2j} = 0.
\]
Returning to equation (47), we first observe that, since $D_1$, $D_2$, and $\delta_2$ are all zero (cf. eqs. [39] and [49]), then, of the $2n$ equations represented by equation (47), those for $m = 1$ and 2 are identically satisfied. Accordingly, we need consider equation (47) only for values of $m = 3, \ldots, 2n$. For odd, respectively even, values of $m$, equation (47) takes the forms

$$
\frac{3}{4} F \sum_{a=1}^{n-1} D_{2j-1, a} \left( e^{-k_a} \frac{dL_a^{(2)}}{d\tau} - e^{+k_a} \frac{dL_a^{(2)}}{d\tau} \right) = \delta_{2j-1} - \frac{1}{2j-1} \delta_1 \quad (j = 2, \ldots, n) \quad (62)
$$

and

$$
\frac{3}{4} F \sum_{a=1}^{n-1} D_{2j, a} \left( e^{-k_a} \frac{dL_a^{(2)}}{d\tau} + e^{+k_a} \frac{dL_a^{(2)}}{d\tau} \right) = \delta_{2j} \quad (j = 2, \ldots, n) \quad (63)
$$

or, substituting for $D_{2j-1, a}$ and $D_{2j, a}$ according to equations (60) and (61), we have

$$
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a^{2j-3}} + \frac{1}{5k_a^{2j-5}} + \ldots + \frac{1}{(2j-1) k_a} \right\} X_a = \delta_{2j-1} - \frac{1}{2j-1} \delta_1 \quad (j = 2, \ldots, n) \quad (64)
$$

and

$$
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a^{2j-2}} + \frac{1}{5k_a^{2j-4}} + \ldots + \frac{1}{(2j-1) k_a^2} \right\} Y_a = -\delta_{2j} \quad (j = 2, \ldots, n) \quad (65)
$$

where, for the sake of brevity, we have written

$$
X_a = F \left( e^{-k_a} \frac{dL_a^{(2)}}{d\tau} - e^{+k_a} \frac{dL_a^{(2)}}{d\tau} \right),
$$

$$
Y_a = F \left( e^{-k_a} \frac{dL_a^{(2)}}{d\tau} + e^{+k_a} \frac{dL_a^{(2)}}{d\tau} \right).
$$

The linear systems represented by equations (64) and (65) can be brought to forms more convenient for their solutions by the following procedure.

Considering, for example, equation (64), we find that the system of equations which this represents is

$$
\begin{align*}
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a} \right\} X_a &= \delta_3 - \frac{1}{3} \delta_1, \\
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a} + \frac{1}{5k_a} \right\} X_a &= \delta_5 - \frac{1}{5} \delta_1, \\
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a} + \frac{1}{5k_a} + \frac{1}{7k_a} \right\} X_a &= \delta_7 - \frac{1}{7} \delta_1, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \\
\frac{3}{2} \sum_{a=1}^{n-1} \left\{ \frac{1}{3k_a} + \frac{1}{5k_a} + \ldots + \frac{1}{2n-1} k_a \right\} X_a &= \delta_{2n-1} - \frac{1}{2n-1} \delta_1.
\end{align*}
$$

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The form of these equations suggests that we construct the following simpler system of equations:

\[
\frac{1}{2} \sum_{a=1}^{n-1} \frac{1}{k_a^j} \delta_a = \delta_3 - \frac{1}{3} \delta_1 = U_1,
\]

\[
\frac{1}{2} \sum_{a=1}^{n-1} \frac{1}{k_a^3} \delta_a = \delta_5 - \frac{1}{3} \delta_1 - \frac{3}{8} U_1 = U_2,
\]

\[
\frac{1}{2} \sum_{a=1}^{n-1} \frac{1}{k_a^5} \delta_a = \delta_7 - \frac{1}{3} \delta_1 - \frac{3}{8} U_2 - \frac{3}{8} U_1 = U_3,
\]

\[
\frac{1}{2} \sum_{a=1}^{n-1} \frac{1}{k_a^{2j-3}} \delta_a = \delta_{2j-1} - \frac{1}{2j-1} \delta_1 - \frac{3}{8} U_{j-2} - \frac{3}{8} U_{j-3} - \ldots - \frac{3}{2j-1} U_1 = U_{j-1},
\]

\[
\frac{1}{2} \sum_{a=1}^{n-1} \frac{1}{k_a^{2n-3}} \delta_a = \delta_{2n-1} - \frac{1}{2n-1} \delta_1 - \frac{3}{8} U_{n-2} - \frac{3}{8} U_{n-3} - \ldots - \frac{3}{2n-1} U_1 = U_{n-1}.
\]

In other words, we can reduce the systems of equations (67) to the simpler one

\[
\sum_{a=1}^{n-1} \frac{1}{k_a^{2j-1}} \delta_a = 2 U_j \quad (j = 1, \ldots, n - 1),
\]

where the \(U_j\)'s are defined as in equation (68).

Similarly, starting from equation (65), we can construct the simpler system

\[
\sum_{a=1}^{n-1} \frac{1}{k_a^{2j}} \delta_a = -2 V_j \quad (j = 1, \ldots, n - 1),
\]

where the quantities \(V_j\) \((j = 1, \ldots, n - 1)\) are defined by the recurrence relation

\[
V_{j-1} = \delta_{2j} - \frac{3}{8} V_{j-2} - \frac{3}{8} V_{j-3} - \ldots - \frac{3}{2j-1} V_1 \quad (j = 3, \ldots, n)
\]

and

\[
V_1 = \delta_4.
\]

We shall now show how it is possible to write down the solution of the equations (69) and (70) quite generally.

Considering first equation (69), we find that the system of equations which this represents can be written down as a single vector equation

\[
KX = 2U,
\]

where \(X\) and \(U\) stand for the vectors

\[
X = (X_1, X_2, \ldots, X_{n-1}) \quad \text{and} \quad U = (U_1, \ldots, U_{n-1})
\]
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and $\mathbf{K}$ for the matrix:

$$
\mathbf{K} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
\frac{1}{k_1} & \frac{1}{k_2} & \frac{1}{k_3} & \ldots & \frac{1}{k_{n-1}} \\
\frac{1}{k_1^3} & \frac{1}{k_2^3} & \frac{1}{k_3^3} & \ldots & \frac{1}{k_{n-1}^3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{k_1^{2n-3}} & \frac{1}{k_2^{2n-3}} & \frac{1}{k_3^{2n-3}} & \ldots & \frac{1}{k_{n-1}^{2n-3}}
\end{pmatrix}.
$$

Equation (73) expresses a relation between $U$ and a linear transform of $X$. This relation can be inverted into a relation between $X$ and the linear transform of $U$ obtained with the inverse of the matrix $\mathbf{K}$.\(^\dagger\) Thus

$$
X = 2 \mathbf{K}^{-1} U,
$$

where $\mathbf{K}^{-1}$ denotes the inverse of $\mathbf{K}$. For the particular matrix $\mathbf{K}$ the inverse can be explicitly written down. But before we do this, it is first necessary to introduce certain definitions.

Consider the expansion of the product

$$
\prod_{j \neq r}^{1 \ldots n-1} (x^2 - k_j^2) = (x^2 - k_1^2) (x^2 - k_2^2) \ldots (x^2 - k_{r-1}^2) (x^2 - k_{r+1}^2) \ldots (x^2 - k_{n-1}^2)
$$

in descending powers of $x^2$. We have

$$
\prod_{j \neq r}^{1 \ldots n-1} (x^2 - k_j^2) = x^{2n-4} - x^{2n-6} \left( \sum_{\mu_1 \neq r}^{1 \ldots n-1} k_{\mu_1}^2 \right) + x^{2n-8} \left( \sum_{\mu_1 \neq \mu_2 \neq r}^{1 \ldots n-1} k_{\mu_1}^2 k_{\mu_2}^2 \right) \\
- x^{2n-10} \left( \sum_{\mu_1 \neq \mu_2 \neq \mu_3 \neq r}^{1 \ldots n-1} k_{\mu_1}^2 k_{\mu_2}^2 k_{\mu_3}^2 \right) + \ldots + (-1)^{n-2} \prod_{\mu \neq r}^{1 \ldots n-1} k_{\mu}^2\tag{77}
$$

or, introducing the $(n - 1)$ independent symmetric functions in the $(n - 2)$ variables $k_{\mu}^2$, ($\mu = 1, \ldots, r - 1, r + 1, \ldots n - 1$),

$$
S_{0, r} = 1,
$$

$$
S_{1, r} = - \sum_{\mu_1 \neq r}^{1 \ldots n-1} k_{\mu_1}^2,
$$

$$
S_{2, r} = + \sum_{\mu_1 \neq \mu_2 \neq r}^{1 \ldots n-1} k_{\mu_1}^2 k_{\mu_2}^2,
$$

$$
\ldots \ldots \ldots
$$

$$
S_{i, r} = (-1)^i \sum_{\mu_1 \neq \mu_2 \neq \ldots \neq \mu_i \neq r}^{1 \ldots n-1} k_{\mu_1}^2 k_{\mu_2}^2 \ldots k_{\mu_i}^2,
$$

$$
\ldots \ldots \ldots
$$

$$
S_{n-2, r} = (-1)^{n-2} k_1^2 k_2^2 \ldots k_{r-1}^2 k_{r+1}^2 \ldots k_{n-1}^2,
$$

\(^\dagger\) The existence of the inverse can be inferred from the linear independence of the equations (67), or more directly from the nonvanishing of the determinant of $\mathbf{K}$. The determinant of $\mathbf{K}$ can be written...
we can re-write the expansion on the right-hand side of equation (78) in the form

$$\prod_{j \neq r}^{1n-1}(x^2 - k_j^2) = \sum_{\lambda=0}^{n-2} x^{2n-2\lambda-4}S_{\lambda, r}. \quad (80)$$

In terms of the functions $S_{\lambda, r}, (\lambda = 0, 1, \ldots, n - 2,$ and $r = 1, \ldots, n - 1), defined in this manner, the inverse of the matrix $K$ can be written down. We have

$$K^{-1} = \begin{bmatrix}
\prod_{j \neq 1}^{2n-3} (k_1^2 - k_j^2) & \prod_{j \neq 1}^{2n-3} (k_1^2 - k_j^2) & \cdots & \prod_{j \neq 1}^{2n-3} (k_1^2 - k_j^2) & \prod_{j \neq 1}^{2n-3} (k_1^2 - k_j^2) \\
\prod_{j \neq 2}^{2n-3} (k_2^2 - k_j^2) & \prod_{j \neq 2}^{2n-3} (k_2^2 - k_j^2) & \cdots & \prod_{j \neq 2}^{2n-3} (k_2^2 - k_j^2) & \prod_{j \neq 2}^{2n-3} (k_2^2 - k_j^2) \\
\prod_{j \neq a}^{2n-3} (k_a^2 - k_j^2) & \prod_{j \neq a}^{2n-3} (k_a^2 - k_j^2) & \cdots & \prod_{j \neq a}^{2n-3} (k_a^2 - k_j^2) & \prod_{j \neq a}^{2n-3} (k_a^2 - k_j^2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\prod_{j \neq n-1}^{2n-3} (k_{n-1}^2 - k_j^2) & \prod_{j \neq n-1}^{2n-3} (k_{n-1}^2 - k_j^2) & \cdots & \prod_{j \neq n-1}^{2n-3} (k_{n-1}^2 - k_j^2) & \prod_{j \neq n-1}^{2n-3} (k_{n-1}^2 - k_j^2)
\end{bmatrix}. \quad (8)$$

That the foregoing matrix does, in fact, define the inverse of the matrix $K$ can be verified directly. Thus,

$$\begin{align*}
(K^{-1}K)_{i, m} &= \sum_{\lambda=0}^{n-2} \frac{k_1^{2n-3}S_{\lambda, i} 1}{\prod_{j \neq i}^{2n-3} (k_j^2 - k_i^2)} \\
&= \left(\frac{k_l^2}{k_m^2}\right)^{2n-3} \frac{1}{\prod_{j \neq i}^{2n-3} (k_j^2 - k_i^2)} \sum_{\lambda=0}^{n-2} \frac{k_1^{2n-2\lambda-4}S_{\lambda, i}}{}
\end{align*} \quad (82)$$

or, according to equation (80),

$$\begin{align*}
(K^{-1}K)_{i, m} &= \left(\frac{k_l^2}{k_m^2}\right)^{2n-3} \frac{1}{\prod_{j \neq i}^{2n-3} (k_j^2 - k_i^2)} \prod_{j \neq i}^{2n-3} (k_j^2 - k_i^2).
\end{align*} \quad (83)$$

down as it is related to the well-known Vandermonde determinant (cf. O. Perron, Algebra, Vol. 1, § 22, pp. 92–94, Leipzig: Gruyter, 1932). We have

$$|K| = \frac{1}{k_1k_2 \ldots k_{n-1}} \prod_{i>j}^{1n-1} \left(\frac{1}{k_i^2} - \frac{1}{k_j^2}\right). \quad (75)$$
If \( l \neq m \), the product which occurs in the numerator of the right-hand side of equation (83) includes the term \( j = m \) and consequently vanishes. On the other hand, if \( l = m \), the right-hand side of equation (83) clearly reduces to 1. Hence,

\[
(K^{-1}K)_{l,m} = 1 \text{ if } l = m \text{ and zero otherwise ,}
\]

proving the inverse relationship between the matrices (75) and (81).

Hence, with \( K^{-1} \) as defined in equation (81), equation (76) determines \( X \). In terms of the components \( X_a \) of \( X \) this equation becomes

\[
X_a = F\left( e^{-k_a r} \frac{dL_a^{(2)}}{d\tau} - e^{+k_a r} \frac{dL_a^{(1)}}{d\tau} \right) + \frac{2k_a^{2n-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} U_{\lambda+1}(\tau) \tag{85}
\]

\[ (a = 1, \ldots, n - 1) . \]

Similarly, the solution of equation (70) is found to be

\[
Y_a = F\left( e^{-k_a r} \frac{dL_a^{(2)}}{d\tau} + e^{+k_a r} \frac{dL_a^{(1)}}{d\tau} \right) - \frac{2k_a^{2n-2}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} V_{\lambda+1}(\tau) \tag{86}
\]

\[ (a = 1, \ldots, n - 1) . \]

From equations (85) and (86) we obtain

\[
F \frac{dL_a^{(2)}}{d\tau} = + \frac{k_a^{2n-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} \left[ U_{\lambda+1}(\tau) - k_a V_{\lambda+1}(\tau) \right] e^{+k_a r} \tag{87}
\]

and

\[
F \frac{dL_a^{(1)}}{d\tau} = - \frac{k_a^{2n-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} \left[ U_{\lambda+1}(\tau) + k_a V_{\lambda+1}(\tau) \right] e^{-k_a r} . \tag{88}
\]

In their integrated forms the foregoing equations are

\[
F L_a^{(2)}(\tau) = \frac{k_a^{2n-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} \int_0^\tau e^{+k_a \tau} \left[ U_{\lambda+1}(\tau) - k_a V_{\lambda+1}(\tau) \right] d\tau \tag{89}
\]

\[ + F (L_a^{(1)} + \Delta L_a) \quad (a = 1, \ldots, n - 1) \]

and

\[
F L_a^{(1)}(\tau) = \frac{k_a^{2n-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda = 0}^{n-2} S_{\lambda, a} \int_\tau^\infty e^{-k_a \tau} \left[ U_{\lambda+1}(\tau) + k_a V_{\lambda+1}(\tau) \right] d\tau \tag{90}
\]

\[ (a = 1, \ldots, n - 1) , \]

where in equation (89) \( \Delta L_a, (a = 1, \ldots, n - 1) \), are \((n - 1)\) constants of integration. However, it will be noticed that in integrating equation (88) in the form (90) we have made a particular choice of the constants of integrations so as to be compatible with the requirement that none of the quantities tend to infinity exponentially as \( \tau \to \infty \) (cf. eq. [34]).
Finally, the constants of integration $\Delta L_a$, $(a = 1, \ldots, n - 1)$, and $\Delta Q$ (cf. eq. [40]) which occur in the solution of the variational equations are to be determined from the boundary conditions

$$I_i = 0 \text{ at } \tau = 0 \text{ for } i = 1, \ldots, n .$$  \hspace{1cm} (91)

Explicitly, these conditions reduce to (cf. II, eq. [21])

$$\sum_{a=1}^{n-1} \frac{L^{(1)}_a + \Delta L_a}{1 - \mu_i k_a} + Q^{(1)} + \Delta Q = \mu_i - \sum_{a=1}^{n-1} \frac{L^{(2)}_a(0)}{1 + \mu_i k_a} \quad (i = 1, \ldots, n) ;$$  \hspace{1cm} (92)

or, since the $L^{(1)}_a$'s and $Q^{(1)}$ satisfy the relation

$$\sum_{a=1}^{n-1} \frac{L^{(1)}_a}{1 - \mu_i k_a} + Q^{(1)} = \mu_i \quad (i = 1, \ldots, n) ,$$  \hspace{1cm} (93)

the equations which determine $\Delta L_a$ and $\Delta Q$ are

$$\sum_{a=1}^{n-1} \frac{\Delta L_a}{1 - \mu_i k_a} + \Delta Q = - \sum_{a=1}^{n-1} \frac{L^{(2)}_a(0)}{1 + \mu_i k_a} \quad (i = 1, \ldots, n) .$$  \hspace{1cm} (94)

This completes the formal solution to the problem in the $(2, n)$ approximation.

5. The solution in the $(2, 1)$ approximation.—The discussion of this case is of particular interest, as in this approximation

$$J^{(2)} = J^{(1)} ,$$  \hspace{1cm} (95)

and equation (19) becomes

$$B^{(2)} = J^{(1)} + \frac{1}{4} \int_0^\infty \delta \frac{dF^{(1)}}{d\tau} \, d\nu .$$  \hspace{1cm} (96)

Substituting for $J^{(1)}$ its solution in the first approximation (II, eqs. [30] and [35]), we have

$$B^{(2)} = \frac{3}{2} F \left( \tau + \frac{1}{\sqrt{3}} + \frac{1}{3F} \int_0^\infty \delta \frac{dF^{(1)}}{d\tau} \, d\nu \right) .$$  \hspace{1cm} (97)

It may be recalled that in the foregoing equations the optical depth $\tau$ is evaluated in terms of the mean absorption coefficient $\kappa$ defined by (cf. eq. [42])

$$\bar{\kappa} = \int_0^\infty \kappa_\nu \left( \frac{F^{(1)}}{F} \right) \, d\nu .$$  \hspace{1cm} (98)

From equations (97) and (98) it is apparent that, when $\delta$ is independent of depth,

$$B^{(2)} = \frac{3}{2} F \left( \tau + \frac{1}{\sqrt{3}} \right) \quad (\delta \text{ independent of } \tau) .$$  \hspace{1cm} (99)

* It is possible to write down explicitly the inverse of the linear transformation which appears on the left-hand side of this system (cf. Ap. J., 101, 320, 1945, eq. [8]). But, as the practical use to which these solutions can be put is limited, we shall not develop further these formalities here.
In other words, we have shown that in a Milne-Eddington type of approximation the temperature distribution obtained on the gray-body assumption continues to be valid in a second approximation if the mean absorption coefficient $\kappa$ (in terms of which $\tau$ has to be defined) is a straight average of $\kappa_\nu$, weighted according to the net monochromatic flux of radiation of frequency $\nu$ in a gray atmosphere, and if, further, $\kappa_\nu/\kappa$ is independent of depth. This answers the question that we asked at the outset (§1), namely, as to the conditions under which an equation for the temperature distribution of the form (9) may be expected to hold even when the material departs from perfect grayness. We may draw particular attention to the fact that our answer has not provided any justification for averaging $\kappa_\nu$ according to the conventional method of Rosseland. Indeed, the Rosseland mean, as customarily defined by

$$\frac{1}{\kappa} \int_0^\infty \frac{dB_\nu}{dT} \, d\nu = \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu,$$

weights $\kappa_\nu$ falsely and overemphasizes the violet region of the spectrum. On the other hand, our present discussion would rather suggest that in all preliminary considerations it would be preferable (and simpler) to average $\kappa_\nu$ directly, weighting with the Planck function corresponding to the effective temperature of the star. One further remark concerning the requirement of the constancy of $\kappa_\nu/\kappa$ with depth may be made. With the negative ion of hydrogen firmly established as the principal source of absorption in stellar atmospheres of the solar and neighboring types, the requirement of the constancy of $\kappa_\nu/\kappa$ with depth is likely to be met fairly satisfactorily.

Returning to equations (97) and (98), we see that a practical use of these equations will require a knowledge of the monochromatic fluxes $F^{(1)}_\nu$ at various depths in a gray atmosphere.

Now, according to equation (26), we can write

$$F^{(1)}_\nu (\tau) = 2 \int_\tau^\infty B_\nu (T^{(1)}_\nu - \tau) \, dt - 2 \int_0^\tau B_\nu (T^{(1)}_\nu) E_2 (\tau - t) \, dt,$$

where

$$T^{(1)}_\nu = T_\nu \left[ \frac{3}{4} (t + q(t)) \right]^{1/4}.$$

Writing

$$a = \frac{\hbar \nu}{k T_\nu},$$

we can express equation (101) in the form

$$\frac{F^{(1)}_\nu (\tau)}{F} = \frac{30}{\pi^4} \frac{a^2}{e^{1.23275 a} - 1} \, f_a (\tau),$$

where

$$f_a (\tau) = (e^{1.23275 a} - 1) \left\{ \int_\tau^\infty \frac{E_2 (t - \tau) \, dt}{e^{a/3 (t+q(t))} 1/4} - \int_0^\tau \frac{E_2 (\tau - t) \, dt}{e^{a/3 (t+q(t))} 1/4} - 1 \right\}$$

and

$$1.23275 \ldots = \left( \frac{4}{\sqrt{3}} \right)^{1/4}.$$

Using for $q(\tau)$ the solution on the fourth approximation given in II, equation (54), Miss Frances Herman and the writer have numerically evaluated the function $f_a (\tau)$ defined as in equation (105) for various values of $a$ and $\tau$. The results of the numerical integrations are summarized in Tables 1 and 2. In Table 1 we give the values of $f_a (\tau)$,
and in Table 2 these are converted according to equation (104) to give the monochromatic fluxes directly. And finally, in Table 3, we give values of $dF_a^{(1)}/dr$ obtained by direct numerical differentiation of the fluxes tabulated in Table 2.

**TABLE 1**

<table>
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<th>$\tau$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
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<td>0.682</td>
<td>0.803</td>
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**TABLE 2**

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In Figure 1 we have illustrated the distribution in frequency of the net flux of radiation at various depths. It is of particular interest to observe how the redistribution in the frequencies takes place as we descend into the atmosphere.
With the data given in Tables 2 and 3 we should be able to use the solution (97) to determine the temperature distribution under a wide range of practical conditions. We shall return to such applications in later papers.

**TABLE 3**

**The Derivatives of the Monochromatic Fluxes: \( dF^{(1)}_\nu /d\tau \) in Units of \( F \)**

<table>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
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<th>12</th>
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<td>-0.11</td>
<td>-0.044</td>
<td>+0.024</td>
<td>+0.0585</td>
<td>+0.0323</td>
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<tr>
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**Fig. 1.—** The frequency distribution of the net flux of radiation at various depths in a gray atmosphere. The abscissa measures the frequency (in units of \( kT_e/h \)), and the ordinate measures the flux (in units of the constant net integrated flux \( F \)). The curves 1, 2, 3, and 4 refer to the depths \( \tau = 0, 0.5, 1, \) and 2, respectively.

6. The solutions for the temperature distribution in higher approximations.—Turning next to solutions in higher approximations, we have (cf. eq. 34 and \( \Pi \), eqs. [28] and [29])

\[
J^{(2)} = \frac{3}{4} F \left\{ \tau + Q^{(2)} + \sum_{\alpha=1}^{n-1} L^{(2)}_{\alpha}(\tau) e^{-2\alpha\tau} + \sum_{\alpha=1}^{n-1} L^{(2)}_{\alpha}(\tau) e^{+2\alpha\tau} \right\};
\]  

(107)
or, substituting for $Q^{(2)}$, $L^{(2)}_a$ and $L^{(2)}_{1a}$ according to equations (40), (89), and (90), we have

$$J^{(2)} = J^{(1)} + \frac{3}{4} F \left( \Delta Q + \sum_{a=1}^{n-1} \Delta L_a e^{-k_a r} \right)$$

$$+ \frac{3}{4} \sum_{a=1}^{n-1} \frac{k^{2a-3}}{\prod_{j \neq a} (k_a^2 - k_j^2)} \sum_{\lambda=0}^{n-2} S_{\lambda a} \left\{ e^{-k_a r} \int_0^r e^{+k_a r} \left[ U_{\lambda+1}(\tau) - k_a V_{\lambda+1}(\tau) \right] d\tau \right\}$$

$$+ e^{+k_1 r} \int_r^\infty e^{-k_1 r} \left[ U_{\lambda+1}(\tau) + k_a V_{\lambda+1}(\tau) \right] d\tau,$$

where it may be recalled that the constants of integration $\Delta Q$ and $\Delta L_a$ ($a = 1, \ldots, n - 1$), are to be determined according to equation (94). Corresponding to the foregoing solution for $J^{(2)}$, there is the temperature distribution

$$\frac{\sigma}{\pi} (T^{(2)})^4 = B^{(2)} = J^{(2)} + \frac{1}{4} \int_0^\infty \delta_\nu (\tau) \frac{dF_\nu^{(1)}}{d\tau} d\nu.$$

Equations (108) and (109) show that, when the material departs from being gray, the corrections which have to be made to the temperature distribution are of two kinds: first, because $B$ is not, in general, equal to $J$, and, second, because the energy density of the radiation does not have the gray-atmosphere value. It is only under very special conditions (which we have already discussed in § 5) that these corrections vanish in a second approximation.

The solution (108) simplifies considerably for the case $n = 2$. For, in this $(2, 2)$ approximation there is only one characteristic root (II, eq. [40]):

$$k_1 = \frac{\sqrt{35}}{3} = 1.97203;$$

and equation (108) becomes (cf. eqs. [68] and [72])

$$J^{(2)} = J^{(1)} + \frac{3}{4} F \left( \Delta Q + \Delta L_1 e^{-k_1 r} \right) + \frac{3}{4} e^{-k_1 r} \int_0^r e^{+k_1 r} \left( \delta_3 - \frac{3}{5} \delta_1 - k_1 \delta_4 \right) d(k_1 \tau)$$

$$+ \frac{3}{4} e^{+k_1 r} \int_r^\infty e^{-k_1 r} \left( \delta_3 - \frac{3}{5} \delta_1 + k_1 \delta_4 \right) d(k_1 \tau),$$

where it may be noted that (II, eqs. [30] and [42])

$$J^{(1)} = \frac{3}{4} F \left( \tau + 0.694025 - 0.116675 e^{-k_1 r} \right).$$

The equations which determine the constants of integrations $\Delta Q$ and $\Delta L_1$ in the solution (111) are (cf. eq. [94])

$$\frac{\Delta L_1}{1 - \mu_1 k_1} + \Delta Q = -\frac{L^{(2)}_1(0)}{1 + \mu_1 k_1}$$

and

$$\frac{\Delta L_1}{1 - \mu_2 k_1} + \Delta Q = -\frac{L^{(2)}_1(0)}{1 + \mu_2 k_1}.$$

From these equations we find that

$$\Delta L_1 = \frac{(1 + \mu_1 k_1)(1 + \mu_2 k_1)}{(1 + \mu_1 k_1)(1 + \mu_2 k_1)} L^{(2)}_1(0)$$

and

$$\Delta Q = -\frac{2}{(1 + \mu_1 k_1)(1 + \mu_2 k_1)} L^{(2)}_1(0).$$
Substituting these values for $\Delta L_1$ and $\Delta Q$ in equation (111), we obtain

$$J^{(2)} = J^{(1)} + \frac{3}{4} \left[ \frac{(1 - \mu_1 k_1)(1 - \mu_2 k_1)}{1 + \mu_1 k_1} \right] \int_0^\infty e^{-k_1 \tau} (\bar{\delta}_3 - \frac{1}{3} \bar{\delta}_1 + k_1 \bar{\delta}_4) \, d(k_1 \tau)$$

(117)

From equation (117) we find that for $\tau = 0$ (cf. II, eq. [71]),

$$J^{(2)} (0) = \frac{\sqrt{3}}{4} \frac{k_1}{(1 + \mu_1 k_1)} \int_0^\infty e^{-k_1 \tau} (\bar{\delta}_3 - \frac{1}{3} \bar{\delta}_1 + k_1 \bar{\delta}_4) \, d(k_1 \tau).$$

(118)

The boundary temperature in the second approximation is therefore given by

$$\frac{\sigma}{\pi} (T_0^{(2)})^4 = \frac{\sigma}{\pi} (T_0^{(1)})^4 + \frac{1}{4} \int_0^\infty \left( \frac{dF^{(1)}}{d\tau} \right)_{\tau=0} d\nu + \frac{\sqrt{3}}{2} \frac{k_1}{(1 + \mu_1 k_1)} \int_0^\infty e^{-k_1 \tau} (\bar{\delta}_3 - \frac{1}{3} \bar{\delta}_1 + k_1 \bar{\delta}_4) \, d(k_1 \tau).$$

(119)

It is seen that a practical use of the foregoing solutions in the $(2, 2)$ approximation will require, in addition to the monochromatic fluxes $F_\nu^{(1)}$, a knowledge also of the quantities

$$M_\nu^{(1)} (\tau) = \frac{1}{2} \int_{-1}^{+1} dI_\nu^{(1)} \mu^3 d\mu \quad \text{and} \quad N_\nu^{(1)} (\tau) = \frac{1}{2} \int_{-1}^{+1} dI_\nu^{(1)} \mu^4 d\mu,$$

(120)

as these will be needed in the evaluation of $\bar{\delta}_3$ and $\bar{\delta}_4$. However, since the first approximation refers only to a gray atmosphere, $M_\nu^{(1)}$ and $N_\nu^{(1)}$ can be expressed in terms of $F_\nu^{(1)}$. For, from the equation of transfer appropriate to this case (eq. [12]), we can readily show that

$$\frac{dK_\nu^{(1)}}{d\tau} = \frac{1}{4} F_\nu^{(1)}; \quad \frac{dM_\nu^{(1)}}{d\tau} = K_\nu^{(1)} (\tau) - \frac{1}{3} D_\nu^{(1)} (\tau),$$

(121)

and

$$\frac{dN_\nu^{(1)}}{d\tau} = M_\nu^{(1)} (\tau).$$

(122)

and, accordingly, $M_\nu^{(1)} (\tau)$ and $N_\nu^{(1)} (\tau)$ can be obtained successively by quadratures involving at each stage only known functions. Numerical work relating to these quantities with applications to practical problems will be given in later papers.

In conclusion, I wish to record my indebtedness to Miss Frances Herman for assistance with the very laborious numerical work which was involved in the preparation of Tables 1, 2, and 3.