On a class of stellar Models.

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In this paper Stellar models in which
\[ \eta = \frac{L(r)}{M(r)/\bar{M}} \propto T^3 \quad \text{and} \quad \eta \propto \rho^a T^3 \]
are studied and the gener-1 nature of the corresponding \((\rho, T)\) relationships are established.

§ 1. In a recent paper\(^1\) the author began the study of a certain new class of stellar models with the object of examining the effect of the distribution of the energy sources on the possible types of density temperature relationships that can occur. A problem of special interest in this connection is to specify at least qualitatively the circumstances under which negative density gradients \(\partial \rho / \partial T < 0\) can arise. To study this problem we introduced two classes of stellar models namely

\[ \eta = \frac{L(r)}{M(r)/\bar{M}} \propto T^3, \quad \text{(A)} \]

and

\[ \eta \propto \rho^a T^3, \quad \text{(B)} \]

If the law of opacity is assumed to be of the form

\[ \kappa = \kappa_1 \frac{\rho}{T^{3+s}}, \quad \text{(1) *)} \]

then in discussing models belonging to the class (A) a certain index \(\gamma\) related to \(\delta\) by

\[ \gamma = \frac{1}{4}(\delta - s), \quad \text{(2)} \]

plays an important role. We shall accordingly refer to these as the \(\gamma\)-models, though when comparing the results based on (A) with those based on (B)

\(^1\) M. N R A. S. Vol 97, p. 132, 1936, referred to as I.

* In I we based the discussion on the usual law of opacity

\[ \kappa = \kappa_1 \frac{\rho}{T^{3.5}}, \quad \text{(1')} \]

(i.e.) the case \(s = \frac{1}{2}\). A glance through I § 3 suffices to show that the use of the more general law (1) only modifies the definition of \(\gamma\) [I equation (15)] and that with this definition of \(\gamma\) the analysis of I §§ 3—10 remains unchanged. For the models belonging to the class (B) the definition of \(\gamma\) according to I equation (85) changes to \(\gamma = \frac{1}{4}(\delta + 3a - s)\). The case \(\gamma = a + 1\) integrated in I § 12 corresponds to \(\delta = 4 + s + a\) instead of the special case \(s = \frac{1}{2}\) considered there.
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it may be more convenient to refer to the $\gamma$-models more simply as the $\delta$-models. Again we saw that among the class of models (B) a case of special interest arises when $\delta = 4 + s + \alpha$ (i.e.) when

$$\eta = \eta_0 \theta^a T^{4+s+\alpha}. \quad (C)$$

For the opacity law (1), (C) is equivalent to the assumption

$$\kappa \eta \propto (\theta T)^{a+1}. \quad (C')$$

We shall accordingly refer to the models (C) as the $\alpha$-models. — In I we treated the case $\gamma = 1$ with some detail. The model $\gamma = 1$, is seen to be the common member of the $\gamma$- and the $\alpha$-models. ($\gamma = 1$ corresponds to the same law for $\eta$ as $\alpha = 0$.)

In this paper we shall attempt to obtain a complete general survey of the properties of the $\gamma$ and the $\alpha$-models. In § 11 a beginning is made in the study of the properties of the general ($\alpha, \delta$) models. The bearing of the results contained in this and the earlier paper on certain general problems of stellar structure are contained in §§ 16—20.

Section 1: The $\gamma$-models.

§ 2. General Formulae. The analytical discussion hinges on the differential equation [I, (14)]

$$\gamma x y \frac{dy}{dx} = y(y + 1) - x \quad (3)$$

where

$$y = \frac{\beta}{1 - \beta}; \quad x = K T^{-4 \gamma}, \quad \begin{cases} 
K = \frac{4 \pi c G M}{\kappa \eta_0 L}; \quad \frac{k}{a \mu H}; & \gamma = \frac{1}{4}(\delta - s) \end{cases} \quad (3')$$

the symbols having the same meaning as in I. (3) has other equivalent forms namely [I, (28), (36)]

$$z q z \frac{dz}{dq} = z q - q', \quad (4)$$

$$z \frac{dz}{dt} = t^{1-\gamma} - z. \quad (4')$$

The variables $x, y; z, q$ or $t$ are related to the physical variables $q$ and $T$ as follows:

Let $\sigma$ and $\tau$ specify $q$ and $T$ in the units introduced in I equations (69), (70) and (71). Then from the definitions in I we verify that

$$z = \sigma \tau; \quad t = q^{-1} = \tau^4, \quad (5)$$

$$y = z q = \sigma \tau^{-3}; \quad x = q' = \tau^{-4 \gamma}. \quad (6)$$
§ 3. The \((\sigma, \tau)\)-differential equation. From (4'), (5) and (6) we have that

\[
\sigma \tau \frac{d(\sigma \tau)}{d(\tau^4)} = \tau^{4(1-\gamma)} - \sigma \tau.
\]

(7)

or

\[
\sigma \tau \frac{d\sigma}{d\tau} + \sigma^2 + 4 \sigma \tau^3 - 4 \tau^{6-4\gamma} = 0.
\]

(8)

The case \(\gamma = 1\) has been fully discussed in I for which case (8) admits of an exact integration. The solution is

\[
\tau^4 = \text{Constant} - \sigma \tau - \log |(1 - \sigma \tau)|.
\]

(9)

For a discussion of the general solution (9) the nature of what we called the "Standard-Solution" is important. This solution satisfies the boundary condition that either \(\sigma\) and \(\tau\) tend to zero simultaneously or \(\tau\) remains finite at \(\sigma = 0\). The importance of the standard solution is simply that once this is known the behaviour of the general solution can easily be described. It is now clear that for any \(\gamma\) there should be a solution which would correspond to the standard solution for the case \(\gamma = 1\) which is

\[
\tau^4 = -\sigma \tau - \log (1 - \sigma \tau).
\]

(10)

We shall first isolate the standard solutions for any \(\gamma\) \(< 2\) and then proceed to discuss the general solution.

§ 4. The Standard Solutions for \(\gamma = 0, 0.75, \text{and } 1.5\). We shall see that \(\gamma = 0, \frac{3}{4}, \text{and } \frac{3}{2}\) represent critical values for the index \(\gamma\).

Case i, \(\gamma = 0\). Our differential equation is

\[
\sigma \tau \frac{d\sigma}{d\tau} + \sigma^2 + 4 \sigma \tau^3 - 4 \tau^6 = 0
\]

(11)

on substituting

\[
\sigma = A \tau^3
\]

(12)

in (11) be find that there is a solution of the form (12) provided \(A\) is chosen so that it satisfies the equation

\[
A^2 + A - 1 = 0
\]

(13)

or

\[
A = \frac{\sqrt{5} - 1}{2}.
\]

(18')

Our standard solution then is

\[
\sigma = \frac{\sqrt{5} - 1}{2} \tau^3.
\]

(14)
The general solution of the differential equation (11) can be explicitly found. If we write \( \sigma = u \tau^3 \) then \( u \) is found to satisfy the differential equation
\[
\frac{1}{4} u \tau \frac{du}{d\tau} = 1 - u - u^2,
\]
the solution of which is
\[
4 \log \tau = -\frac{A}{A + B} \log (|A - u|) - \frac{B}{A + B} \log (|B + u|) + C,
\]
where \( C \) is a constant of integration, and \( A \) and \(-B \) are the roots of the equation (13) so that
\[
A = \frac{\sqrt{5} - 1}{2}; \quad B = \frac{\sqrt{5} + 1}{2}.
\]
From (14') we see that any solution of (11) asymptotically tends to the standard solution. If we consider a solution which satisfies the boundary condition \( \sigma = 0 \) at \( \tau = \tau_0 \) then (14') can be written as
\[
4 \log \left( \frac{\tau}{\tau_0} \right) = \frac{A}{A + B} \log \left( \frac{A}{A - u} \right) + \frac{B}{A + B} \log \left( \frac{B}{B + u} \right),
\]
which shows again that as \( \tau \to \infty, u \to A \).

**Case ii, \( \gamma = \frac{3}{4} \).** Our differential equation is
\[
\sigma \tau \frac{d\sigma}{d\tau} + \sigma^3 + 4 \tau^3 (\sigma - 1) = 0.
\]
From the form of (15) it follows that \( \sigma \) tends to a finite limit as \( T \to \infty \). One derives the following asymptotic series when \( \tau \to \infty \)
\[
\sigma = 1 - \frac{1}{4 \tau^3} - \frac{1}{16 \tau^6} - \frac{1}{32 \tau^9} - \frac{7}{256 \tau^{12}} - \cdots \tag{16}
\]
The analytic continuation of (16) corresponds to the singular solution of the differential equation (15) (analogous to the solution \( \sigma\tau = 1 \) for the case \( \gamma = 1 \)). Consequently it is clear that along any solution of (15)
\[
\sigma \to 1, \quad \tau \to \infty.
\]
The series (16) is very rapidly convergent for \( \tau > 1 \). For \( \tau \to 0 \) we obtain the following series which satisfies the boundary condition \( \sigma = 0, \tau = 0 \):
\[
\sigma = A \tau^{3/2} - \frac{8}{18} \tau^3 + \frac{20}{18^2 A} \tau^{4/3} + \frac{600}{18^3 \cdot 19} \tau^4 - \frac{25 \cdot 181}{19 \cdot 18^4 \cdot 11 A} \tau^{7/4} - \cdots, \tag{18}
\]
where
\[
A = \sqrt{\frac{8}{5}}. \tag{18'}
\]
The series (18) suffices to calculate \( \sigma \) sufficiently accurately for \( \tau \leq 1 \). For \( \tau \geq 1.3 \) (16) enables the calculation of \( \sigma \). Thus the two series together determine our standard solution for \( \gamma = 0.75 \). The solution monotonically
increases tending to a finite limit as $\tau \to \infty$. The standard solution therefore corresponds to a positive density gradient for all values of $\tau$. The solution is tabulated in Table 1, and the $\sigma - \tau$ variation is shown in Fig. 1 (b).

Case iii, $\gamma = 1.5$. Our differential equation is

$$\sigma \frac{d\sigma}{d\tau} + \sigma^2 + 4 \sigma \tau^2 - 4 = 0. \quad (19)$$

As $\tau \to 0$ we find that we have the series

$$\sigma = 2 - \frac{4}{5} \tau^2 + \frac{12}{25} \tau^4 + \frac{12}{1875} \tau^6 + \frac{71}{125 \cdot 28 \cdot 11} \tau^9 + \ldots, \quad (20)$$

which is again rapidly convergent for $\tau \leq 1$. Hence our standard solution is such that as $T \to 0$ the density tends to a finite limit. Further it is clear that along the standard solution we always have a negative density gradient. For $\tau > 1$ we can in the same way obtain an asymptotic series expansion for $\sigma$:

$$\sigma = \frac{1}{\tau^3} \left\{ 1 + \frac{1}{2} \frac{1}{\tau^6} + \frac{5}{4} \frac{1}{\tau^{12}} + \frac{11}{2} \frac{1}{\tau^{18}} + \frac{589}{16} \frac{1}{\tau^{24}} + \ldots \right\}, \quad (21)$$

which defines $\sigma$ sufficiently accurately for $\tau \geq 1.4$. All solutions of (19) eventually become asymptotic to (21). Thus (20) and (21) again determine the nature of the standard solution for practically the whole range of $\gamma$. The solution is tabulated in Table 1 and the $\sigma - \tau$ variation is shown in Fig. 1 (e).

It may be noticed finally that $\gamma = 0, 0.75, 1.5$ correspond to $\delta = s, 3 + s$ and $6 + s$ respectively.

§ 5. Remembering that the physically interesting solutions are those for which $\sigma \to 0$ for $\tau$ finite or zero the results of § 4 are seen to be equivalent to the following:

If $\gamma \leq 3/4$ (i.e. if $\delta \leq 3 + s$) all solutions of the differential equation (8) which satisfy the boundary condition $\sigma = 0, \tau = \tau_0 \geq 0$ are characterized by positive density gradients for all values of $\tau$. If $3/4 < \gamma < 3/2$ (i.e. if $3 + s < \delta < 6 + s$) then a solution of (8) satisfying the boundary condition $\sigma = 0, \tau = \tau_0 \geq 0$ has a maximum along any particular solution and for $\tau$ exceeding a certain definite value (depending on $\tau_0$) we have only negative density gradients. For $\gamma = 3/2$ (or $\delta = 6 + s$) the "Standard Solution" (i.e.) a solution which satisfies the boundary condition $\sigma \to \sigma_0 \geq 0$ ($\sigma_0$ finite) as $\tau \to 0$ is characterized by a negative density gradient for all values of $\tau$, and for $\gamma > 3/2$ there is no proper solution which has a finite value at the origin.

The physical meaning of the above is explained in § 17.
§ 6. Asymptotic series for the standard solution for $2 > \gamma > 0$ ($8 + s > \delta > s$). In § 4 we saw that the asymptotic solution for $\tau \to \infty$ can be found for the differential equation (8). The series for $\gamma = 0.75$ and 1.5 were given. Quite generally one has

$$\sigma = \frac{1}{\tau^{22}} \left[ 1 - \frac{(\gamma - 1)}{\tau^{2}} + \frac{(\gamma - 1)(3\gamma - 2)}{\tau^{3}} + \frac{(\gamma - 1)(2\gamma - 1)(7\gamma - 5)}{\tau^{12}} + \frac{(\gamma - 1)(5\gamma - 2)(17\gamma^{3} - 22\gamma + 7)}{\tau^{16}} + \frac{(\gamma - 1)(3\gamma - 1)(207\gamma^{5} - 371\gamma^{3} + 218\gamma^{4} - 42)}{\tau^{20}} + \ldots \right] \quad (22)$$

we see that for $\gamma = 1$, (22) reduces to the single term

$$\sigma \tau = 1, \quad (23)$$

which is therefore the exact form for the singular solution for this case. Our present result that all solutions of (8) must tend asymptotically to the solution defined by (22) corresponds to the result established in I that when $\gamma = 1$, all solutions tend to the hyperbola $\sigma \tau = 1$ for $\tau \to \infty$.

For $\gamma < 2$, we have as $\tau \to 0$

$$\sigma = \tau^{3-2\gamma} \left[ A - \frac{2}{4 - \gamma} \tau^{2} + \frac{2 - \gamma}{(4 - \gamma)^2} A \tau^{4} + \frac{2\gamma(2 - \gamma)^2}{(4 + \gamma)(4 - \gamma)^3} \tau^{6} + \frac{(2 - \gamma)^2(\gamma^2 + 28\gamma^3 - 4\gamma - 16)}{4(4 + \gamma)(1 - \gamma)^4(2 + \gamma)A} \tau^{8} + \ldots \right] \quad (24)$$

where

$$A = \sqrt{\frac{2}{2 - \gamma}}. \quad (25)$$

(24) defines our standard solution. When $\gamma = 1$ we see that

$$\sigma \sim \sqrt{2} \tau, \quad \tau \to 0 \quad (26)$$

thus recovering our result in I (cf. equation (73)).

The series (22) and (24) again enable us to isolate the critical natures of $\gamma = 0$, $\gamma = \frac{3}{4}$ and $\gamma = \frac{3}{2}$. Further from (25) we see that the coefficient of $\tau^{3-2\gamma}$ in (24) tends to infinity for $\gamma = 2$. Hence $\gamma = 2$, (or $\delta = 8 + s$) is another critical value for the index. The reason for this is clear from the analysis in § 7.

The standard solutions for $\gamma = \frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ and $1\frac{1}{2}$ have been computed and are tabulated in Table 1). The $\sigma - \tau$ relationships are illustrated in Fig. 1.

1) The standard solution for $\gamma = 1$, known explicitly has been tabulated in I.
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Table 1. Standard solutions for the $\gamma$-models.

<table>
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<th>$\tau$</th>
<th>$\gamma = 1/2$</th>
<th>$\gamma = 3/4$</th>
<th>$\gamma = 1/4$</th>
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§ 7. Consider the differential equation

$$\gamma x y \frac{dy}{dx} = y(y + 1) - x. \quad (27)$$

If $y$ be large compared with unity we can write the above approximately as

$$\gamma x y \frac{dy}{dx} = y^2 - x. \quad (28)$$

Put

$$y^2 = \varphi. \quad (29)$$

Then (28) reduces to

$$\frac{1}{2} \gamma x \frac{d\varphi}{dx} = \varphi - x. \quad (28')$$

The general solution of (28') is easily found to be

$$y^2 = \frac{2}{2 - \gamma} x + B x^\gamma, \quad (\gamma \neq 2), \quad (30)$$

$$y^2 = -x \log x + B x, \quad (\gamma = 2). \quad (31)$$
where $B$ is a constant of integration. Of course (30) and (31) are approximate solutions of (27) provided only $y \gg 1$. In terms of $\sigma$ and $\tau$ (30) and (31) are equivalent to

\[
\sigma^2 = \frac{2}{2-\gamma} \tau^{6-4\gamma} + \frac{B}{\tau^{2\gamma}}, \quad (\gamma = 2), \tag{32}
\]

\[
\sigma^2 = \frac{8}{\tau^2} (\log \tau + B), \quad (\gamma = 2). \tag{33}
\]

(32) and (33) are approximate solutions only if

\[
\sigma \gg \tau^3. \tag{34}
\]

Equations (30)—(33) while bringing into relief the critical nature of $\gamma = 2$ have some further interesting features. We first notice that for $\gamma < 2$ the standard solutions correspond to putting $B = 0$ in which case (32) gives the dominant term of the series (24). Again if $\gamma < 2$ and $B > 0$, then as $\tau \to 0$

\[
\sigma \sim \frac{\text{Constant}}{\tau}, \quad (\gamma < 2). \tag{35}
\]

Thus the general solution of (9) has a singularity at the origin, and it is interesting that the behaviour near the origin is independent of $\gamma$ if $\gamma < 2$. We shall return to these formulae again in §9.

§ 8. The regions of positive and negative density gradients in the $(\sigma-\tau)$ plane. In §1 we saw that the $(x, y)$ plane could be divided into two regions, the region of positive and the region of negative density gradients. A similar kind of division of the $(\sigma, \tau)$ plane should of course be possible. From (8) we see that the locus of the maxima and the minima is given by

\[
\sigma^2 + 4 \sigma \tau^3 - 4 \tau^{6-4\gamma} = 0 \tag{36}
\]

or

\[
\sigma = 2 \tau^3 \left[ \sqrt{1 + \frac{1}{\tau^{4\gamma}}} - 1 \right]. \tag{37}
\]

It is easily seen that the region bounded by the curve (37) and the $\tau$-axis is the region of the positive density gradient, while the region bounded by the curve (37) and the $\sigma$-axis is the region of the negative density gradient. For $\gamma = \frac{3}{4}$ and $\frac{3}{2}$ the locus (37) takes the simpler forms

\[
\tau^3 = \frac{1}{4} \left( 1 - \frac{\sigma}{1 - \sigma} \right), \quad (\gamma = \frac{3}{4}), \tag{38}
\]

\[
\tau^3 = \frac{1}{\sigma} \left( 1 - \frac{1}{4} \sigma^2 \right), \quad (\gamma = \frac{3}{2}). \tag{39}
\]
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(38) is a monotonic increasing function of \( \sigma \), \( \sigma \) increasing from 0 to 1 as \( \tau \) varies from 0 to \( \infty \). (39) on the other hand is a monotonic decreasing function of \( \sigma \), \( \sigma \) decreasing from \( \sigma = 2 \) at \( \tau = 0 \) to \( \sigma = 0 \) for \( \tau \to \infty \).

The loci (37) for different values of \( \gamma \) are shown in Fig. 1.

From (37) it follows that along the locus
\[ \sigma \sim 2 \tau^{3-2\gamma}, \quad (\tau \to 0), \]
\[ \sigma \sim \tau^{3-4\gamma}\left[1 - \frac{1}{4 \tau^{4\gamma}} + \cdots \right], \quad (\tau \to \infty). \]  

(40) and (41) it follows that for \( \frac{3}{4} < \gamma < \frac{3}{2} \), \( \sigma \) tends to zero both when \( \tau \to 0 \) and when \( \tau \to \infty \). Hence the locus (37) must have a maximum for \( \frac{3}{4} < \gamma < \frac{3}{2} \) and the solution of (8) which passes through the maximum of (37) must have an inflex at this point. It is easily found that at the maximum of (37)
\[ \sigma = \frac{2}{3} (3-2\gamma) \tau^{3-4\gamma}. \]  

(42) Substituting (42) in (36) we find that at the maximum of the locus (37)
\[ \sigma = \frac{2}{3} (3-2\gamma)^{3-2\gamma} \frac{1}{[3(4\gamma-3)]^{\frac{3-4\gamma}{\gamma}}}, \]
\[ \tau = \left[\frac{(3-2\gamma)^{2\gamma}}{3(4\gamma-3)}\right]^{\frac{1}{\gamma}}. \]  

(43) From (5) and (6) we find the corresponding values of the other variables
\[ z = \frac{2}{3} \frac{(3-2\gamma)^{2-\gamma}}{[3(4\gamma-3)]^{1-\gamma}}, \quad q = \left[\frac{3(4\gamma-3)}{(3-2\gamma)^2}\right]^{\gamma}, \]
\[ x = q^2 = \frac{3(4\gamma-3)}{(3-2\gamma)^2}, \quad y = z q = \frac{2(4\gamma-3)}{3-2\gamma}. \]  

(45) We see that when \( \gamma = 1 \), the above formulae give
\[ \sigma = 2 \cdot 3^{-3/4} = 0.878; \quad \tau = 3^{-1/4} = 0.760, \]
\[ x = 3, \quad y = 2, \quad z = \frac{2}{3}, \quad q = 3. \]

It is clear that the solution which passes through \( x = 3, \ y = 2 \) is the one which touches the parabola \( x = y (1 + \frac{1}{4} y) \) in the \((x, y)\) plane. The existence of such a solution was stated in I § 8 (case II (c)). We can numerically specify \( B^* \) introduced there. Since the general solution is
\[ 1/x = B - z - \log (1 - z) \]
and the particular solution we are now considering passes through \( x = 3, \ y = 2 \), we have
\[ B^* = 1 - \log 3 = -0.0986. \]
It may finally be noticed that the area $S$ of the positive region in the $(\sigma, \tau)$ plane is bounded for $1 < \gamma < 2$. One has in fact

$$S = \frac{1}{4\sqrt{\pi}} \Gamma\left(\frac{1}{\gamma} - \frac{1}{2}\right) \Gamma\left(1 - \frac{1}{\gamma}\right).$$  \(\text{(47)}\)

§ 9. The nature of the general solutions. We have seen that along the standard solution

$$\sigma \sim \sqrt{\frac{2}{2 - \gamma}} \tau^{3 - 2\gamma}, \quad (\tau \to 0), \quad \text{(48)}$$

and

$$\sigma \sim \tau^{3 - 4\gamma} \left[1 + \frac{(\gamma - 1)}{\tau^{4\gamma}} + \cdots\right], \quad (\tau \to \infty). \quad \text{(49)}$$

On the other hand along the locus of the maxima and minima given by

$$\sigma = 2\tau^{3} \left[\sqrt{1 + \frac{1}{\tau^{4\gamma}}} - 1\right], \quad \text{(50)}$$

we have

$$\sigma \sim 2\tau^{3 - 2\gamma}, \quad (\tau \to 0), \quad \text{(51)}$$

$$\sigma \sim \tau^{3 - 4\gamma} \left[1 - \frac{1}{4\tau^{4\gamma}} + \cdots\right], \quad (\tau \to \infty). \quad \text{(52)}$$

From (48) and (51) it follows that (50) lies initially above the standard solution if

$$2 > \sqrt{\frac{2}{2 - \gamma}} \quad \text{or} \quad \gamma < \frac{3}{4}. \quad \text{(53)}$$

In the same way (50) lies above the standard solution eventually (i.e. as $\tau \to \infty$) if

$$\frac{1}{4} < (1 - \gamma) \quad \text{or} \quad \gamma < \frac{3}{4}. \quad \text{(54)}$$

Hence from (53) and (54) we conclude that (50) lies entirely above the standard solution if $\gamma \leq \frac{3}{4}$, while it lies entirely below the standard solution if $\gamma \geq \frac{3}{2}$. If $\frac{3}{4} < \gamma < \frac{3}{2}$ then (50) has an intersection with the standard solution, lying above the standard solution near the origin and lying below it as $\tau \to \infty$.

The discussion of the nature of the general solution of (8) is now simple.

Case a, $\gamma \leq \frac{3}{4}$. (i) All solutions below the standard solution start from the $\tau$-axis (vertically) at a finite $\tau$, monotonically increase becoming asymptotic to the standard solution.

(ii) Any solution above the standard solution has a singularity at the origin:

$$\sigma \sim \text{Constant} \quad \tau \quad (\tau \to 0).$$
These solutions have a minimum, and after reaching the minimum, monotonically increase converging towards the standard solution.

Case b, $\frac{3}{4} < \gamma < \frac{3}{2}$. The general arrangement of the solutions is the same as for the case $\gamma = 1$ which has been fully described in I.

(i) All solutions below the standard solution start (vertically) from the $\tau$-axis at a finite $\tau$ reach a maximum and decrease again becoming asymptotic to the standard solution.

(ii) Among the solutions above the standard solution there is just one which touches the locus (50) at its maximum, having an inflex at this point. Denote this solution by I.

(iii) Any solution between the standard solution and I has a minimum followed by a maximum.

(iv) Any solution above I are monotonically decreasing functions.

The behaviour near the origin of the solutions considered in (ii), (iii) and (iv) above is the same as in case (a), (ii).

Case c, $\gamma = \frac{3}{2}$. (i) The standard solution has a negative density gradient for all values of $\tau$ except at $\tau = 0$.

(ii) All solutions below the standard solution have the same character as Case (b) (i) above.

(iii) All solutions above the standard solution are characterized by negative density gradients for all values of $\tau$.

Case d, $2 > \gamma > \frac{3}{2}$. The arrangement of the solutions is the same as in Case (c) except that even the standard solution has a singularity at the origin:

$$\sigma \sim \sqrt{\frac{2}{2-\gamma}} \tau^{-(2\gamma-5)}.$$  (55)

Case e, $\gamma \geq 2$. The solutions have rather complicated behaviour at the origin and we shall not consider them in this paper.

This completes the formal discussion of the $\gamma$-models. We shall return to the physical aspect of the situation in section IV.

Section II: The $\alpha$-models.

§ 10. The $\alpha$-models were introduced in I as special cases of the general $(\alpha, \delta)$-models namely, when

$$\delta = 4 + \alpha + s.$$  (56)
As pointed out in § 1 a single assumption concerning the behaviour of \( \kappa \eta \) — namely that \( \kappa \eta \) varies as some power of \( \varrho T \) — would have been sufficient to derive the solution given in I namely [cf. I equation (92)],

\[
\frac{1}{q} = C - z - \int_{0}^{z} \frac{d z}{z^{\alpha + 1} - 1},
\]

where \( C \) is a constant of integration. The variables are

\[
\varrho = \varrho_0 \sigma; \quad T = T_0 \tau; \quad \sigma = z \varrho^{1/4}; \quad \tau = q^{-1/4},
\]

where \( \varrho_0 \) and \( T_0 \) are defined as follows:

\[
\varrho_0 = \frac{1}{3} a \frac{\mu H}{k} K^{1/2}; \quad T_0 = K^{1/2}.
\]

The \( K \) occurring in (59) is defined as in I(82).

The \( \alpha \)-models have the following fundamental characteristic:

*All the solutions are asymptotic to the hyperbola*

\[ \sigma \tau = 1. \]

\( \sigma \tau = 1 \) is in fact a singular solution of the appropriate \( \sigma, \tau \) differential equation.

In I we further derived that for the standard solution [for which \( C = 0 \) in (57)]

\[
\frac{1}{q} \sim \frac{1}{\alpha + 2} z^{\alpha + 2}, \quad (z \to 0),
\]

so that by (58)

\[
\tau \sim (\alpha + 2)^{-1/4} z^{1/4}, \quad (z \to 0),
\]

\[
\sigma \sim (\alpha + 2)^{1/2} z^{1/2}. \quad (z \to 0).
\]

From (62) and (62') we have that

\[
\sigma \sim (\alpha + 2)^{\frac{1}{2} - \alpha} \tau^{\frac{1}{2} + \alpha}.
\]

From (63) it follows that for \( \alpha = 2 \)

\[ \sigma = \sqrt{2} \quad \text{for} \quad \tau = 0 \quad (\alpha = 2). \]

Hence \( \alpha = 2 \) plays the same role for the \( \alpha \)-models as \( \gamma = \frac{3}{2} \) plays for the \( \gamma \)-models.
On a class of stellar Models.

The law for \( \eta \) corresponding to \( \alpha = 2 \) is
\[
\eta = \eta_0 \xi^2 T^{6 + \delta},
\]  
(65)

or
\[
x \eta = x_1 \eta_0 (\xi T)^3.
\]  
(65')

We notice that the exponent of \( T \) in (65) for which the \((\sigma, \tau)\) variations are such that along the standard solution, \( \sigma \) tends to a constant as \( T \to 0 \),

\[
\begin{align*}
\alpha &= -\frac{1}{2}: & \text{The Solution is} & 
\tau^4 = C - \left( \sigma \tau + 2 \sqrt{\sigma \tau + 2 \log (1 - \sqrt{\sigma \tau})} \right). \\
1 & \rightarrow \text{The Standard solution (} C = 0 \text{).} \\
6 & \rightarrow C = 0.1. \\
\alpha &= 0: & \text{The Solution is} & 
\tau^4 = C - \left( \sigma \tau + \log (1 - \sigma \tau) \right). \\
2 & \rightarrow \text{The Standard solution (} C = 0 \text{).} \\
7 & \rightarrow C = 0.1. \\
\alpha &= 1: & \text{The Solution is} & 
\tau^4 = C - \left( \sigma \tau + \frac{1}{2} \log \left( \frac{1 - \sigma \tau}{1 + \sigma \tau} \right) \right). \\
3 & \rightarrow \text{The Standard solution (} C = 0 \text{).} \\
8 & \rightarrow C = 0.1. \\
\alpha &= 2: & \text{The Solution is} & 
\tau^4 = C - \left( \sigma \tau + \frac{1}{4} \log \left( \frac{(\sigma \tau - 1)^2}{\sigma^2 \tau^2 + \sigma \tau + 1} \right) - \frac{1}{2} \tan^{-1} \frac{2 \sigma \tau + 1}{\sqrt{3}} + \frac{1}{2} \tan^{-1} \frac{1}{\sqrt{3}} \right). \\
4 & \rightarrow \text{The Standard solution (} C = 0 \text{).} \\
9 & \rightarrow C = 0.1. \\
\alpha &= 3: & \text{The Solution is} & 
\tau^4 = C - \left( \sigma \tau + \frac{1}{4} \log \left( \frac{1 - \sigma \tau}{1 + \sigma \tau} \right) \right) - \frac{1}{2} \tan^{-1} \sigma \tau. \\
5 & \rightarrow \text{The Standard solution (} C = 0 \text{).} \\
10 & \rightarrow C = 0.1. \\
\end{align*}
\]

is the same as that which occurred for a similar situation in the \( \gamma \)-models.

A more general result of this kind is proved in Section III.
When \( z > 2 \) then the standard solution has a singularity at the origin from (63) it follows that as \( z \to \infty \)

\[
\sigma \to \tau^{-1} \quad (\alpha \to \infty, \tau = 0)
\]  

(66)

(i.e.) the standard solution approaches the singular solution \( \sigma \tau = 1 \) as \( \alpha \) increases, indefinitely.

In I we gave the explicit solutions for \( \alpha = -\frac{1}{2}, 1, 2 \) and 3. The corresponding \((\sigma, \tau)\) variations are tabulated in Table 2. In Fig. 2 the results are graphically illustrated. A comparison of these results with those obtained on the basis of the \( \gamma \)-models brings out the remarkable similarity between them.

The description of the general arrangements of the solutions for the \( \alpha \)-models follows on lines similar to that for the \( \gamma \)-models with minor differences in detail arising from the fact that all the solutions approach the same singular solution \((\sigma \tau = 1)\) which is independent of \( \alpha \). We shall not go into the details here — the reader can easily fill in the gaps.

<table>
<thead>
<tr>
<th>Table 2. Standard Solutions for the ( \alpha )-models.</th>
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<tbody>
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*) For \( \alpha = -\frac{1}{2} \) the argument is not \( z \) by \( z^{1/2} \).
On a class of stellar Models.

Section III: The \((\alpha, \delta)\)-models.

§ 11. For these models we have

\[ \eta = \eta_0 \theta^\alpha T^\delta. \]  \hspace{1cm} (67)

The equations of the problem are given in I § 11. The discussion now hinges on the differential equation

\[ \gamma x y^{\alpha+1} \frac{dy}{dx} = y^{\alpha+1} (y + 1) - x, \]  \hspace{1cm} (68)

or its equivalent forms

\[ z^\alpha q^{\gamma+1} \frac{dz}{dq} = z^\alpha q^\gamma - q^\gamma, \]  \hspace{1cm} (69)

\[ z^\alpha \frac{dz}{dt} = t^{\gamma-\gamma} - z^\gamma, \]  \hspace{1cm} (70)

where

\[ \gamma = \frac{1}{4} (\delta + 3 \alpha - \theta); \quad \nu = \alpha + 1. \]  \hspace{1cm} (71)

The variables \(x, y; z, q\) or \(t\) are related to the physical variables \(\sigma\) and \(T\) as follows:

Let

\[ \sigma = \sigma_0 \sigma; \quad T = T_0 t \]  \hspace{1cm} (72)

where

\[ \sigma_0 = \frac{1}{3} a \frac{\mu H}{k} \frac{3}{K^{\frac{3}{\gamma}}}; \quad T_0 = \frac{1}{K^{\frac{1}{\gamma}}}. \]  \hspace{1cm} (73)

and

\[ K = \frac{4 \pi c G M}{\alpha \eta_0 L} \left[ -\frac{3 k}{a} \right] \frac{\gamma^\alpha + 1}{\mu H}. \]  \hspace{1cm} (74)

Then

\[ z = \sigma^3; \quad t = q^{3-1} = t^4, \]  \hspace{1cm} (75)

\[ y = \sigma^\gamma; \quad x = q^\gamma = t^{-4\gamma}. \]  \hspace{1cm} (76)

§ 12. By arguments similar to that of I § 14 we can show that the curve

\[ x = y^{\alpha+1} (1 + \frac{1}{4} y), \]  \hspace{1cm} (77)

divides the \((x, y)\) plane in such a way that for all points below the curve (77) we have positive density gradients while for all points above the curve (77) we have negative density gradients. It follows then that to determine the conditions under which we can have negative density gradients we have simply to determine the conditions under which a solution curve of (68) crosses (or lies entirely above) the curve (77).
§ 13. Consider the differential equation (68). If \( y \) is very large compared to unity we can write it approximately as
\[
\gamma x y^{\alpha + 1} \frac{dy}{dx} = y^{\alpha + 2} - x.
\]  
(78)

Put
\[
y^{\alpha + 2} = \varphi.
\]  
(79)

Then
\[
\frac{\gamma}{\alpha + 2} x \frac{d\varphi}{dx} = \varphi - x,
\]  
(80)

which is of the same form as our earlier (28'). The general solution of (78) then is
\[
y^{\alpha + 2} = \frac{\alpha + 2}{\alpha + 2 - \gamma} x + B x^{\gamma}, \quad (\alpha + 2 \neq \gamma),
\]  
(81)

\[
y^{\alpha + 2} = -x \log x + B x, \quad (\alpha + 2 = \gamma).
\]  
(82)

From (81) it follows that if \( \gamma < \alpha + 2 \) and if \( B \) were negative, then the analytic-continuation of (81) will intersect the \( x \)-axis, and hence such solutions enter the positive region. The limiting solution which intersects the \( x \)-axis at infinity (and this is our standard solution) clearly corresponds to
\[
y^{\alpha + 2} \sim \frac{\alpha + 2}{\alpha + 2 - \gamma} x. \quad (x \to \infty).
\]  
(83)

Comparing (77) and (83) we infer that if
\[
\frac{\alpha + 2}{\alpha + 2 - \gamma} < 4,
\]  
(84)

then the standard solution enters the positive region as \( x \) increases. (84) is equivalent to [by (71)]
\[
\delta < 6 + s.
\]  
(85)

When \( \delta = 6 + s \), then the standard solution touches the curve (77) at infinity. It is clear then that for \( \delta = 6 + s \) the solution which has no singularity at the origin tends to a finite value for \( \sigma \) as \( \tau \to 0 \).

Thus for all laws of the type
\[
\eta = \eta_0 \varrho^\alpha T^{\delta + s},
\]  
(86)

where \( \alpha \) is arbitrary\(^1\), the standard solution has a negative density gradient for all values of \( \tau \) except at \( \tau = 0 \) where \( \varrho \) takes a constant value.

\(^1\) We shall see that it is necessary that \( \alpha \neq -2 \).
On a class of stellar Models.

It is remarkable that the occurrence of the above "critical" solution should not depend upon the exponent of the density in the $\eta$ law. We shall prove a more general result of this kind in § 14. Meantime we notice that (81) and (82) in terms of $\sigma$ and $\tau$ take the forms

$$\sigma^{\alpha+2} = \frac{\alpha + 2}{\alpha + 2 - \gamma} \tau^{3(\alpha + 2) - 4\gamma} + \frac{B}{\tau^{\alpha + 2}}, \quad (\gamma = \alpha + 2), \quad (87)$$

$$\sigma^{\alpha+2} = \frac{4(\alpha + 2)}{\tau^{\alpha + 2}} (\log \tau + B), \quad (\gamma = \alpha + 2). \quad (88)$$

From (87) it follows that if $B > 0$ then the solution has a singularity at the origin:

$$\sigma \sim \frac{\text{Constant}}{\tau}, \quad (\tau \to 0). \quad (89)$$

We may notice that $\gamma = \alpha + 2$ is equivalent to the condition

$$\delta = \alpha + s + 8. \quad (90)$$

§ 14. From (70) and (75) we have

$$(\sigma \tau)^{\gamma - \eta} \frac{d(\sigma \tau)}{d(\tau^4)} = 1 - (\sigma \tau)^{\gamma - \eta}, \quad (91)$$

which simplifies into

$$\sigma^2 \tau^\delta \frac{d\sigma}{d\tau} + \sigma^{\alpha + 2} + 4\sigma^2 \tau^3 - 4\tau^{3(\alpha + 1) - 4\gamma} = 0. \quad (92)$$

Now

$$3(\gamma + 1) - 4\gamma = 3(\alpha + 2) - (\delta + 3\alpha - s) = 6 + s - \delta. \quad (93)$$

Hence (91) can be written as

$$\sigma^{\alpha + 1} \tau \frac{d\sigma}{d\tau} + \sigma^{\alpha + 2} + 4\sigma^2 \tau^3 - 4\tau^{3(\alpha + 1)} = 0. \quad (94)$$

Form (94) it follows that two critical cases arise, when $\delta = 3 + s$ and $\delta = 6 + s$. In the former case we can rewrite (94) as

$$\sigma^{\alpha + 1} \tau \frac{d\sigma}{d\tau} + \sigma^{\alpha + 2} + 4\tau^3 (\sigma^{\alpha + 1} - 1) = 0, \quad (95)$$

from which it is clear that along all solutions of (95)

$$\sigma \to 1 \quad \text{as} \quad \tau \to \infty. \quad (96)$$

The above result is independent of $\alpha$. If $\delta = 6 + s$, (94) can be written as

$$\sigma^{\alpha + 1} \tau \frac{d\sigma}{d\tau} + \sigma^{\alpha + 2} + 4\sigma^{\alpha + 1} \tau^3 - 4 = 0. \quad (97)$$
From (97) it follows that a solution of the equation which has no singularity at the origin must tend towards a finite value for $\sigma$ as $\tau \rightarrow 0$:

$$\sigma \rightarrow \frac{1}{4^2 + \delta}, \quad (\tau \rightarrow 0). \tag{98}$$

More generally we find that all solutions of (92) tend asymptotically towards a certain singular solution whose behaviour at infinity is determined by an asymptotic series similar to (22). The dominant term now is

$$\sigma \sim \tau^{3 - \frac{4\gamma}{\nu}} \quad (\tau \rightarrow \infty) \tag{99}$$
or expressing $\gamma$ and $\nu$ in terms of $\alpha$ and $\delta$ we have

$$\sigma \sim \tau^{\frac{3 + \varepsilon - \delta}{\alpha + 1}} \quad (\tau \rightarrow \infty). \tag{100}$$

In the same way for the standard solutions we have

$$\sigma \sim \left(\frac{\nu + 1}{\nu + 1 - \gamma}\right)^{\frac{1}{\alpha + 2}} \tau^{\frac{3 - \frac{4\gamma}{\nu + 1}}{\nu + 1}}, \quad (\tau \rightarrow 0), \tag{101}$$
or again expressing $\gamma$ and $\nu$ in terms of $\alpha$ and $\delta$, we have

$$\sigma \sim \left[\frac{4(\alpha + 2)}{\alpha + \delta + 8}\right]^{\frac{1}{\alpha + 2}} \tau^{\frac{3 + \varepsilon - \delta}{\alpha + 2}}, \quad (\tau \rightarrow 0). \tag{102}$$

From (100) and (102) we see that if

$$\eta = \eta_0 \varrho^\alpha T^{3 + \delta} \tag{103}$$

where $\alpha \neq -1$ but arbitrary otherwise, then along any solution of the appropriate $(\varrho, T)$ differential equation $\varrho$ tends to a constant value as $T \rightarrow \infty$.

Again, if

$$\eta = \eta_0 \varrho^\alpha T^{6 + \delta} \tag{104}$$

where $\alpha \neq -2$ but arbitrary otherwise, then a solution of the appropriate $(\varrho, T)$ differential equation which is free from singularity at the origin is a monotonic decreasing function of $\varrho$.

It is remarkable that the existence of the two critical cases (which is intuitively obvious) should not depend on the exponent to which the density occurs in the law for $\eta$. This result must have important astrophysical implications (cf. §§ 16, 17).

For the $\alpha$-models $\delta = 4 + \alpha + \varepsilon$ and (100) and (102) now respectively reduce to

$$\sigma \sim \tau^{-1}, \quad (\tau \rightarrow \infty), \tag{105}$$

$$\sigma \sim (\alpha + 2)^{\frac{1}{\alpha + 2}} \tau^{\frac{2 - \varepsilon}{\alpha + 2}}, \quad (\tau \rightarrow 0), \tag{106}$$

thus recovering our earlier results.
§ 15. \((-i, \delta)-Models\). Apart from the \(\alpha\)-models there is one other class of the \((\alpha, \delta)\)-models for which an explicit integration of the \((\sigma, \tau)\) differential equation is possible. This occurs when \(\alpha = -1\). For the corresponding law for \(\eta\) we have

\[
\kappa \eta = \kappa_1 \eta_0 \tau^{\delta - 3 - s}.
\]  

(107)

When \(\alpha = -1\), (94) reduces to

\[
\frac{d\sigma}{d\tau} + \sigma + 4 \tau^3 - 4 \tau^3 + s - \delta = 0,
\]  

(108)

which is easily integrated. We have

\[
\sigma = \frac{C}{\tau} - \tau^3 + \frac{4}{7 + s - \delta} \tau^{7 + s - \delta}, \quad (\delta \neq 7 + s),
\]  

(109)

\[
\sigma = \frac{C}{\tau} - \tau^3 + \frac{4 \log \tau}{\tau}, \quad (\delta = 7 + s),
\]  

(110)

where \(C\) is a constant of integration. A detailed discussion of the above equations does not seem to be of much interest.

Section IV: Physical Discussion.

§ 16. As stated in I the fundamental problem underlying these studies is to obtain some general information concerning the dependence of the possible character of the density-temperature relation-ships on the distribution of the energy sources — in particular to determine the circumstances under which negative density gradients become possible. It was conjectured in I that "for a given degree of the concentration of the energy sources towards the centre it should be possible to set a lower limit to the central radiation pressure such that a configuration with a higher value for the central radiation pressure should be characterized by a negative density gradient in the neighbourhood of the centre". We have now to examine as to in how far we can substantiate the above conjecture on the basis of our present results.

Firstly, it has to be noticed that the object of studying models based on the assumption

\[
\eta \propto T^\delta \quad \text{or} \quad \eta \propto \varrho^\alpha T^\delta
\]  

(111)

is simply that both the assumptions provide methods — even if some what special — of concentrating the energy sources towards the centre and consequently one might expect the general features to hold good qualitatively.
under less specialized circumstances. There is however one further reason why these models have a wider meaning. From the definitions

\[ \eta = \frac{L(r)}{M(r)} \frac{M}{L}; \quad \varepsilon = \frac{d L(r)}{d M(r)}, \]

(112)

where \( \varepsilon \) is the rate of liberation of energy per gram of the stellar material, we can derive that

\[ \varepsilon = \bar{\varepsilon} \eta \left[ 1 + \frac{d \log \eta}{d \log M(r)} \right] \]

(113)

where \( \bar{\varepsilon} = L/M \), is the mean rate of liberation of energy per gram of the star. (113) can be alternately written as

\[ \varepsilon = \bar{\varepsilon} \eta \left[ 1 + \frac{1}{8} r \frac{\bar{\varrho}(r)}{\varrho(r)} \frac{d \log \eta}{d r} \right], \]

(114)

where \( \bar{\varrho}(r) \) is the mean density interior to \( r \). For an assumed law of the type (111) we have from (114) that

\[ \varepsilon = \bar{\varepsilon} \eta \left[ 1 + \frac{1}{8} r \frac{\bar{\varrho}(r)}{\varrho} \left( \frac{\alpha}{\varrho} \frac{d \varrho}{d r} + \frac{\delta}{T} \frac{dT}{dr} \right) \right]. \]

(115)

The terms in the curly brackets in the above expression are negative and it is also clear that the right hand side of (115) will vanish for some value of \( r \) (say \( r^* \)) inside the configuration. For \( r \) greater than \( r^* \), (115) will give negative values for \( \varepsilon \). Consequently for models of the type we have been considering we should have to “break off” our solution at \( r^* \) and consider a point source envelope for \( r > r^* \). The significant feature, however, of (115) is that in the neighbourhood of the centre

\[ \varepsilon = \bar{\varepsilon} \eta + O \left( r^2 \right). \quad (r \to 0). \]

(116)

Hence in the immediate neighbourhood of the centre of the star, the assumption (111) becomes equivalent to an assumption of the same type for \( \varepsilon \). Consequently when we are concerned with the gradient \( d \varrho /dT \) at the centre, the results derived on the \( \eta \)-basis would be the same as those which would be derived on the \( \varepsilon \)-basis — the reason for this being that in (116) the “error-term” is of the second order in \( r \). Clearly, this has important consequences in the interpretation of the results.

§ 17. We have seen in I § 10 that the physically important solution for the case \( \gamma = 1 \) was the standard solution. The same arguments apply for the general case. The main reason, however being that any other solution rapidly converges towards the standard solution. We shall therefore
— in the first instance — consider only the standard solutions — the "modifications" arising from a consideration of the general solutions are taken up in §§ 18, 19, 20.

First of all we notice that on the basis of the $\gamma$-models, the standard solutions have no maxima for $\gamma \leq \frac{3}{4}$. This means that if $\eta$ did not increase more rapidly than $T^3 + s$ then we have positive density gradients for all values for the central radiation pressure. On the other hand if $\frac{3}{4} < \gamma < \frac{3}{2}$ the solution has a maximum. Let $\sigma^*$ and $\tau^*$ be the value of $\sigma$ and $\tau$ at the the maximum of the standard solution. Since one has quite generally that

$$1 - \beta = \frac{\tau^4}{\sigma \tau + \tau^4},$$

(117)

$\sigma^*$ and $\tau^*$ will define a certain value for $(1 - \beta)$ say, $(1 - \beta^*)$. It is now clear that if the central radiation pressure in a configuration (built on the model under consideration) be greater than $(1 - \beta^*)$ then we should have negative density gradients in the immediate neighbourhood of the centre.

The following table gives the values of $(1 - \beta^*)$ for the $\gamma$-models, for the values of $\gamma$ for which we have data.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$1 - \beta^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq \frac{3}{4}$</td>
<td>$3 + s$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$4 + s$</td>
<td>0.52</td>
</tr>
<tr>
<td>1.25</td>
<td>$5 + s$</td>
<td>0.23</td>
</tr>
<tr>
<td>1.5</td>
<td>$6 + s$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$1 - \beta^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$3 + s$</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$3.5 + s$</td>
<td>0.73</td>
</tr>
<tr>
<td>0</td>
<td>$4 + s$</td>
<td>0.52</td>
</tr>
<tr>
<td>1</td>
<td>$5 + s$</td>
<td>0.19</td>
</tr>
<tr>
<td>2</td>
<td>$6 + s$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4 gives similar results for the $\alpha$-models.

We see that the $\alpha$- and the $\gamma$-models both give very nearly the same variation of $(1 - \beta^*)$ with $\delta$.

From the remarks in § 16 it is clear that the above maximum values for $(1 - \beta)$ for the non-occurrence of negative density gradients continue to be true on the $\varepsilon$-basis — that is for assumptions of the type

$$\varepsilon \propto \rho^\alpha T^\delta,$$

— when we restrict our discussion to the immediate neighbourhood of the centre. More precisely the physical meaning of the results are that if the rate of generation of energy $\varepsilon$ varied according to the law

$$\varepsilon = \varepsilon_0 \rho^\alpha T^\delta$$

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then in the immediate neighbourhood of the centre we do not have negative density gradients provided \( \delta < 3 + s \) (\( \alpha \) can be arbitrary), where \( s \) is the exponent occurring in the opacity law,

\[
z = \kappa_1 T^{1-s}.
\]

In the same way, if \( 6 + s > \delta > 3 + s \) then for the occurrence of negative density gradients at the centre the central radiation pressure will have to exceed a certain limit \((1 - \beta^*)\) depending on \( \alpha \) and \( \delta \) — generally \((1 - \beta^*)\) decreases with increasing \( \delta \). If \( \delta \geq 6 + s \) then we always have negative density gradients\(^1\). Thus our analysis of the \( \alpha \) and the \( \gamma \) models completely proves the conjecture, — of course, only for the models we have considered.

\section{18.} In our discussion in \$ 17\) we restricted ourselves to the standard solution. If \( \delta > 6 + s \) then the only possibility for the occurrence of positive density gradients is to use the solutions which start vertically at the \( \tau \)-axis at some finite \( \tau = \tau_0 \) (say). But we see from an examination of Fig. 2 \((\alpha = 2, 3)\) that along these solutions the density increases extremely rapidly even if the temperature increases only very slightly. But after such a terrifically steep increase in density, the phenomenon of the negative density gradient occurs. Thus if \((1 - \beta) > (1 - \beta^*)\) and if the density gradient is positive then we must be in a region where the density increases extremely rapidly with temperature — so rapidly that we should very soon reach a point where the density gradient becomes negative.

\section{19.} If \( 6 + s > \delta > 3 + s \) then positive density gradients are possible for the solutions which tend to infinity along the \( \sigma \)-axis as \( \tau \to 0 \). But along these solutions the maximum \((1 - \beta)\) is less than the possible maximum \((1 - \beta^*)\) along the standard solution. The \textit{minimum} of the possible \textit{maximum} values of \((1 - \beta)\) along such solutions (before negative density gradients occur) would correspond to the maximum of the locus of the maxima and minima in the \((\sigma, \tau)\) plane. Let \((1 - \beta^*)\) denote the the minimum value of the possible maximum \((1 - \beta)\) along the different solutions \textit{above} the standard solution. Then for the \( \gamma \)-models — for instance—\((1 - \beta^*)\) would have to be calculated for \( \sigma \) and \( \tau \) given by equations (43) and (44). We find that

\[
(1 - \beta^*)_{\text{min}} = \frac{3 - 2 \gamma}{3(2 \gamma - 1)}, \quad (118)
\]

or in terms of \( \delta \)

\[
(1 - \beta^*)_{\text{min}} = \frac{6 + s - \delta}{3(\delta - s - 2)}. \quad (119)
\]

\(^1\) With the reservations mentioned in \$\$ 18, 19, 20.
It is of course clear that for the general ($\alpha$, $\delta$) models we can in the same way define $(1 - \beta^*)_{min}$. It is remarkable that the expression (119) should be true for the general ($\alpha$, $\delta$) model — (i.e., $(1 - \beta^*)_{min}$ does not depend on $\alpha$. The proof of this is as follows.

From (92) it follows that the locus of the maxima and minima is given by

$$\sigma^{v+1} + 4v^3 \tau^2 - 4\tau^{v+1} - 4\gamma = 0. \quad (120)$$

The maximum along the curve (120) is found by direct differentiation. It is found that at the maximum of (120) we have

$$\sigma^r = \frac{1}{3} [3v + 1 - 4\gamma] \tau^{v+1} - 4\gamma. \quad (121)$$

Substituting the above for the $\sigma^r$ occurring in the second term of (120) we have

$$\sigma^{v+1} = \frac{4}{3} (4\gamma - 3v) \tau^{v+1} - 4\gamma. \quad (122)$$

From (121) and (122) we find that

$$\sigma = \frac{4(4\gamma - 3v)}{3(v + 1) - 4\gamma} \tau^3. \quad (123)$$

Hence by (117) we have

$$(1 - \beta^*)_{min} = \frac{3(v + 1) - 4\gamma}{3(4\gamma - 3v + 1)}, \quad (124)$$

or using the definitions of $\gamma$ and $v$ in terms of $\alpha$ and $\delta$ [cf. equations (71)], (124) reduces to

$$(1 - \beta^*)_{min} = \frac{6 + s - \delta}{3(\delta - s - 2)}, \quad (125)$$

which is the same as (119). From (125) we have the following table.

<table>
<thead>
<tr>
<th>$\delta - s$</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
<th>5.5</th>
<th>6.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 - \beta^*)_{min}$</td>
<td>1</td>
<td>$5/9$</td>
<td>$1/3$</td>
<td>$1/5$</td>
<td>$1/9$</td>
<td>$1/21$</td>
<td>0</td>
</tr>
</tbody>
</table>

(125) again brings out the greater importance of the dependence of $\varepsilon$ on temperature (than on density) in determining the character of the density-temperature relationship. Indeed, certain aspects of the situation (like the ones we have just considered) depends only on how $\eta$ (or $\varepsilon$ in the neighbourhood of the centre) varies with temperature.
§ 20. Finally we have to consider the character of the situation arising from the solutions for the case \((3 + s < \delta < 6 + s)\) which start from the \(\tau\)-axis at a finite \(\tau\). The remarks in § 18 hold good now. Hence the general conclusion that (for the types of models considered) we can fix an upper limit \((1 - \beta^*)\) to the central radiation pressure \((1 - \beta_c)\) depending on the degree of the concentration of the energy sources towards the centre, such that if \((1 - \beta_c) > (1 - \beta^*)\) we would have negative density gradients in the immediate neighbourhood of the centre is valid, with however the reservation that if we have a positive density gradient even with a value for \((1 - \beta_c)\) appreciably different from and greater than \((1 - \beta^*)\) then the density gradient must be in general very steep indeed — the net result being the appearance of the negative density gradient after a slight increase in the temperature.

This completes our discussion of the first aspect of the problem. It still remains to investigate Rosseland’s suggestion as to how “a state of frequent collapsing of matter towards the centre of a star” is to be regarded as a possibility when negative density gradients arise. This will be the subject of a future communication.

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