THE PRESSURE IN THE INTERIOR OF A STAR.

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1. In a recent paper * Professor E. A. Milne has established by direct methods certain inequalities which should be true for stellar configurations in hydrostatic equilibrium. A fundamental theorem which is proved in Milne’s paper is the following:

If $P_c$ denotes the central pressure in any equilibrium configuration, and $P_1$ the pressure at a conventionally assigned boundary where the radius is $R$ and the mass enclosed is $M$, then

$$P_c > P + \frac{3}{8\pi} \frac{GM^2(r)}{r^4} > P_1 + \frac{3}{8\pi} \frac{GM^2}{R^4}. \quad (1)$$

In the above formula $M(r)$ denotes the mass enclosed inside a sphere of radius $r$. The inequalities expressed in (1) are true, provided $\bar{\rho}(r)$, the mean density inside $r$, always exceeds the actual density $\rho(r)$ at $r$.

2. The outer members in the above inequality yield

$$P_c > \frac{3}{8\pi} \frac{GM^2}{R^4}. \quad (2)$$

Formula (2) provides the minimum pressure at the centre of an equilibrium configuration of assigned mass and radius. But in certain stellar applica-

![Fig. 1.](image)

† the inequality which is required is in the opposite direction to (2), i.e. one needs an inequality which would give an upper bound to the central pressure in an equilibrium configuration of assigned mass and central density.

Essentially, inequalities of the kind proved by Milne arise from a "comparison" § of the given equilibrium configuration with another configuration of the same mass and radius but at a uniform density equal to the

† The outer members of (1) form an inequality which is the statement of a theorem due to Eddington.
§ The word "comparison" is not to be taken too literally.
mean density of the original configuration (see fig. 1; compare (a) and (b)). One gets a different set of inequalities by a corresponding "comparison" of the given configuration with another homogeneous configuration of the same mass but at a uniform density now equal to the central density in the original configuration (see fig. 1; compare (a) and (c)).

3. Theorem 1.—In any equilibrium configuration in which the mean density inside \( r \) decreases outwards we have the inequality

\[
\frac{1}{2}G\left(\frac{4}{3}\pi\right)^{1/3}\bar{\rho}^{4/3}(r)M^{2/3}(r) < P_c - P < \frac{1}{2}G\left(\frac{4}{3}\pi\right)^{1/3}\rho_c^{4/3}M^{2/3}(r),
\]

where \( \bar{\rho}(r) \) denotes the mean density inside \( r \), and \( \rho_c \) the central density, and \( P_c \) the central pressure.

Proof.—The equation of hydrostatic equilibrium is

\[
\frac{dP}{dr} = -\frac{GM(r)}{r^2}. 
\]

Integrating this we have

\[
P_c - P = G\int_0^r \frac{M(r)}{r^2} \rho dr,
\]

or since

\[
dM(r) = 4\pi r^2 \rho dr,
\]

we have

\[
P_c - P = \frac{G}{4\pi} \int_0^r \frac{M(r) dM(r)}{r^4},
\]

By definition

\[
\frac{4}{3}\pi r^3 \bar{\rho}(r) = M(r).
\]

Hence

\[
r^4 = \left[ \frac{M(r)}{\frac{4}{3}\pi \bar{\rho}(r)} \right]^{4/3}.
\]

Substituting (9) in (7) we have

\[
P_c - P = \frac{1}{4\pi} \left(\frac{4}{3}\pi\right)^{4/3} G \int_0^r \bar{\rho}^{4/3}(r) M^{-1/3}(r) dM(r).
\]

Since by hypothesis \( \bar{\rho}(r) \) decreases outwards we have

\[
P_c - P < \frac{1}{4\pi} \left(\frac{4}{3}\pi\right)^{4/3} G \rho_c^{4/3} \int_0^r M^{-1/3}(r) dM(r),
\]

\[
< \frac{1}{2} G\left(\frac{4}{3}\pi\right)^{1/3}\rho_c^{4/3}M^{2/3}(r),
\]

Again from (10) we have

\[
P_c - P > \frac{1}{4\pi} \left(\frac{4}{3}\pi\right)^{4/3} G \bar{\rho}^{4/3}(r) \int_0^r M^{-1/3}(r) dM(r),
\]

\[
> \frac{1}{2} G\left(\frac{4}{3}\pi\right)^{1/3}\bar{\rho}^{4/3}(r)M^{2/3}(r).
\]

Combining (12) and (14) we have the required result.

Corollary 1.—If we put \( r = R \) in (12) and (14) we have

\[
\frac{1}{2} G\left(\frac{4}{3}\pi\right)^{1/3}\rho_c^{4/3}M^{2/3} \leq P_c - P_1 \leq \frac{1}{2} G\left(\frac{4}{3}\pi\right)^{1/3}\rho_c^{4/3}M^{2/3},
\]

where \( P_1 \) is the boundary pressure.
Corollary 2. — If \((1 - \beta_c)\) is the ratio of the radiation pressure to the total pressure at the centre of a wholly gaseous configuration satisfying the conditions of theorem 1, then

\[1 - \beta_c < 1 - \beta^*,\]  

(16)

where \((1 - \beta^*)\) satisfies the quartic equation

\[M = \left( \frac{6}{\pi} \right)^{1/2} \left[ \left( \frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta^*}{\beta^{*4}} \right]^{1/2} \frac{1}{G^{3/2}} \]  

(17)

In terms of \((1 - \beta_c)\) the central pressure \(P_c\) is given by

\[P_c = \left[ \left( \frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta_c}{\beta_c^4} \right]^{1/3} \rho_c^{4/3}.\]  

(18)

Comparing (18) with the inequality on the right-hand side of (15) we have

\[M^{2/3} > \left( \frac{6}{\pi} \right)^{1/3} \left[ \left( \frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta_c}{\beta_c^4} \right]^{1/3} \frac{1}{G}.\]  

(19)

From (19) the required inequality (16) follows, since \((1 - \beta) / \beta^4\) is a monotonically increasing function of \(1 - \beta\).

In the same way one can prove that

\[\frac{1}{G^{3/2}} \left( \frac{6}{\pi} \right)^{1/2} \left[ \left( \frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta_c}{\beta_c^4} \right]^{1/2} \geq M \cdot \left( \frac{\rho_c}{\rho_c} \right)^2.\]  

(19')

4. We shall now establish certain inequalities similar to those in §§ 5, 6 in Milne's paper.

Theorem 2. — If \(I_v\) is the integral defined by

\[I_v = \frac{\int_0^R GM(r)dM(r)}{r^v}, \quad (v < 6),\]  

(20)

then under the conditions of Theorem 1

\[\frac{3GM^2}{6 - v} \leq I_v \leq \frac{3GM^2}{6 - v} \xi^v,\]  

(21)

where \(\xi\) is defined by the equation

\[\frac{4}{3} \pi \xi^3 \rho_c = M.\]  

(22)

(The first part of the inequality (21) is proved by a different method by Milne (loc. cit., § 5).)

Proof. — We have

\[I_v = G \frac{\int_0^R M(r)dM(r)}{r^v},\]  

\[= G \left( \frac{\xi}{\rho_c} \right)^{v/3} \int_0^R \rho^{v/3}(r)M^{(3-v)/3}(r)dM(r),\]  

(23)

\[\leq G \left( \frac{\xi}{\rho_c} \right)^{v/3} \int_0^R M^{(3-v)/3}(r)dM(r),\]  

\[\leq \frac{3}{6 - v} G \left( \frac{\xi}{\rho_c} \right)^{v/3} M^{(6-v)/3},\]  

\[\leq \frac{3}{6 - v} G \left( \frac{\xi}{\rho_c} \right)^{v/3} M^{(6-v)/3}.\]  

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