which is precisely our earlier expression (2) and therefore leads to the same final result.

3. Since Eddington prefers to work with standing waves, it may be mentioned here that we could just as well have taken for the set of orthogonal functions the following set of standing waves instead of our $\psi_{p,s}$:

$$
\phi_{p,s} = \frac{1}{\sqrt{2}} (\psi_{p,s} + \psi_{-p,s})
$$

$$
\phi_{-p,s} = \frac{1}{\sqrt{2}} (\psi_{p,s} - \psi_{-p,s})
$$

(16)

It is easily verified that (12) continues to be true if the $\psi_{p,s}$'s are replaced by the $\phi_{p,s}$'s. Thus (15) continues to be true even if the fundamental set of orthogonal functions were taken to correspond to standing waves.

4. In conclusion we wish to state that we do not intend this note as a reply in any sense to Eddington's papers. We thought it of interest, however, to point out that, starting with the energy-stress tensor as is defined in relativistic quantum mechanics and following Eddington's own procedure for calculating the pressure, we are simply led back to the relation between $P$ and $N$ one had earlier derived from (2) by directly inserting in it the relation between $E$ and $p$ given by relativistic mechanics.


STELLAR CONFIGURATIONS WITH DEGENERATE CORES.

(SECOND PAPER.)

S. Chandrasekhar, Ph.D.

1. In a previous communication * the general problems of stellar structure as they present themselves on the standard model were rediscussed, using the exact relativistic equation of state to describe degenerate matter.† The method developed in I is, however, quite general and consists essentially in relating the completely degenerate gas spheres governed by the differential equation

$$
\frac{1}{\eta^2} \frac{d}{d\eta} \left( \eta^2 \frac{d\phi}{d\eta} \right) = - \left( \phi^2 - \frac{I}{y_0^2} \right)^{3/2}
$$

(1) ‡

* M.N., 95, 226–260, 1935. This paper will be referred to as I.
† In a recent paper, M.N., 95, 297, 1935, Eddington has questioned the validity of the relativistic equation of state for degenerate matter which is still generally accepted. There are, however, grounds for not abandoning the accepted form of the equation of state—the arguments are presented in the preceding paper by Dr. Christian Møller and the writer.
‡ This equation was established in the author's paper, M.N., 95, 207–226, 1935. This paper will be referred to as H.C. II. The earlier paper, M.N., 91, 456, 1931, will be referred to as H.C. I.
with the wholly gaseous configurations. Since on the standard model approximation for the gaseous configurations the ratio \((1 - \beta_1)\) of the radiation pressure to the total pressure is a function of the mass only, the study of the curves of constant mass in the \((R, 1 - \beta_1)\)-diagram allows a convenient approach to the problem. In this diagram wholly gaseous configurations are represented by lines parallel to the \(R\)-axis, while the completely degenerate configurations are represented by points on the \(R\)-axis. The relation between these two sets of configurations was obtained by starting with a wholly gaseous configuration of prescribed mass and infinite extension and slowly contracting it and considering whether deviations from perfect gas laws towards degeneracy set in at all and if so when. In this way a domain of degeneracy in the \((R, 1 - \beta_1)\)-diagram was defined in which the configurations must be composite. To fix the precise nature of the curves of constant mass in the domain of degeneracy one requires further assumptions regarding the opacity of the degenerate core, but the problem of relating the degenerate spheres with the gaseous configurations was in principle solved.

2. But the discussion in I was incomplete in so far as the explicit appearance of the physically important parameter, namely, the luminosity \(L\) was suppressed by the use of \((1 - \beta_1)\) as the main variable. To gain further physical insight it is necessary therefore to transform the discussion of the curves of constant mass in the \((R, 1 - \beta_1)\)-plane to a discussion of the curves of constant mass in the \((\log L, \log R)\)-plane. This is done in Section I of this paper.

3. To complete the discussion we shall have to verify that the general results are not dependent on the very special nature of the model on which they have been obtained. As Jeans has more than once emphasised,* considerable caution is required in interpreting results based on stellar models which make gaseous configurations Emden polytropes of index 3. The more general analysis in which \((1 - \beta_1)\) was allowed to vary through the configuration was provided by Jeans.† It follows from his analysis that for fairly general stellar models the ratio \((1 - \beta_1)\) of the radiation pressure to the total pressure at the centre of the configuration is a function of the mass only and is independent of the radius. It is therefore clear that the whole discussion of I (especially that in Section I of that paper) can now be repeated on this more general analysis by considering the curves of constant mass in the \((R, 1 - \beta_1)\)-plane. In this plane the gaseous configurations are represented by lines parallel to the \(R\)-axis, and the relation between these “Emden-Jeans” polytropes to the completely degenerate configurations can be examined as before. This is done in Section II of this paper.

4. Lastly, in Section III various miscellaneous problems which arise are briefly considered.

* Astronomy and Cosmogony (Cambridge), chap. iii. Also M.N., 85, 201, 1925.
† Astronomy and Cosmogony (Cambridge), §§ 78–86, 88–92.
Section I.

5. The Physical Variables.—As shown in I, equation (55), Eddington’s quartic equation can be written in the form

\[ M = M_3 \left( \frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/2}, \]  

(2)

where

\[ M_3 = -4\pi \left( \frac{2 A_2}{G} \right)^{3/2} \frac{I}{B^3} \left( \frac{\xi^2 d\xi}{d\xi} \right), \]  

(3)

where \( A_2, B \) and the other symbols have the same meaning as in I. \( M_3 \) of course represents the upper limit to the mass of a completely degenerate configuration.

If \( R \) is the radius of the configuration, then one easily finds that the central density \( \rho_0 \) is given by

\[ \rho_0 = B \cdot \frac{M \left( \frac{l}{l_\infty} \right)^3}{M_3}, \]  

(4)

where \( l \) is the unit of length introduced in I, equation (33), namely,

\[ l = \left( \frac{2 A_2}{\pi G} \right)^{1/2} \frac{\xi_1(\theta_0)}{B}. \]  

(5)

Again the central temperature \( T_0 \) of the configuration can be determined from the equation

\[ T_0 = -\frac{\beta_1 \mu H}{4k} \frac{GM}{R(\xi_1(\theta_0))}. \]  

(6)*

Using (2) and (3), (6) can be rewritten as

\[ 4kT_0 \frac{M l}{mc^2} = M_3 R \beta_1. \]  

(7)

If \( M^*, R^*, \rho^*, T^* \) denote the mass, the radius, the density and the temperature when expressed in units of \( M_3, l, B \) and \( (mc^2/4k) \) respectively then we have

\[ M^* = \left( \frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/2}; \quad \rho_0^* = \frac{M^*}{R^*}; \quad T_0^* = \frac{M^*}{R^* \beta_1}. \]  

(8)

Also we notice the relation

\[ \frac{T_0^{*3}}{\rho_0^*} = \frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}. \]  

(9)

6. Luminosity.—We start with the equation †

\[ L = \frac{4\pi c GM(1 - \beta_1)}{a \kappa c}, \]  

(10)

---

* See, for instance, Milne, Handbuch der Astrophysik, Band III/1, p. 209.
where $\kappa_c$ is the central opacity. We shall assume that

$$\kappa = \kappa_1 \frac{\rho^*}{T^{8/7} \eta^2} \tag{11}$$

where $\kappa_1$ is the opacity at $\rho = B$ and $T = me^2/4k$.

From (8), (9), (10) and (11) we derive that

$$L = \frac{4\pi c GM_3}{a\kappa_1}\left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}\right)^{7/4} \beta_1^{7/2}(1 - \beta_1)R^{*-1/2}. \tag{12}$$

If we now introduce the unit of luminosity $L_1$ defined by

$$L_1 = \frac{4\pi c GM_3}{a\kappa_1}, \tag{13}$$

then (12) can be rewritten in the form

$$L^* = (M^*\beta_1)^{7/2}(1 - \beta_1)R^{*-1/2}, \tag{14}$$

where $L^*$ is used to denote the luminosity expressed in units of $L_1$. If we further introduce the quantity $L^*(\beta_1)$ defined by

$$L^*(\beta_1) = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}\right)^{7/4} \beta_1^{7/2}(1 - \beta_1), \tag{15}$$

then we have from (14) that

$$\log L^* = \log L^*(\beta_1) - \frac{1}{2} \log R^*. \tag{16}$$

The first term on the right-hand side of (16) is a function of the mass only, and hence the curves of constant mass in the $(\log L^*, \log R^*)$ diagram are straight lines.

7. The Domain of Degeneracy in the $(\log L^*, \log R^*)$-Diagram.—In I, § 5, we showed at what stage a configuration of a prescribed mass less than $\mathcal{M}$ (contracting from infinite extension) would “just begin to develop degeneracy” * at the centre. This occurs when the radius $R_0^*$ of the configuration is given by (I, equation (34))

$$R_0^* = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}\right)^{1/6} \frac{1}{x_0(\beta_1)} \tag{17}$$

where $x_0(\beta_1)$ is such that

$$f(x_0) = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}\right)^{1/3} \tag{18}$$

This value of $R_0^*$ substituted in (12) defines the corresponding luminosity $L_0^*$:

$$L_0^* = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4}\right)^{5/3} \beta_1^{7/2}(1 - \beta_1)(x_0(\beta_1))^{-1/2}. \tag{19}$$

(17) and (19) together define a curve in the $(\log L^*, \log R^*)$-plane,

* What is here meant by “degeneracy just beginning to develop” is stated on p. 229 of my last paper (I).
corresponding to the \((R_0^*, 1 - \beta_1)\)-curve in the \((R^*, 1 - \beta_1)\)-plane. Thus for any given mass less than \(\mathcal{M}\) the intersection of the line
\[
\log L^* = \log L^*(\beta^*) - \frac{1}{2} \log R^*
\]
with the \((\log L_0^*, \log R_0^*)\)-curve defines the stage at which degeneracy would just begin to develop at the centre. In (20), \(\beta^*\) represents the value \(\beta_1\) has for the prescribed mass in the wholly gaseous state. In Table I a set of corresponding pairs of values for \(\log L_0^*\) and \(\log R_0^*\) is given and the corresponding locus is shown in fig. 1. The part of the plane below this curve defines our domain of degeneracy in this plane.

<table>
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<th>(\alpha)</th>
<th>(1 - \beta_1)</th>
<th>(\log L_0^*)</th>
<th>(\log R_0^*)</th>
<th>(\alpha)</th>
<th>(1 - \beta_1)</th>
<th>(\log L_0^*)</th>
<th>(\log R_0^*)</th>
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<td>(+\infty)</td>
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It is of course clear that the \((\log L_0^*, \log R_0^*)\)-curve asymptotically approaches the line
\[
\log L^* = \log L^*(\beta_\Omega) - \frac{1}{2} \log R^* = 1.0378 - \frac{1}{2} \log R^*.
\]
where, as in I, equation (15), \(\beta_\Omega\) is such that
\[
\frac{g60}{\pi^4} \frac{1 - \beta_\Omega}{\beta_\Omega} = 1.
\]
That the \((\log L_0^*, \log R_0^*)\)-curve asymptotically approaches the line (21) simply corresponds to the fact that \(\mathcal{M}\) represents the upper limit to the masses for which degeneracy can set in on contraction.

When \(M \ll M_8\) one easily obtains the asymptotic relation (cf. I, equation 48)
\[
L_0^* \sim \frac{\pi^4}{g60} \left(\frac{4}{5}\right)^{33/2} R_0^*^{-17},
\]
or
\[
\log L_0^* = 3.4073 - 17 \log R_0^*, \quad (M^* \rightarrow 0).
\]
The line (24) is also shown in fig. 1.
—Consider a mass less than $M_3$. Then for this mass there exists an equili-

![Diagram of stellar configurations with degenerate cores]

**Fig. 1.** — The nature of the curves of constant mass in the $(\log L^*, \log R^*)$-plane on the usual standard model.

For a general description of the results summarized in the above diagram, see 8 and also 23. On the generalized standard model the system of the curves will look slightly different: thus the continuation of the curves of constant mass marked (11 . . . 14) in the perfect gas region will all tend asymptotically to the line $\text{MA}$. But the continuation of the curves of constant mass for those marked (1 . . . 10) in the perfect gas region must on all models eventually become asymptotic to the lines (1 . . . 10) in the domain of degeneracy (which is the region below the $(\log L^*, \log R^*)$-curve).

brium configuration in which it is completely degenerate and has a radius $R_1^*$ given by (cf. I, equation (46))

$$R_1^* = \frac{I}{y_0(M)} \frac{\eta_1(\psi_0(M))}{\xi_1(\theta_0)}.$$  \hspace{1cm} (25)

Hence if we start with this mass and contract it from infinite extension,
then its luminosity increases according to (20) till this line intersects the 
\((\log L_0^*, \log R_0^*)\) locus. On further contraction the configuration develops
a degenerate core of finite dimensions and the luminosity must ultimately
decrease, and as \(L^* \to 0\) the curve must tend asymptotically to

\[
\log R^* = \log R_1^* = \text{constant}. \tag{26}
\]

In H.C. II we have obtained the values of \(\eta_1, M^*, \) etc. for ten different
values of \(y_0, \) and for these configurations the values of \(\log R_1^*\) and \(\log L^*(\beta^*)\)
can be evaluated. The results of such calculations are given in Table II.
The corresponding lines are shown in fig. 1 (the lines marked 1 to 10 in
the domain of degeneracy and also in the perfect gas region).

The precise nature of the curves of constant mass in the domain of
degeneracy will depend on the assumptions one makes regarding the opacity
of the degenerate core. We shall indicate the qualitative results for the
two extreme models \((\beta_1 = \beta_2 \) and \(\beta_2 = 1)\) discussed in I, §§ 18–21.

(A) The Usual Standard Model \((\beta_1 = \beta_2)\).—(i) For \(M < M_3,\) since the
composite configurations are all of the collapsed type, it is clear that as
soon as the configuration begins to develop degeneracy at the centre the luminosity should begin to increase less rapidly than \(R^{-\frac{1}{2}}.\) The curves of
constant mass must therefore be of the nature shown by the dotted curves in
fig. 1.

\[
\begin{array}{|c|c|c|c|}
\hline
\frac{1}{y_0^2} & \frac{M}{M_3} & \log L^*(\beta_1) & \log R_1^* \\
\hline
0 & 1.0 & 2.7543 & -\infty \\
0.01 & 0.95733 & 2.6670 & 2.8903 \\
0.02 & 0.92419 & 2.5957 & 1.0096 \\
0.05 & 0.84709 & 2.4172 & 1.1602 \\
0.1 & 0.7543 & 2.1810 & 1.2708 \\
0.2 & 0.61389 & 2.7374 & 1.3832 \\
0.3 & 0.51218 & 3.3286 & 1.4538 \\
0.4 & 0.42660 & 4.9116 & 1.5095 \\
0.5 & 0.35933 & 4.4620 & 1.5590 \\
0.6 & 0.28137 & 5.9321 & 1.6072 \\
0.8 & 0.15316 & 6.5173 & 1.7198 \\
\hline
\end{array}
\]

(ii) The curve for \(M = M_3\) bends inwards as it enters the domain of
degeneracy and goes to \((-\infty, -\infty).\)

(iii) For \(M_3 < M < M_8\) the luminosity initially increases less rapidly than
\(R^{-1/2}\) on entering the domain of degeneracy, but should again ultimately
increase as \(R^{-1/2},\) since for these masses the curves of constant mass should
be asymptotic to the line

\[
\log L^* = \log L^*(\beta^*) - \frac{1}{2} \log R^*, \tag{27}
\]
where $\beta^*$ is related to $\beta^t$ (the value $\beta_1$ has in the wholly gaseous state) by the equation (cf. I, equation (56)),

$$\beta^* = \left( \frac{\pi^4}{960} \frac{\beta^t \gamma}{1 - \beta^t} \right)^{1/3}. \quad (28)$$

In fig. 1 this feature of the usual standard model is indicated.

(B) The Generalized Standard Model ($\beta_2 = 1$). — (i) For $M < M_a$ ($\sim 0.87 M_8$) the composite configurations are all of the collapsed type and the qualitative nature of the curves of constant mass for these masses must therefore be of the same nature as A (i) above.

(ii) For $M_a < M < M_3$, the composite configurations are initially of the centrally condensed type, and hence for these masses the luminosity will begin to increase more rapidly than $R^{-1/2}$ on developing degeneracy at the centre. However, the luminosity must begin to decrease after attaining a certain maximum, since eventually the configurations must tend towards the completely degenerate state.

(iii) For $M_3 < M < M_B$ the luminosity increases more rapidly than $R^{-1/2}$ in the domain of degeneracy and the curves of constant mass for all these masses must asymptotically tend to the line (cf. equations (21), (22))

$$\log L^* = \log L^*(\beta_\omega) - \frac{1}{2} \log R^*. \quad (29)$$

9. So far we have restricted ourselves to the standard model approximation, and in relating the completely degenerate configurations with $M < M_3$ with the wholly gaseous configurations we have seen how such a stellar mass tends to the completely degenerate state when the luminosity tends to zero through a sequence of equilibrium configurations all conforming to the standard model. In addition to $M_3$ there appeared another mass $\mathfrak{m}$ which played an important rôle in the theory. $\mathfrak{m}$ was initially defined as one for which $\beta^t = \beta_\omega$ (cf. I, equations (15), (29)), but the relation between $M_3$ and $\mathfrak{m}$, namely (I, equation (52)),

$$\mathfrak{m} = M_3 \beta_\omega^{-3/2}, \quad (30)$$

merely means that the existence of an upper limit $M_3$ to the mass of a completely degenerate configuration and the upper limit $\mathfrak{m}$ to the mass of a configuration for which degeneracy can at all set in on contraction are closely related to one another. Thus configurations in the mass range $M_3 < M < \mathfrak{m}$ bridge the gap between masses for which we have equilibrium configurations with zero luminosity and those which cannot develop degenerate cores however far the contraction may proceed.

The existence of a mass $\mathfrak{m}$ is by no means surprising, for, as we have already emphasised in I, § 22, the increased dominance of the radiation pressure for large stellar masses is quite a general result,* and the possibility of degeneracy is entirely excluded if only the radiation pressure is greater than a tenth of the total pressure throughout the entire mass. Hence it is clear that the general features that can be inferred from fig. 1 of this paper,

* An elementary proof of this result is given in the author's report in Nordisk Astronomisk Tidsskrift, 16, 37, 1935.
for instance, should to a large extent be independent of the model on the
basis of which the discussion has been carried out. It is, however, of interest
to verify that this is so by discussing the relation between the completely
degenerate configurations and the wholly gaseous configurations on the basis
of a more general scheme than what the standard model provides. As the
analysis required for this verification is given in Section II below, we shall
postpone to Section III some general considerations which arise from a
closer examination of fig. 1.

Section II

10. The starting-point of our present discussion is provided by Jeans's
investigations on gaseous configurations in which "1 - \( \beta_1 \)" is allowed to
vary. We shall briefly recapitulate Jeans's analysis in our present notation.

It is clear, of course, that \( (1 - \beta_1) \) must decrease inwards but the precise
law of variation will depend on various factors. If one assumes that
\[
\frac{ML(r)}{LM(r)} \propto T^3,
\]
and that further the coefficient of opacity varies according to the law
\[
\kappa \propto \frac{\rho}{T^{2/3}},
\]
then one can easily show that to a fair degree of approximation we have
\[
\frac{\beta_1}{(1 - \beta_1)^2} = \frac{\beta_c}{(1 - \beta_c)} \left( \frac{T}{T_0} \right)^{(4 - \delta)},
\]
where \( \beta_1 \) as usual defines the ratio of the gas pressure to the total pressure,
and \( \beta_c \) the value of \( \beta_1 \) at the centre of the configuration where the temperature
is assumed to be equal to \( T_0 \). Equation (33) is valid for layers not imme-
diately near the surface. Further, we have quite generally that
\[
\frac{\rho}{\rho_0} = \frac{\beta_1}{1 - \beta_1} \frac{T^{3/2}}{T_0},
\]
From (33) and (34) we deduce that
\[
\frac{\beta_1}{(1 - \beta_1)^2} \left( \frac{\beta_1}{1 - \beta_1} \right)^{4(4 - \delta)} = \frac{\beta_c}{(1 - \beta_c)^2} \left( \frac{\beta_c}{1 - \beta_c} \right)^{4(4 - \delta)} \left( \frac{\rho}{\rho_0} \right)^{4(4 - \delta)}.
\]
From (35) we obtain that
\[
[(1 + \beta_1) + \frac{4}{3}(4 - \delta)] \frac{d\beta_1}{\beta_1(1 - \beta_1)} = \frac{4}{3}(4 - \delta) \frac{d\rho}{\rho}.
\]
Since, however, the total pressure \( P \) is given,
\[
P = \left[ \frac{\hbar}{\mu H} \frac{3}{a} \frac{1 - \beta_1}{\beta_1^4} \right]^{1/3} \rho^{4/3},
\]
* This equation is due to Jeans and Woltjer.
we have (assuming $\mu$ constant *)

$$\frac{dP}{P} = -\frac{1}{3} \frac{4 - 3\beta_1}{\beta_1(1 - \beta_1)} \frac{d\beta_1}{3} + \frac{4}{3} \frac{4\rho}{\rho}; \quad (38)$$

or using (36),

$$\frac{dP}{P} = \frac{1}{3} \left[ 4 - \frac{(\frac{1}{2} - \delta)(4 - 3\beta_1)}{3(1 + \beta_1) + (\frac{1}{2} - \delta)} \right] \frac{d\rho}{\rho}. \quad (39)$$

On the other hand, if

$$P = K \rho^{\frac{1}{n} + \frac{1}{n}} \quad (40)$$

we should have

$$\frac{dP}{P} = \left( 1 + \frac{1}{n} \right) \frac{d\rho}{\rho}. \quad (41)$$

Comparing (39) and (41) we have for the “effective polytropic index” $n$ the expression

$$n = 3 + (1 - 2\delta) \frac{4 - 3\beta_1}{1 + 3\beta_1 + 2\delta(1 - \beta_1)}. \quad (42)$$

Equation (42) shows how the effective polytropic index $n$ varies through the star. (With $\delta = \frac{1}{2}$, $n = 3$ = constant, and we go back to the standard model.) However, Jeans considers that a fair approximation is obtained by regarding the whole configuration as a complete Emden polytrope with an index $n_{\beta_e}$ given by

$$n_{\beta_e} = 3 + (1 - 2\delta) \frac{4 - 3\beta_e}{1 + 3\beta_e + 2\delta(1 - \beta_e)}. \quad (43)$$

II. If one assumes (43), then (37) can be rewritten as ($n = n_{\beta_e}$),

$$P = \left( \frac{k}{\mu H/ \alpha \beta_e^4} \right)^{1/3} \frac{1}{\rho_0^{(3-n)/(3n)^{1}}}, \quad (44)$$

Or in terms of our $A_2$ and $B$ we have

$$P = \frac{2A_2}{B^{1/3} \beta_e^4} \frac{1}{\pi^4} \rho_0^{(3-n)/(3n)^{1}}. \quad (45)$$

The justification for (44) is simply that it gives the same initial variation of $P$ with $\rho$ at the centre as is jointly predicted by (35) and (37) taken together.

With (45) as the "equation of state" the structure of the configuration is completely specified by the Emden function $\theta_{n_{\beta_e}}$ with index $n_{\beta_e}$. We easily find that the mass of the configuration is given by

$$M = M_3 \left( \frac{960}{\pi^4} \frac{1 - \beta_e}{\beta_e^4} \right)^{1/2} \mathcal{F}(n_{\beta_e}), \quad (46)$$

where

$$\mathcal{F}(n) = \left( \frac{n + 1}{4} \right)^{3/2} \left( \frac{\epsilon^2 \theta_n}{\epsilon^2 \theta_n} \right) \frac{1}{1}. \quad (47)$$

* Variation of $\mu$ according to the law $\mu a T^4$ can easily be taken into account, but we shall not consider these refinements here.
If $\delta = 1/2$, $n_{\beta c} = 3$ and $J_M(3) = 1$, and (46) reduces to our earlier equation (2). Hence we can rewrite (46) as

$$M(\delta)(\beta_c) = M_{(1/2)}(\beta_c) \cdot J_M(n_{\beta c}),$$  \hspace{1cm} (48)$$

in an obvious notation.

12. The Domain of Degeneracy in the $(R, 1 - \beta_c)$-diagram.—For a given mass $M$ equations (43), (46) and (47) uniquely determine a $\beta_c$. In particular there will be a mass for which $(1 - \beta_c) = (1 - \beta_c o)$. We shall denote this mass by $\mathfrak{M}(\delta)$. By (46)

$$\mathfrak{M}(\delta) = M_{3\beta_c^o - 3/2}J_M(n_{\beta o}) = \mathfrak{M}(1/2)J_M(n_{\beta o}),$$  \hspace{1cm} (49)*$$

where

$$n_{\beta o} = 3 + (1 - 2\delta) \frac{4 - 3\beta_c}{1 + 3\beta_c + 2\delta(1 - \beta_c o)}.$$  \hspace{1cm} (50)$$

Arguing as in I, § 5, we now see that all configurations with $M > \mathfrak{M}(\delta)$ are necessarily wholly gaseous, and that therefore for these masses the curves of constant mass in the $(R, 1 - \beta_c)$-diagram are fully represented by the lines parallel to the $R$-axis. However, for $M < \mathfrak{M}(\delta)$ degeneracy would begin to develop when the central density is given by

$$\rho_0 = Bx_0^3,$$  \hspace{1cm} (51)$$

where $x_0$ is such that

$$f(x_0) = \frac{960}{\pi^4} \frac{1 - \beta_c}{\beta_c} \left( \frac{\rho_0}{\theta_{n\beta c}} \right)^{1/3}.$$  \hspace{1cm} (52)$$

The radius $R_0(\beta_c ; \delta)$ of this configuration can be determined as in I, § 5, and we find that

$$R_0(\beta_c ; \delta) = \left( \frac{n + 1}{4} \right)^{1/3} 960 \frac{1 - \beta_c}{\beta_c^4} \frac{1}{\pi^4} \frac{\xi_1(\theta_{n\beta c})}{\xi_1(\theta_{3})}.$$  \hspace{1cm} (53)$$

Comparing this with I, equation (34), we deduce that

$$R_0(\beta_c ; \delta) = R_0(\beta_c ; 1/2)J_M(n_{\beta c}),$$  \hspace{1cm} (54)$$

where

$$J_M(n) = \left( \frac{n + 1}{4} \right)^{1/3} \xi_n(\theta_n).$$  \hspace{1cm} (55)$$

(53) now defines a curve in the $(R, 1 - \beta_c)$-plane. The region enclosed by the two axes and the $(R_0, 1 - \beta_c)$-curve now defines our domain of degeneracy in this plane.

13. In our numerical work we shall confine ourselves exclusively to the case $\delta = 0$. Then we have

$$n_{\beta c} = 3 + \frac{4 - 3\beta_c}{1 + 3\beta_c}.$$  \hspace{1cm} (56)$$

The quantities $J_M(n)$ and $J_M(n)$ are known for certain values of $n$, and for the intermediate values recourse was made to methods of interpolation.

* It may be noticed here that $\mathfrak{M}(1/2)$ is our original "$\mathfrak{M}"."
14. From (56) we now deduce that
\[ n_{\beta_0} = 3.343; \quad J_0(3.343) = 1.080. \]  
Hence
\[ \mathfrak{M}_{(0)} = 1.080 \mathfrak{M}_{(1/2)} = 1.249 M_3, \]  
or putting in numerical values
\[ \mathfrak{M}_{(0)} = 7.153 \mu^{-2}. \]  

15. In Table III we have tabulated for this model (\( \delta = 0 \)) the corresponding sets of values for \( x, 1 - \beta_c, R_0 \) and \( M \). The \( (R_0, 1 - \beta_c) \)-curve is shown in fig. 2.

**Table III**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 1 - \beta_c )</th>
<th>( n_{\beta_c} )</th>
<th>( R_0/l )</th>
<th>( M/M_3 )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
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<td>( \infty )</td>
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<tr>
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<td>1.249</td>
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16. The Nature of the Curves of Constant Mass for \( M \ll M_3 \) in the Domain of Degeneracy.—Let \( \beta_c \) be the value of \( \beta_c \) for a wholly gaseous configuration (of mass less than \( M_3 \)) which in its completely collapsed state has a central density corresponding to \( y = y_0 \). Then by equation (46)
\[ \frac{\Omega(y_0)}{\omega_3^0} = \left( \frac{960}{\pi^4} \frac{1 - \beta_c}{\beta_c} \right)^{1/2} J_0(n_{\beta_c}). \]
Now the line through \((1 - \beta_0^+)^\parallel\) parallel to the \(R\)-axis will intersect the \((R_\theta, 1 - \beta_\theta^+)-\)curve at \((R_\theta(M(y_0)), 1 - \beta_\theta^+)\). In the domain of degeneracy the continuation of the curve must in some way connect the point \((R_\theta(M(y_0)), 1 - \beta_\theta^+)\) and the point \(R_1\) on the \(R\)-axis where

\[
\frac{R_1}{I} = \frac{1}{y_0(M)} \frac{\eta_1(\phi(y_0(M)))}{\xi_1(\theta_3)}.
\]

In I (Table II) we have already tabulated the values of \(R_1\) for ten different values for \(y_0\) and for these configurations \((1 - \beta_\theta^+)\) was obtained by interpolating among the figures given in Table III. The corresponding pairs of points on the \((R_\theta, 1 - \beta_\theta^+)-\)curve and the \(R\)-axis are shown in fig. 2. (The points marked 5 to 15 on the \(R\)-axis and also on the \((R_\theta, 1 - \beta_\theta^+)-\)curve.) It is of course clear that for \(M = M_3\) the associated curve of constant mass must pass through the origin of our system of axes. If we denote by \(\beta_\theta(0)\) the value of \(\beta_\theta\) which \(M_3\) has in the wholly gaseous state then we should have by (60) that

\[
es_{1/2} = \left(\frac{960}{\pi^4} \frac{1 - \beta_\theta(0)}{\beta_\theta^4(0)}\right)^{1/2} j_M(n_{\beta_\theta(0)}) = I.
\]

Numerically \((1 - \beta_\theta(0))\) is found to be 0.0668.

17. The Nature of the Curves of Constant Mass for \(M_3 < M < \mathfrak{M}_{(1/2)}\) in the Domain of Degeneracy.—The discussion of this case will naturally depend on the assumption one makes regarding the opacity in the degenerate core.

We shall assume that \("(\kappa\eta)_\text{a}\)" is constant in the core. Then we have (cf. I, equation (63))

\[
P = \frac{I}{\beta_2} A_2 f(x) - D_2 \frac{1 - \beta_2}{\beta_2},
\]

where \(D_2\) is a constant and

\[
\beta_2 = \left(1 - \frac{(\kappa\eta)_\text{a} L}{4\pi c GM}\right).
\]

The reduction to our differential equation (1) for the degenerate core follows at once.

In our present scheme \"\(\beta\)\" is of course allowed to vary in the gaseous envelope, and we shall in the first instance consider the case where \(\beta_2\) is just equal to the value of \(\beta_1\) at the interface between the degenerate core and the gaseous envelope. For this case the discussion can be carried out as in I, §9.

A completely relativistically degenerate configuration has a mass given by (H.C. I, page 463)

\[
M = M_3 \beta^{-3/2},
\]

and is of zero radius. Since we have the further relation

\[
\mathfrak{M}_{(1/2)} = M_3 \beta_\omega^{-3/2},
\]

it follows that the curve of constant mass for \(\mathfrak{M}_{(1/2)}\) must connect the point \((R_\theta, 1 - \beta_\theta^+(\mathfrak{M}_{(1/2)}))\) on the \((R_\theta, 1 - \beta_\theta^+)-\)curve and the point \((\sigma, 1 - \beta_\omega)\) on the \((1 - \beta_\omega)\)-axis. It is therefore clear that for \(M_3 < M < \mathfrak{M}_{(1/2)}\) the
Fig. 2.—The curve running from $1 - \beta_c = 0.92 \ldots$ to infinity along the $R_1$-axis is the $(R_0(\beta_c; \delta = 0), 1 - \beta_c)$-curve (see equation (53)). The points marked (5 \ldots 15) on the $(R_0, 1 - \beta_c)$-curve and on the $R_1$-axis are the end points in the domain of degeneracy for the curves of constant mass for the values of $M$ for which the $\psi$-integrals are known (H.C. II). The points marked (o, 1 \ldots 4) on the $(R_0, 1 - \beta_c)$-curve and on the $(1 - \beta_1)$-axis are the corresponding end points for some curves of constant mass in the domain of degeneracy on the model discussed in 17 (see equation (67)). The point $o$ corresponds to $\mathfrak{M}(\delta = 1)$, i.e. our earlier “$\mathfrak{M}$.”
curves of constant mass must cross the $(1 - \beta_c)$-axis at a point $(1 - \beta^*)$ such that

$$M = M_0 \beta^{*-3/2}. \quad (66)$$

If $\beta^+_c$ denotes the value of $\beta_c$ in the wholly gaseous state then by equating (66) and (46) we have

$$\beta^* = \left( \frac{\pi^4}{960} \frac{\beta^+_c^4}{1 - \beta^+_c^2} \right)^{1/3} (J_M(n_{\beta_c^+}))^{-2/3}. \quad (67)$$

Some corresponding pairs of points on the $(R_0, 1 - \beta_c)$-curve and the $(1 - \beta_c)$-axis are shown in fig. 2.

Finally, if $\mathfrak{M}_{(1/2)} < M < \mathfrak{M}_{(4)}$ it is immediately clear that the curves of constant mass consist simply of segments connecting the point $(R_0, 1 - \beta^+_c(M))$ on the $(R_0, 1 - \beta_c)$-curve to the point $(o, 1 - \beta_w)$ on the $(1 - \beta_c)$-axis.

We thus see that on this model the curves of constant mass in the $(R_0, 1 - \beta_c)$-plane combine in the same diagram some of the features of both figs. 3 and 4 of I, obtained on the basis of the two extreme cases of the generalized standard model.

18. It may finally be pointed out that if one assumed that the opacity of the degenerate core is zero then the general qualitative features of the system of the curves of constant mass in the $(R, 1 - \beta_c)$-plane must be exactly the same as in I, fig. 4.

19. A complete discussion on the basis of Jeans’s model will require a study of the composite configurations. The formal theory (which would run similar to I, §§ 11 to 15) can easily be sketched, but as such discussions are not of much interest without the necessary numerical work (which would be considerable) we shall not go into these details here. However, it is clear that the general results derived on the basis of the standard model are fully retained even in this more general analysis.

Section III

20. The Wolf-Rayet Phenomenon.—It has already been suggested in I that the Wolf-Rayet phenomenon of the radial ejection of matter may be indirectly due to the fact that the stars of mass greater than $\mathfrak{M}$ (or its equivalent $\mathfrak{M}(\delta)$ on the more general stellar models discussed in Section II) cannot pass directly into the white-dwarf stage.* This suggestion is confirmed by observation in so far as general estimates do indicate that Wolf-Rayet stars are massive and dense. On the theoretical side the suggestion gains further support from the following argument:

Consider a mass greater than $\mathfrak{M}_{(1/2)}$. On the standard model the star must necessarily be wholly gaseous, and we have in a certain system of “natural units” (cf. equation (14))

$$L^* = (M^* \beta_1)^{7/2}(1 - \beta_1)R^{*-1/2}. \quad (68)$$

* This suggestion (in a rather different form) was independently made to the writer by Dr. W. H. McCrea, to whom the author had earlier communicated a preliminary statement of the main results in the form later published in Observatory, 57, 373, 1934.
Further, the value of the surface gravity \( g \) is given by

\[
g = \frac{GM}{R^2}.
\]  

(69)

From (68) and (69) we have for the ratio \( X \) between the integrated flux of radiation \( \pi F \) at the surface of the star to the value of gravity, the expression

\[
X = \frac{\pi F}{g} = \frac{L}{4\pi cGM} = \frac{L_1}{4\pi cGM} \frac{M^{5/2} \beta_1^{1/2}(1 - \beta_1)R^*)^{-1/2}}{R^*}. 
\]  

(70)

From our definition of \( L_1 \) in (13) we have

\[
X = \left( \frac{1}{\alpha \kappa_1} \right) \frac{M^{5/2} \beta_1^{1/2}(1 - \beta_1)R^*)^{-1/2}}{R^*}. 
\]  

(71)

From (71) we see that for a given mass \(( > M_0)\) the ratio \( X \) steadily increases with decreasing \( R \) and in fact tends to infinity. This suggests that at some stage in the process of contraction the radiation pressure (a measure of which is given by \( \pi F \)) must overbalance gravity. Ejection of matter must necessarily ensue. In drawing this inference it is of course realized that the deduction from (71) cannot be regarded as a rigorous proof in so far as in our analysis the equations of equilibrium have been integrated up to the boundary. But if one takes this last boundary condition seriously then one cannot also strictly speak of a “mass-luminosity-effective temperature” relation as the temperature has been made zero at the boundary. That this involves no real contradiction was shown in the early writings of Eddington, Jeans and Russell and more recently by Cowling. Bearing this in mind it is now clear that the fact that \( X \rightarrow \infty \), as \( R \rightarrow 0 \) merely means that the approximations underlying the deduction of the mass-luminosity-radius relation (68) should cease to be valid at some stage. Our conclusion that the ejection of matter must ensue since \( X \rightarrow \infty \), as \( R \rightarrow 0 \) is now seen to be equivalent to the suggestion that the Wolf-Rayet phenomenon should set in precisely in the region of the Russell diagram where the mass-luminosity-radius relation for the massive wholly gaseous configurations ceases to be valid on the perfect-gas hypothesis itself. It is of course necessary that the star should be massive \(( M > M_0) \) or its equivalent on more general stellar models) for otherwise we could not extrapolate (68) to high mean densities—degeneracy would have set in earlier for the less massive stars.

21. The Hydrogen Content of the Massive Stars.—In § 20 we have used the term “massive stars” to denote those with \( M > M_0(\delta) \). It was found on Jeans’s model \(( \delta = 0) \) that we have

\[
M_0(\delta) = 7.15\mu^{-2} \odot. 
\]  

(72)

To define \( M_0(\delta) \) more precisely we need to know the hydrogen content. Depending on the hydrogen content \( M_0(\delta) \) can be varied numerically by a factor 16 \(( \mu = \frac{1}{2} \) to \( \mu = 2) \), and it becomes necessary therefore to know at least the minimum hydrogen contents of stars as a function of their mass.
Fortunately we have for our guidance here Strömgren’s systematic investigations of this problem.* From Strömgren’s work it appears that the molecular weight of the massive B stars already tends towards the lower limit $0.5$. Thus we can conclude that for our purposes a star of mass greater than about $250 \odot$ can be regarded as "massive."

It is of interest to recall in this connection that in his "Interpretation of the Hertzsprung-Russell diagram" Strömgren says: "With an appreciable overcompressible nucleus the predicted luminosities would be appreciably larger than the observed, and increasing the hydrogen content—as is usually possible to remove the difference—is not possible in these cases, as the limit has already been reached. We conclude then that for the B stars in question there cannot be any appreciable nucleus." We now see that Strömgren’s conclusions receive further indirect confirmation from our analysis.

22. The Hydrogen Content of the White Dwarfs.—The hydrogen content of the white dwarfs had been investigated earlier by various writers on the Emden polytrope $n = 3/2$ approximation for them. In our H.C. II we have made an exact study of these completely degenerate configurations, and it is now possible to make a more reliable estimate for the appropriate molecular weights for the white dwarfs.

The necessary data required for this calculation are given in Table III of H.C. II. The following table is due to Strömgren:

**Table IV**

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<th>$\mu$ for Sirius B</th>
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<td>8300$^0$</td>
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From the above table Strömgren concludes that the value of $\mu$ for Sirius B should be about 1.6, which means relatively low hydrogen content. The low hydrogen content of the white dwarfs has already been discussed by Strömgren (loc. cit.).

23. Some Remarks on Figure 1.—Figure 1 is of course the domain of the Hertzsprung-Russell diagram. From an examination of this diagram it is immediately clear that the white dwarfs are placed in their right positions in the Russell diagram. The two essential observational results concerning the white dwarfs, namely, their small mass and low luminosity, receive their natural explanations. The region of the diagram in which we should expect the Wolf-Rayet stars is also indicated. The region of the ordinary stars is indicated by "perfect gas stars." Presumably stars like Krueger 60 are representatives of the "incipiently degenerate" region of our diagram.

The above general conclusions, so far as they go, should clarify the present position regarding stellar structure.

24. Deviations from Perfect Gas Laws arising from Causes other than Degeneracy.—In our discussion we have so far considered only deviations from perfect gas laws which are due to degeneracy. However, Dirac’s theory of the electron predicts a further different type of deviation from the perfect gas laws due to the production of electron pairs at very high temperatures. The bearing of this phenomenon on the theory of stellar structure has been examined in a preliminary communication by L. Rosenfeld and the writer.* As we have indicated in that letter, the deviations from the perfect gas laws arising from this cause are of quite negligible importance for stars with $M < \mathfrak{M}(\cdot)$; however, they become increasingly important for the very massive stars. The detailed results of this study will be published separately by Rosenfeld and the writer, but it may be mentioned here that it follows from that study that the production of electron pairs will be of importance in considerations of the structure of stars of masses about $80\odot$ and more. The existence of such very massive stars is indicated by the work of J. S. Plaskett, O. Struve, Bottlinger, Trumpler and others, and it seems very probable that the discussion of their structure will lead to some essentially new considerations in the studies on stellar structure.

Finally, it is necessary to point out in this connection that J. von Neumann has recently shown that the very ultimate equation of state for matter should always be

$$P = \frac{1}{3}c^2\rho. \quad (73)$$

The considerations of this new equation of state does not, however, introduce any essential modifications in our present scheme.

Concluding Remarks.—In two earlier papers (M.N., 95, 207–260, 1935) a first systematic attack was made on the problem of how the conclusions regarding stellar constitution and stellar evolution that have been drawn on the perfect gas hypothesis for the stars have to be modified by the physical possibility of degeneracy in stellar interiors. In this paper the discussion is carried one stage further. Firstly, the physical results have been made more explicit by considering the curves of constant mass in the $(\log L, \log i^2)$-diagram, which is essentially the domain of the Hertzsprung-Russell diagram. Secondly, the analysis has been extended to include other stellar models more general than the standard model. Thirdly, the bearing of the Wolf-Rayet phenomenon on the evolution of massive stars is examined a little more closely. Certain other miscellaneous questions have also been considered.

In conclusion, I wish to record here my best thanks to Dr. W. H. McCrea, Professor J. von Neumann, Dr. L. Rosenfeld and Dr. B. Strömgren for the encouraging interest they have taken in these studies and for many stimulating discussions.

Trinity College, Cambridge:
1935 June 7.