THE HIGHLY COLLAPSED CONFIGURATIONS OF A STELLAR MASS. (SECOND PAPER.)

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1. A study of the equilibrium of degenerate gas spheres has a twofold significance in the analysis of stellar structure, namely, in providing an approach to a proper theory of white dwarfs, and also, we shall see, in providing a certain limiting sequence of configurations to which all stars must tend eventually. A beginning in the study of these configurations was made by the author in a previous communication,* where for convenience the equation of state of degenerate matter was taken to correspond to one or other of the two limiting forms \( \rho = K_1 \rho^{5/3} \) or \( \rho = K_2 \rho^{4/3} \) according as the density was less than or greater than a certain density \( \rho' \) where

\[
\rho' = \left( \frac{K_2}{K_1} \right)^{3/2},
\]

\( \rho' \) itself being such that both the equations of state yield the same calculated value for the pressure. Actually in the analysis a certain small temperature gradient was allowed for. Working on the standard model it was assumed that the ratio \( \beta \) of the gas pressure to the total pressure was a constant, but by hypothesis ("highly collapsed") \( \beta \) was taken to be very nearly unity. On these assumptions it followed that stars of mass less than a certain specified \( M_{3/2} \) (see I, § 6, page 462) were complete Emden polytropes with index \( n = 3/2 \), and further that configurations of greater mass must be composite, i.e. must have inner regions where degeneracy is predominantly relativistic. Lastly, and this was the most important conclusion reached, these composite configurations have a natural limit: On the standard model a completely relativistically degenerate configuration has a mass given by (cf. I, equation (36))

\[
M = -\frac{4}{\pi^{1/2}} \left( \frac{K_2}{G} \right)^{3/2} \left( \xi^2 \frac{d \theta_3}{d \xi} \right)_{1} \cdot \beta^{-3/2} = M_3 \beta^{-3/2} \text{ (say)},
\]

where \( \theta_3 \) is the Emden function with index \( n = 3 \). These configurations have zero radius (cf. the remarks in I following the equations (45), (46), page 463).†

* M.N., 91, 456, 1931 (referred to as I). See also the earlier papers of the author in Phil. Mag., II, 592, 1931, and Astrophysical Journal, 64, 92.
† In I we denoted by \( M_3 \) what we have now defined as \( M_3 \beta^{-3/2} \). It is convenient to separate out the term involving \( \beta \) from the purely "mass factor."
† In I this "singularity" was formally avoided by introducing a state of "maximum density" for matter, but now we shall not introduce any such hypothetical states, mainly for the reason that it appears from general considerations that when the central density is high enough for marked deviations from the known gas laws (degenerate or otherwise) to occur the configurations then would have such small radii that they would cease to have any practical importance in astrophysics.
Apart from the above results of a general character, the analysis in I did not lead to any further quantitative results. To obtain by the methods of I anything more exact would have meant very considerable numerical work to "fit" an appropriate solution of Emden's equation with index \( n = 3/2 \) (to describe the outer ordinarily degenerate envelope) with an *Emden function* of index 3 (to describe the inner relativistically degenerate core).

It would be very much more satisfactory to take the exact equation describing the degenerate state and treat the whole degenerate parts of a star on the same footing instead of as in I, further subdividing it to correspond to one or other of the two limiting forms of the equation describing the degenerate state. By a very remarkable coincidence the differential equation (governing the structure of a degenerate gas sphere in hydrostatic equilibrium) based on the exact equation of state takes an extremely simple form. We show, in fact, that the structure of the configuration is governed by a solution of the differential equation,

\[
\frac{1}{\eta^2} \frac{d}{d\eta} \left( \frac{d\phi}{d\eta} \right) = -\left( \frac{\phi^2 - \frac{1}{y_0^2}}{y_0^2} \right)^{3/2}. \tag{2}^*
\]

It is to be noticed that there is only one parameter occurring in the equation, and a single system of integrations should suffice to obtain a clear insight into these configurations. Equation (2) has a formal similarity with Emden's equation. Indeed, we shall show that under certain circumstances \( \phi \) can be expressed in terms of the Emden functions with appropriate indices. It is the derivation of the above equation that has led to the developments summarised in this and the following paper. In this paper we shall establish this equation and provide tables of solutions. In the analysis we shall omit all references to radiation pressure, i.e. this paper strictly deals with configurations having \( \beta = 1 \). The introduction of radiation in these configurations involves quite delicate considerations, and all these find a proper treatment in the paper following this one.

2. The Differential Equation governing the Structure of Degenerate Matter in Gravitational Equilibrium.—The pressure-density relation for a degenerate gas can be written parametrically as follows:

\[
\rho = \frac{\pi m c^5}{3h^3} \left[ x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x \right],
\]

\[
\rho = \frac{8\pi m c^5 \mu H}{3h^3} x^3,
\]

where \( m = \) mass of the electron, \( c = \) velocity of light, \( h = \) Planck's constant, \( H = \) mass of the proton, \( \mu = \) molecular weight. Equation (3) is established in Appendix I to this paper, where also \( f(x) \) is tabulated. We rewrite (3) as

\[
\rho = A_2 f(x); \quad \rho = Bx^3, \tag{4}
\]

* This equation was given without proof in the author's preliminary note in the *Observatory*, 57, 373, 1934.
where
\[ A = \frac{\pi m^4 c^5}{3 h^2}; \quad B = \frac{8\pi m^2 c^3 \mu H}{3 h^3}, \]
\[ f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x. \]

The equations of equilibrium are, as usual,
\[ \frac{dp}{dr} = -\frac{GM(r)}{\rho^2}, \]
\[ \frac{dM(r)}{dr} = 4\pi G \rho^2. \]

From (6) we have
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dp}{\rho dr} \right) = -4\pi G \rho. \]

Substitute for \( p \) and \( \rho \) from (4). We have
\[ \frac{A_2}{B} \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 df(x)}{x^3 dr} \right) = -4\pi GBx^3. \]

From the definition of \( f(x) \) in (5) we easily verify that
\[ \frac{df(x)}{dr} = \frac{8x^4}{(x^2 + 1)^{1/2}} \frac{dx}{dr}, \]

or
\[ \frac{1}{x^3} \frac{df(x)}{dr} = \frac{8x}{(x^2 + 1)^{1/2}} \frac{dx}{dr} = \frac{8}{2A_2} \frac{d\sqrt{x^2 + 1}}{dr}. \]

Hence (8) can be rewritten as
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 d\sqrt{x^2 + 1}}{dr} \right) = -\frac{\pi GB^2}{2A_2} x^3. \]

Put
\[ y^2 = x^2 + 1. \]

Then
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dy}{dr} \right) = -\frac{\pi GB^2}{2A_2} \left( y^2 - 1 \right)^{3/2}. \]

Let \( x \) take the value \( x_0 \) at the centre.

Further, let \( y_0 \) be the corresponding value of \( y \) at the centre. Introduce the new variables \( \eta \) and \( \phi \) defined as follows:
\[ r = a \eta; \quad y = y_0 \phi, \]
where
\[ a = \left( \frac{2A_2}{\pi G} \right)^{1/2} \frac{1}{By_0}, \]
\[ y_0^2 = x_0^2 + 1. \]

Our differential equation finally takes the form
\[ \frac{1}{\eta^3} \frac{d}{d\eta} \left( \eta^3 \frac{d\phi}{d\eta} \right) = -\left( \phi^2 - \frac{1}{y_0^2} \right)^{3/2}. \]

By (14) we have to seek a solution of (16) such that \( \phi \) takes the value unity at the origin. Further, from symmetry the derivative of \( \phi \) must
vanish at the origin. The boundary is defined at the point where the density vanishes, and this by (12) means that if $\eta_1$ specifies the boundary

$$\phi(\eta_1) = \frac{1}{y_0}.$$  

(17)

3. From our definitions of the various quantities we find that

$$\rho = \rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \left( \frac{\phi^2 - \frac{1}{y_0^2}}{y_0^2} \right)^{3/2},$$  

(18)

where

$$\rho_0 = Bx_0^3 = B(y_0^2 - 1)^{3/2}$$  

(18')
specifies the central density. Also we may notice that the scale of length $a$ introduced in (15) has in terms of the physical quantities the form

$$a = \frac{1}{4\pi m\mu H y_0} \left( \frac{3h^3}{2cG} \right)^{1/2},$$  

(19)
or putting in numerical values

$$a = \frac{7.720 \times 10^8}{\mu y_0} = l_1 y_0^{-1} \text{ cm. (say)}. \quad (20)$$

4. The Potential.—The function $\phi$ itself has a physical meaning. If $V$ is the inner gravitational potential, then from general theory we have

$$\frac{dV}{dr} = \frac{1}{\rho} \frac{d\rho}{dr}. \quad (21)$$

From (5) and (10) we see that

$$\frac{dV}{dr} = \frac{8A_2}{B} \frac{d\phi}{y_0^3 \frac{d\rho}{d\eta}},$$  

(22)
or integrating

$$V = \frac{8A_2}{B} y_0 \phi + \text{constant}. \quad (23)$$

If we choose the arbitrary zero of the potential on the boundary of the configuration we have by (17) that the “constant” in (23) is $(8A_2/B)$. Hence finally

$$V = \frac{8A_2}{B} y_0 \left( \phi - \frac{1}{y_0^2} \right). \quad (24)$$

5. The Mass Relation.—The mass of the material enclosed up to a point $\eta$ is clearly

$$M(\eta) = 4\pi \int_0^\eta \rho r^2 dr = 4\pi \alpha A_2 \int_0^\eta \rho \eta^2 d\eta. \quad (25)$$

By (18),

$$M(\eta) = 4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \left( \frac{\phi^2 - \frac{1}{y_0^2}}{y_0^2} \right)^{3/2} \eta^2 d\eta, \quad (26)$$

or using our differential equation (16)

$$M(\eta) = -4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta d\eta \frac{d}{\eta} \left( \eta^2 \frac{d\phi}{d\eta} \right) d\eta. \quad (27)$$

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Collapsed Configurations of a Stellar Mass

Remembering that $\rho_0$ is given by (18) we have explicitly

$$M(\eta) = -4\pi \left( \frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \eta^2 \frac{d\phi}{d\eta}. \quad (28)$$

The mass of the whole configuration is therefore

$$M = -4\pi \left( \frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left( \eta^2 \frac{d\phi}{d\eta} \right)_{\eta = \eta_1}. \quad (29)$$

We notice that in (28) and (29) $\rho_0$ does not explicitly occur. It is of course implicitly present inasmuch as in the differential equation defining $\phi$, $\rho_0$ occurs.

6. The Relation between the Mean and the Central Density.—Let $\bar{\rho}(\eta)$ be the mean density of the material inside $\eta$. Then

$$M(\eta) = \frac{4}{3} \pi a^3 \eta^3 \bar{\rho}(\eta). \quad (30)$$

Comparing (28) and (30), we have

$$\frac{1}{3} \eta^3 \bar{\rho}(\eta) = -\rho_0 \frac{y_0^2}{(y_0^2 - 1)^{3/2}} \eta^2 \frac{d\phi}{d\eta}, \quad (31)$$

or

$$\frac{\bar{\rho}(\eta)}{\rho_0} = -\frac{3}{(y_0^2 - 1)^{3/2}} \frac{1}{\eta} \frac{d\phi}{d\eta}. \quad (32)$$

From (32) we deduce that the relation between the mean and the central density of the whole configuration is

$$\rho_0 = -\bar{\rho} \left( 1 - \frac{1}{y_0^2} \right)^{3/2} \eta \frac{d\phi}{d\eta} \quad (33)$$

($\phi'$ denoting the derivative)—a relation analogous to the corresponding relation in the theory of polytropes.

7. An Approximation for Configurations with Small Central Densities.—When the central density is small we should have the law $p = K \eta^{5/3}$ holding approximately, and the corresponding configurations must have structures which can approximately be represented by an Emden polytrope with index $n = 3/2$. We establish this on our differential equation in the following way:

Now by definition $y_0^2 = x_0^2 + 1$, and we need a first-order approximation when $x_0^2$ is small. We shall neglect all quantities of order $x_0^4$ or higher. Then

$$y_0 = 1 + \frac{1}{2} x_0^2. \quad (34)$$

Put

$$\frac{\phi^2 - 1}{y_0^2} = \theta. \quad (35)$$

In our approximation we have

$$\phi = 1 - \frac{1}{2} (x_0^2 - \theta). \quad (36)$$

At the origin $\phi$ takes the value unity. Hence

$$\theta(0) = x_0^2. \quad (37)$$
From (16) we derive the following differential equation for $\theta$:

$$\frac{1}{2} \frac{d^2 \theta}{d\eta^2} + \frac{1}{\eta} \frac{d \theta}{d\eta} = -\theta^{3/2}. \quad (38)$$

Put

$$\xi = 2^{1/2}\eta. \quad (39)$$

Then

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d \theta}{d\xi} \right) = -\theta^{3/2}, \quad (40)$$

which is Emden’s equation with index $n = 3/2$, but the solution we need is not the Emden function in the usual normalisation* with $\theta = 1$ at $\xi = 0$. By (37) our $\theta$ takes the value $x_{0}^2$ at the origin. Denote by $\theta_{3/2}$ the Emden function. Now it is a property of the differential equation (40) that if $\theta$ is any solution then $C^2 \theta(C\xi)$ is also a solution where $C$ is any arbitrary constant. Hence if we put

$$C = x_{0}^{1/2}, \quad (41)$$

and take for $\theta, \theta_{3/2}$, we would obtain the solution we need. Hence

$$\theta = x_{0}^{2} \theta_{3/2}(x_{0}^{1/2}\xi) = x_{0}^{2} \theta_{3/2}(\sqrt{2x_{0}\eta}). \quad (42)$$

By (37) then

$$\phi = 1 - \frac{1}{2} x_{0}^{2}(1 - \theta_{3/2}(\sqrt{2x_{0}\eta})) + O(x_{0}^4), \quad (43)$$

which relates $\phi$ with $\theta_{3/2}$. From (43) we see that for these configurations the boundary $\eta_1$ must be such that

$$(\theta_{3/2} \sqrt{2x_{0}\eta_1}) = 0. \quad (44)$$

Let $\xi_1(\theta_{3/2})$ be the boundary of the Emden function. Then from (44) we deduce that

$$\eta_1 = \frac{\xi_1(\theta_{3/2})}{\sqrt{2x_0}}. \quad (45)$$

From (45) we see that as $y_0 \to 1, x_0 \to 0, \eta_1 \to \infty$. The radius tends to infinity with the same singularity.

Again from (43) we have

$$\frac{d\phi}{d\eta} = \frac{1}{2} x_{0}^{2} \sqrt{2x_{0}} \frac{d\theta_{3/2}(\xi)}{d\xi}. \quad (46)$$

Combining (45) and (46) we have a relation we shall need later:

$$\left( \frac{\eta_1}{d\eta} \frac{d\phi}{d\eta} \right) = \left( \frac{x_0}{2} \right)^{3/2} \left( \xi \frac{d\theta_{3/2}}{d\xi} \right) \left( \frac{1}{\xi} \frac{d\theta_{3/2}}{d\xi} \right). \quad (47)$$

Further,

$$\left( \frac{x^3}{\eta} \frac{d\phi}{d\eta} \right) = x_{0}^{3} \left( \frac{1}{\xi^2} \frac{d\theta_{3/2}}{d\xi} \right). \quad (48)$$

We shall find the above expressions useful when we come to discuss “highly”

* In the sequel by “Emden function” we shall always mean the one which takes the value unity at the origin. We shall denote the Emden function with index $n$ by $\theta_n$.  

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collapsed configurations \((i - \beta)\) finite but small, but now we verify that the scheme is consistent. From (48) and (33) we have

\[
\rho_0 = -\beta \left( \frac{\xi^3}{\theta^{3/2}_0} \right),
\]

(49)

which is precisely the formula for an Emden polytrope with index \(n = 3/2\). Again from (29) and (47)

\[
M = -4\pi \left( \frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^3} \frac{\xi^{3/2}}{2} \left( \xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1.
\]

(50)

To compare the above with the formula derived on the law \(p = K_1 \rho \gamma^{5/3}\) we note that the degenerate constant \(K_1\), given by

\[
K_1 = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}}.
\]

(51)

is related to our \(A_1\) and \(B\) by the relation

\[
K_1 = \frac{8}{5} \frac{A_2}{B^{3/5}}.
\]

(52)

Combining (50) and (52) and setting \(\lambda_2\) to denote the central density \(= Bx_0^0\) we find that

\[
M = -4\pi \left( \frac{5K_1}{8\pi G} \right)^{3/2} \lambda_2^{1/2} \frac{\xi^2 \frac{d\theta_{3/2}}{d\xi}}{d\xi}.
\]

(53)

which is the usual formula since on the law \(p = K_\rho^{1/2}\) the polytropic relation is

\[
M = -4\pi \left( \frac{(n + 1)K}{4\pi G} \right)^{3/2} \lambda_2 \left( \frac{\xi^2 \frac{d\theta_n}{d\xi}}{d\xi} \right).
\]

(53')

8. The Limiting Mass.—From our differential equation (16) we see that \(\phi \to \theta_\infty\) as \(y_0 \to \infty\).

(54)

But from (20) we see that at the same time the radius tends to zero. From (28) then

\[
M \to -4\pi \left( \frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^3} \frac{\xi^2 \frac{d\theta_2}{d\xi}}{d\xi}.
\]

(55)

To see that we have now simply recovered our earlier result in I (equation (36)) we have only to notice that the relativistic degenerate constant \(K_2\), defined by

\[
K_2 = \left( \frac{3}{\pi} \right)^{1/3} \frac{hc}{8(\mu H)^{4/3}},
\]

(56)

is related to our \(A_2\) and \(B\) by the relation

\[
K_2 = \frac{2A_2}{B^{4/3}}.
\]

(57)

9. As mentioned in § 1, we shall denote by \(M_3\) the mass

\[
M_3 = 4\pi \left( \frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^3} \omega_3^6, \]

(58)
where following Milne we have introduced the quantity \( \omega_s^0 \) defined by

\[
\omega_s^0 = -\left( \frac{\xi^2 d\theta_3}{d\xi} \right)_{1}. 
\]

If we define correspondingly that

\[
\Omega(y_0) = -\left( \frac{\eta^2 d\phi}{d\eta} \right)_{\eta=\eta_1}
\]

for our function \( \phi \), then the mass relation can be written as

\[
M(y_0)\omega_s^0 = M_3\Omega(y_0). \tag{61}
\]

As the mass of the configuration increases monotonically with increasing \( y_0 \), we have the useful inequality

\[
\Omega(y_0) > \omega_s^0 \quad (y_0 \text{ finite}). \tag{62}
\]

Finally we may note that the insertion of numerical values in our formula for \( M_3 \) yields

\[
M_3 = 5.728 \mu^{-2} \times \odot, \tag{63}
\]

where \( \odot \) represents the mass of the Sun.

10. The General Results.—In the previous sections, §§ 7, 8, 9, we have merely related our present treatment with the results obtained in I on the basis of the polytropic theory. Those results appear as simple limiting cases. However, the exact treatment on the basis of our differential equation

\[
\frac{1}{\eta^2} \frac{d}{d\eta} \left( \eta^2 \frac{d\phi}{d\eta} \right) = -\left( \phi^3 - \frac{1}{y_0^2} \right)^{3/2} \tag{64}
\]

at the same time provides much more quantitative information. The boundary conditions

\[
\phi = 1, \quad \frac{d\phi}{d\eta} = 0 \text{ at } \eta = 0, \tag{65}
\]

combined with a particular value for \( y_0 \), would determine \( \phi \) completely, and therefore the mass of the configuration as well. The equation (64) does not admit of a "homology constant," and hence each mass has a density distribution characteristic of itself which cannot be inferred from the density distribution in a configuration of a different mass. This difference between our configurations governed by (64) and polytropes has, as we shall see, an important bearing in the theory of general stellar models considered in the following paper.

Each specified value for \( y_0 \) determines uniquely the mass \( M \), the radius \( R_1 \) and the ratio of the mean to the central density. We have (collecting together our earlier results):

\[
\frac{M}{M_3} = \Omega(y_0)/\omega_s^0, \tag{66}
\]

\[
\frac{R_1/l_1} = \eta_1/y_0, \tag{67}
\]

\[
\rho_0/B = (y_0^2 - 1)^{3/2}, \tag{68}
\]
In (67) we have introduced a new unit of length \( l_1 = a y_0 \),

\[
l_1 = \frac{1}{4\pi m_\mu H} \left( \frac{3h^3}{2cG} \right) = 7\cdot720\mu^{-1} \times 10^8 \text{ cm.},
\]

and which therefore does not involve factors in \( y_0 \). Further, the physical variables determining the structure of the configuration are:

\[
\rho = \rho_0 \left( \frac{1}{I - \frac{1}{y_0^2}} \right)^{3/2} \phi^2 - \frac{1}{y_0^2},
\]

\[
\bar{\rho} = -\rho_0 \left( \frac{1}{I - \frac{1}{y_0^2}} \right)^{3/2} \frac{3}{\eta} d\phi
\]

\[
M(\eta) \propto -\eta^2 \frac{d\phi}{d\eta}.
\]

In § 10 we have reduced the problem of the structure of degenerate gas spheres to a study of our functions \( \phi \) for different initially prescribed values for the parameter \( y_0 \). The integration has been numerically effected for the following ten different values of the parameter:

\[
1/y_0^2 = 0.8, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, 0.01.
\]

The following expansion for \( \phi \) near the origin may be noted here for further reference:

\[
\phi = \frac{q^3}{6} \xi^2 + \frac{q^4}{40} \xi^4 - q^6(5q^4 + 14) \xi^6 + q^8(339q^8 + 280) \xi^8
\]

\[
- \frac{q^9(1425q^4 + 11436q^8 + 4256)}{5 \times 11!} \xi^{10} + \ldots
\]

where

\[
q^2 = 1 - \frac{1}{y_0^2}.
\]

The important quantities of interest are the boundary quantities occurring in equations (66), (67), (69). These are tabulated in Table I for the different values of \( y_0 \).

12. From the figures of Table I it is easy to calculate the mass in units of \( M_3 \), the radius in units of \( l_1 \) and the central density \( (= \rho_0) \) in units of \( B \).

* When \( y_0 \rightarrow \infty \), \( q \rightarrow 1 \) and the series (74) goes over into the expansion for Emden \( \theta_3 \) near the origin (cf. British Association Tables, 2, Introduction, equation on top of page v).
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These express the chief physical characteristics of these configurations in the “natural system” of units occurring in the theory of these configurations. In Table III they are converted into the more conventional system of units expressing the radius and the density in C.G.S. units and the mass in units of the Sun. The actual figures tabulated are for $\mu = 1$. The figures for other values of $\mu$ can be obtained by multiplying $M$ by $\mu^{-2}$, $R_1$ by $\mu^{-1}$ and $\rho$ by $\mu$. To see the order of magnitudes involved here it is of interest to point out that the mass $4.852 \circ \mu^{-2}$ has a radius only slightly over the radius of the Earth (radius of the Earth $6 \times 10^8$ cm. compared to $7.7 \times 10^8$ cm. for the radius of $4.852 \circ$). The mass $0.957 M_\odot$ has a radius considerably less than the radius of the Earth.

### Table I

The Physical Characteristics of Degenerate Spheres in the “Natural” Units

<table>
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<tr>
<th>$\frac{1}{y_0^2}$</th>
<th>$\eta$</th>
<th>$-\eta^2 \phi'(\eta)$</th>
<th>$\rho_0 / \rho$</th>
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### Table II

<table>
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<tr>
<th>$\frac{1}{y_0^2}$</th>
<th>$M/M_\odot$</th>
<th>$R_1/l_1$</th>
<th>$\rho_0 / B$</th>
</tr>
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<tr>
<td>0</td>
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<td>0.95733</td>
<td>0.53571</td>
<td>985.038</td>
</tr>
<tr>
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<td>0.92400</td>
<td>0.70587</td>
<td>343.000</td>
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<td>0.84709</td>
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<td>82.8191</td>
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</table>

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Table III

The Physical Characteristics of Degenerate Spheres in the Usual Units

(Calculations are for $\mu = 1$. For other values $\mu$, $M$ should be multiplied by $\mu^{-2}$, $R_1$ by $\mu^{-1}$, $\rho_c$ by $\mu$)

<table>
<thead>
<tr>
<th>$\frac{1}{\gamma_0}$</th>
<th>$M/\bigcirc$</th>
<th>$\rho_0$ in grm./cm.$^{-3}$</th>
<th>$\rho_{\text{mean}}$ in grm./cm.$^{-3}$</th>
<th>Radius in cm.</th>
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</thead>
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<td>$5.728$</td>
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<td>$4.716 \times 10^7$</td>
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<td>$0.01$</td>
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<td>$5.111 \times 10^8$</td>
<td>$4.136 \times 10^8$</td>
</tr>
<tr>
<td>$0.02$</td>
<td>$5.294$</td>
<td>$3.391 \times 10^7$</td>
<td>$2.114 \times 10^6$</td>
<td>$5.443 \times 10^8$</td>
</tr>
<tr>
<td>$0.05$</td>
<td>$4.852$</td>
<td>$8.187 \times 10^7$</td>
<td>$7.960 \times 10^6$</td>
<td>$7.699 \times 10^8$</td>
</tr>
<tr>
<td>$0.1$</td>
<td>$4.310$</td>
<td>$2.669 \times 10^7$</td>
<td>$4.065 \times 10^6$</td>
<td>$9.936 \times 10^8$</td>
</tr>
<tr>
<td>$0.2$</td>
<td>$3.528$</td>
<td>$7.908 \times 10^6$</td>
<td>$2.302 \times 10^6$</td>
<td>$1.287 \times 10^9$</td>
</tr>
<tr>
<td>$0.3$</td>
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<td>$3.523 \times 10^6$</td>
<td>$1.345 \times 10^6$</td>
<td>$1.514 \times 10^9$</td>
</tr>
<tr>
<td>$0.4$</td>
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<td>$1.816 \times 10^6$</td>
<td>$7.741 \times 10^4$</td>
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<td>$5.381 \times 10^5$</td>
<td>$7.793 \times 10^4$</td>
<td>$2.793 \times 10^9$</td>
</tr>
<tr>
<td>$0.8$</td>
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<td>$1.236 \times 10^5$</td>
<td>$1.936 \times 10^4$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Now if we define that matter is "relativistically degenerate" for densities greater than $\rho' = (K_s/K_1)^{\frac{3}{2}}$, then we can from our results easily find the masses which are characterised by central regions of "relativistic degeneracy." The value of $x$ corresponding to $\rho'$ is readily seen to be $1.25$. Hence

$$\frac{1}{\gamma_0^2} = \frac{1}{x'^2 + 1} = 0.39024.$$  \hspace{1cm} (76)

From fig. 1 we now see that for $M < 0.43M_8$ there are no regions which are "relativistically degenerate" on this convention. For $M > 0.43M_8$ there are regions in which $x > x' = (1.25)$, and the fraction of the whole radius inside which $x > x'$ rapidly increases to unity. In the mass-radius curve we can therefore draw circles about each point with radii proportional to the actual radii of the corresponding configurations, and draw inside each a concentric circle to represent the "relativistic" region. This has been done in fig. 2 at a few points. We see that even for $M = 0.75M_8$ there is barely a "fringe" of ordinarily degenerate regions. This diagram clearly illustrates a general principle that degeneracy never usually sets in without being relativistic.

13. Comparison with the Results on Emden Polytrope $n = 3/2$.—It is of interest to see in how far the results of the above exact treatment differ from what one would obtain on the law $p = K\rho^{n/3}$. We have already shown in § 7 that one gets these Emden configurations as limiting cases for zero density and therefore for small masses (expressed in units of $M_8$). Our comparison here therefore amounts to a comparison of the results based on an exact treatment of the equation (64) with the limiting form for $\gamma_0 \to 1$ extrapolated for all masses. For this purpose it is convenient to rewrite the formulæ for the case of the polytrope $n = 3/2$ in the following way.
From (45) and (50) we have now
\[
R_1 = \frac{l_1 \xi_1(\theta_{3/2})}{\sqrt{2r_0}}, \quad (77)
\]
\[
M/M_3 = \left(\frac{x_0}{2}\right)^{3/2} \frac{1}{\omega_3^0} \left(\xi_3^2 d\theta_{3/2}/d\xi\right)_1. \quad (77')
\]

From (77) and (77') we have on eliminating \(x_0\)
\[
2R_1 = \left(\frac{\omega_{3/2}^0 M_2}{\omega_3^0 M}\right)^{1/3} I_1,
\]
where following Milne we have introduced the "invariant" \(\omega_{3/2}^0\) defined by
\[
\omega_{3/2}^0 = -\left(\xi_3^2 d\theta_{3/2}/d\xi\right)_1 = 132.3843. \quad (79)
\]

**Fig. 1.**—\((\Omega/\omega_3^0, 1/\rho_0^4)\)-relation.
It is of interest to notice that the two invariants \( \omega_3^0 \) and \( \omega_{3/2}^0 \) of the Emden equation with the indices \( n = 3 \) and \( 3/2 \) occur in (78) in a "symmetrical way." Numerically (78) is found to be

\[
R_1 = 2.01647 \left( \frac{M_3}{M} \right)^{1/3} l_1. \tag{80}
\]

(80) expresses the mass-radius relation for the polytropic limit, the radius and the mass expressed in the same units as the quantities in Table II. Similarly the mass-central density relation now reads

\[
x_0^3 = 4.4238 l(M/M_3)^2. \tag{81}
\]
The results calculated on the basis of (80) and (81) for the same masses as in Table II are summarised in Table IV. The corresponding curves are shown dotted in figs. 2 and 3.

<table>
<thead>
<tr>
<th>$M/M_3$</th>
<th>$R_1/l_1$</th>
<th>$x_0^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>0.1532</td>
<td>3.769</td>
<td>0.1038</td>
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</table>

One notices clearly from these two curves how marked the deviations from the limiting curves become even for quite small masses. Thus for $M = 0.15M_3$ the central density predicted by our exact treatment is about 25 per cent. greater and the radius about 5 per cent. smaller. The relativistic effects are therefore quite significant even for small masses. They certainly cannot be ignored for masses greater than $0.2M_3$. Of course the extrapolation of the $n=3/2$ configurations for masses (in units of $M_3$) approaching unity is quite misleading. These completely collapsed configurations have a natural limit, and our exact treatment now shows how this limit is reached.

It is of interest to compare the full-line curve in fig. 2 representing our exact (mass-radius) curve with what one would obtain by the methods of I, where the degenerate spheres of mass greater than a certain limit $M_{3/2}$ were considered as "composite configurations." The mass $M_{3/2}$ was defined as one in which the Emden polytrope with $n=3/2$ * would have a central density $\rho' = (K_2/K_1)^3$. In our present notation we have by (81)

$$M_{3/2} = \sqrt[3]{\frac{(1.25)^3}{4.42381}} M_3 = 0.66446M_3.$$  (82)

This particular point is marked as B in fig. 2 on the - - - - - curve. A treatment of the composite configurations by the methods of I would have led to some kind of curve like the dotted one in fig. 2 conjecturally drawn. But fortunately it is now not necessary to go into the very elaborate numerical work that would have been involved to fix the part BA by the methods of I. By a single system of integrations we have now fixed the exact nature of the (mass-radius) curve for these completely collapsed configurations.

* The equation of state being $p = K_1\rho^{5/3}$. 

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14. The Relative Density Distributions in the Different Configurations.—
Our main diagram (fig. 4) now illustrates the relative density distributions
in the configurations studied. Here we have plotted \((\rho/\rho_0)\) against \((\eta/\eta_4)\) for
the different masses for which we have numerical results. The two limiting
density distributions specified by Emden, \(\theta_3\) and \(\theta_{3/2}\), are also shown (dotted)
in the same figure. Fig. 4, which is the principal outcome of our studies,
presents a set of ten out of a continuous family of density distributions

covering the range specified by the polytropic distributions of indices
\(3/2\) and \(3\).

15. Concluding Remarks.—In this paper we have strictly confined our-
selves to the case \(\beta = 1\). But in stellar problem the radiation pressure
(even if small) necessarily plays a deciding rôle, and the question as to in what
sense we have to understand the completely degenerate spheres studied
here as representing “the limiting sequence of configurations to which all
stars must tend eventually” can be answered only by introducing radia-
tion in these configurations. To do this properly we have first to develop
adequate methods to treat composite configurations consisting of degenerate
cores (of the structures studied here) surrounded by gaseous envelopes.
These and related problems are studied in the following paper (p. 226).

16. Manuscript Copy of Tables.—The functions \(\phi\) and their derivatives
\(\phi'\) (to six and five significant figures respectively) have been computed by
the author for the values of \(1/y_0^2\) specified in (73). In addition to \(\phi\) and \(\phi'\)
the auxiliary functions \(\rho/\rho_0\), \(\rho_0/\rho\), \(-\eta^2\phi'\) and two other functions \(U\) and \(V\)
deefined in equation (91) of the following paper) have also been tabulated.
The auxiliary functions were calculated correct to five significant figures. All
the functions were tabulated for steps of \(0.1\) for the argument \(\eta\). A manuscript copy of these tables has been deposited in the Library of the Society.*

---

**APPENDIX**

*The Equation of State for a Degenerate Gas.*—The equation has been derived by Stoner (among others),† but we shall give a simpler derivation of the same.

In a completely degenerate electron assembly all the electrons have momenta less than a certain “threshold” value \(p_0\), and in the region of the

---

* Dr. Chandrasekhar’s Tables can be consulted by Fellows on application to the Assistant Secretary (Editors).
† M.N., 92, 444, 1931.
available phase space of volume $\frac{4}{3}\pi r_0^3 V$ every cell of volume $h^3$ contains just two electrons. Clearly then we have

$$n = \frac{8\pi \rho_0^3}{h^3} \int_0^p p^3 dp,$$

(1)

$$\mathcal{E} = \frac{8\pi V}{h^3} \int_0^{r_0} E p^2 dp,$$

(2)

$$P = \frac{8\pi \rho_0^3}{3h^3} \int_0^p \frac{dE}{dp} dp,$$

(3)

where $n$ is the number of electrons per unit volume in the assembly of volume $V$, $\mathcal{E}$ the total energy and $E$ the kinetic energy of a free electron. We have now denoted the pressure by $P$ instead of by $p$ as in the text of the paper to avoid confusion with the momentum, which has to be denoted by $p$. From (1) and from (2) and (3) we have respectively

$$p_0^3 = \frac{3h^3 n}{8\pi}; \quad P = \frac{8\pi}{3h^3} E \rho_0 \frac{\mathcal{E}}{V}.$$

(4)

Equations (1) to (4) are quite general. Now in the relativistic mechanics we have

$$E = mc^2 \left\{ \left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2} - 1 \right\},$$

(5)

or

$$p^2 = \frac{E(E + 2mc^2)}{c^2}.$$

(5')

Using (5') in (3) we have, after some minor transformations, that

$$P = \frac{8\pi m_0 c^6}{3h^3} \int_0^{\theta_0} \sinh^4 \theta d\theta,$$

(6)

where

$$\sinh \theta = \frac{p}{mc}; \quad \sinh \theta_0 = \frac{p_0}{mc}.$$

(7)

(7) yields at once that

$$P = \frac{8\pi m_0 c^6}{3h^3} \left[ \sinh^3 \theta \cosh \theta - \frac{3}{16} \sinh 2\theta + \frac{3}{8} \right] \bigg|_{\theta = \theta_0}.$$

(8)

Writing $x$ for ($p_0/mc$) we have

$$P = \frac{\pi m_0 c^6}{3h^3} \left[ x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x \right],$$

(9)

* This equation follows directly from the expression for the number of waves associated with electrons whose energies lie between $E$ and $E + dE$ given by Dirac (P.R.S., 112, 660, 1926, his unnumbered equation on p. 671). Actually Dirac obtains this result using the Klein-Gordon relativistic wave equation. That the same result would follow from Dirac's relativistic wave equation (on neglecting the states of kinetic energy—which is permissible when no external perturbations are present) is clear from J. von Neumann, Z. f. Physik, 48, 868, 1928.

† $\theta$ here introduced will not be confused with the Emden function.
\[ \rho = n \mu H = \frac{8\pi m^2 c^3 \mu H}{3h^3} x^3, \]  

which are the equations quoted in the text. Our derivation now shows "why" we are able to reduce the differential equation for degenerate gas spheres in gravitational equilibrium to such a simple form. The "reason" is that we have such an elementary integral for \( P \) as in (6).

The function \( f(x) \) on the right-hand side of (9) has the following asymptotic forms:—

\[ f(x) \sim \frac{5}{2} x^5 - \frac{5}{4} x^7 + \frac{3}{2} x^9 - \frac{5}{2} x^{11} + \ldots \quad x \to 0, \]  

\[ f(x) \sim 2x^4 - 3x^2 + \ldots \quad x \to \infty. \]

Finally we notice that

\[ \frac{f(x)}{2x^4} < 1 \quad \text{for all finite} \quad x. \]  

The inequality in (13) is a strict one. If only the first terms in the expansions (11) and (12) are retained, we can easily eliminate \( x \) from (9) and (10) for these limiting cases and obtain, as we should expect, that

\[ P = K_1 \rho^{5/3} \quad (x \to 0) ; \quad P = K_2 \rho^{5/3} \quad (x \to \infty), \]

with

\[ K_1 = \left( \frac{1}{20\pi} \right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}} ; \quad K_2 = \left( \frac{3}{20\pi} \right)^{1/3} \frac{hc}{8(\mu H)^{1/3}}. \]

If we write our "equation of state" (9) and (10) parametrically as (changing to "\( \rho \)" to denote pressure),

\[ \rho = A_2 f(x) ; \quad \rho = Bx^3, \]

we find, on putting in the numerical values for the constants, that (in C.G.S. units)

\[ A_2 = 6.0406 \times 10^{22} ; \quad B = 9.8848 \times 10^6 \mu, \]

or

\[ \log \rho = 5.9950 + 3 \log x + \log \mu, \quad \log \rho = 22.7811 + \log f(x). \]

Stoner has previously made some calculations concerning the \((\rho, \rho)\) relation for a degenerate gas, but for the study in the following paper more accurate tables for \( f(x) \) were needed. Accordingly the whole computation was re-
done and the results are tabulated in Table V. I am indebted to Dr. Comrie and Mr. Sadler for the loan of a manuscript copy of a seven-figure table for sinh⁻¹ x, which was valuable in the computations of f(x).

### Table V

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<th>x</th>
<th>f(x)</th>
<th>f(x)/2x^4</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
</tr>
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<td>0.000505</td>
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<tr>
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Trinity College, Cambridge:
1935 January 1.