I. The Fundamental Equations and the Methods of Approximation

§ 2. The Flux-Integral.—Let \( I \) be the specific intensity of the radiation at a point \( P \) and at a distance \( r \) from the origin \( O \) and in a direction making an angle \( \theta \) with \( OP \) in the positive direction of the radius vector. Let \( \rho \) and \( T \) be the density and temperature at any point and \( \kappa \) the coefficient of mass-absorption supposed independent of wave-length. Lastly, let \( B \) (a function of \( r \) only) be the intensity of the black body radiation corresponding to the temperature of the matter at any point \( r \). The equation of transfer in polar co-ordinates is

\[
\cos \theta \frac{\partial I}{\partial r} \sin \theta \frac{\partial I}{\partial \theta} = \kappa \rho (B - I). \tag{1}
\]

The equation of radiative equilibrium is, as usual,

\[
2B = \int_{0}^{\pi} I \sin \theta d\theta. \tag{2}
\]

Multiplying (1) by \( \sin \theta d\theta \) and integrating between \( 0 \) and \( \pi \) we get after some transformations that

\[
\frac{d}{dr} \int_{0}^{\pi} I \cos \theta \sin \theta d\theta = 0. \tag{3}
\]
If we write
\[ F = 2 \int_0^\pi I \sin \theta \cos \theta \, d\theta, \] (4)
then \( \pi F \) is merely the net flux of radiation at the point \( r \). Hence by (3) the first integral of the equation of transfer is
\[ Fr^2 = \text{constant} = F_0 \, (\text{say}). \] (5)

§ 3. The Methods of Approximation.—The problem is to obtain the solution of (1) subject to (2) or its equivalent (5). Even in the case where the material is regarded as being stratified in parallel planes it has not been possible to integrate the equation of transfer exactly, and the two main methods of approximation that have been used in the “plane case” are the following:

(a) The Schuster-Schwarzschild method of approximation in which we simply integrate the equation of transfer between \( 0 < \theta < \frac{1}{2} \pi \) and \( \frac{1}{2} \pi < \theta < \pi \) and replace \( I \) in the two respective hemispheres by their average values \( I_1 \) and \( I_2 \).

(b) A second type of approximation which is due to Milne* is to seek a solution of the equation of transfer consistent with the flux integral. The approximations obtained by this method are much more accurate than those obtained by the method (a). A less rigorous but a much more rapid method of obtaining solutions (identical with those obtained by Milne’s method) is the Eddington† type of approximation, which consists in the following two steps. If
\[ J = \frac{1}{2} \int_0^\pi I \sin \theta \, d\theta, \] (6)
and
\[ K = \frac{1}{2} \int_0^\pi I \cos^2 \theta \sin \theta \, d\theta, \] (7)
then we put
\[ K = \frac{1}{3} J, \] (8)
suggested by the mean value of \( \cos^2 \theta \) over the sphere. Secondly, one uses the boundary condition that at the boundary
\[ F(\pi) = 2 J(\pi). \] (9)

§ 4. In treating the problems of radiative equilibrium in polar co-ordinates it is however not clear that one could use the same methods of approximation. Indeed, as McCrea‡ first showed, the Schuster-Schwarzschild method of approximation applied to our case without any modification leads to inconsistencies. But, as will be shown subsequently, if one uses the Milne-Eddington type of approximation without any modification then one is not led to such inconsistencies. This, however, does not mean that this method of approximation is the proper one to work with when treating the problem in polar co-ordinates.

* E. A. Milne, M.N., 81, 382, 1921, which is the standard paper on the subject.
Actually the type of approximation one finally adopts in a special case has, of course, to be chosen from physical considerations. If, for instance, we are considering the radiative equilibrium of a tenuous atmosphere (like the chromosphere) enveloping a parent star, then, as was first pointed out by McCrea, a consistent method of approximation is always to average over \( \phi < \theta < \sin^{-1}(a/r) \) and \( \sin^{-1}(a/r) < \theta < \pi \) (instead of \( \phi < \theta < \pi/2 \) and \( \pi/2 < \theta < \pi \)), where \( a \) is the radius of the parent star. On this basis McCrea indicated the method of approximation which should replace the usual Schuster-Schwarzschild method of approximation. Here we shall also develop methods for an approximation which should replace the usual Milne-Eddington type of approximation. Of course, this method of averaging is suitable only when one has an unambiguous \( \phi \) occurring in the problem. If we are considering extended photospheres, for instance, then this method of averaging can no longer be valid. We shall then use the Milne-Eddington type of approximation which, as we have already pointed out, does not lead us to any inconsistency.

II. The Schuster-Schwarzschild Problems

§ 5. We shall first consider the case where the McCrea type of averaging has to be adopted.

At a point \( r \) we define \( \theta_r \) to be \( \sin^{-1}(a/r) \). Now, multiply the equation of transfer by \( \frac{1}{r} \sin \theta d\theta \) and integrate from \( \phi \) to \( \theta_r \). On the left-hand side we have

\[
\frac{1}{2} \int_0^{\theta_r} \left\{ \cos \theta \sin \frac{\partial I}{\partial r} - \frac{\sin^2 \theta}{r} \frac{\partial I}{\partial \theta} \right\} d\theta,
\]

which, after partially integrating the second term, transforms into

\[
\frac{1}{2} \int_0^{\theta_r} \cos \theta \frac{\partial I}{\partial r} \left( \frac{r^2 I}{r^2} \right) d\theta - \frac{a^2}{r^2} I(r, \theta_r),
\]

which, after another partial integration, yields

\[
\frac{1}{2r^2} \frac{\partial}{\partial r} r^2 \int_0^{\theta_r} \cos \theta \sin \theta d\theta - \frac{1}{2} I(r, \theta_r) \left[ \cos \theta \frac{\partial^2 \theta}{\partial r^2} \right] \sin \theta_r - \frac{a^2}{r^2} I(r, \theta_r). \quad (10)
\]

But

\[
\sin \theta_r = a/r; \quad \cos \theta_r \frac{\partial \theta_r}{\partial r} = -\frac{a}{r^2}. \quad (10')
\]

The last two terms in (10) cancel and we are left with

\[
\frac{1}{2r^2} \frac{d}{dr} r^2 \int_0^{\theta_r} \cos \theta \sin \theta d\theta. \quad (11)
\]

Defining

\[
F_1 = 2 \int_0^{\theta_r} \cos \theta d\theta; \quad F_2 = 2 \int_0^{\pi} \cos \theta d\theta, \quad (12)
\]

\[
\mathcal{J}_1 = \frac{1}{2} \int_0^{\theta_r} \cos \theta d\theta; \quad \mathcal{J}_2 = \frac{1}{2} \int_0^{\pi} \cos \theta d\theta, \quad (13)
\]
we have finally

\[ \frac{I}{4r^2} \frac{d}{dr} r^2 F_1 = \frac{1}{2} \kappa \rho [(1 - \cos \theta_r) B - 2 F_1]. \]  

(14)

Similarly

\[ \frac{I}{4r^2} \frac{d}{dr} r^2 F_2 = \frac{1}{2} \kappa \rho [(1 + \cos \theta_r) B - 2 F_2]. \]  

(15)

The equation of radiative equilibrium now takes the form

\[ B = F_1 + F_2 = \mathcal{F} \text{ (say)}. \]  

(16)

§ 6. A First Approximation.—Equations (14), (15) and (16) are exact. Now replace \( I \) in the ranges \((\tau, \theta_r)\) and \((\theta_r, \pi)\) by their average values \( I_1 \) and \( I_2 \). We then have

\[ F_1 = 2 \int_{\theta_r}^{\pi} I \sin \theta \cos \theta d\theta = I_1 \frac{a^2}{r^2}, \]  

(17)

\[ F_2 = 2 \int_{\theta_r}^{\pi} I \sin \theta \cos \theta d\theta = -I_2 \frac{a^2}{r^2}. \]  

(17')

Also

\[ F_1 = \frac{1}{2} I_1 (1 - \cos \theta_r); \quad F_2 = \frac{1}{2} I_2 (1 + \cos \theta_r). \]  

(18)

By (17) and (17') we have

\[ F = F_1 + F_2 = (I_1 - I_2) \frac{a^2}{r^2}. \]  

(19)

From the flux-integral (5) we easily deduce from the above that,

\[ (I_1 - I_2) = F_a, \]  

(20)

where \( F_a \) is the flux of radiation at \( r = a \). The equation of radiative equilibrium now takes the form

\[ B = \frac{1}{2} I_1 (1 - \cos \theta_r) + \frac{1}{2} I_2 (1 + \cos \theta_r). \]  

(21)

Solving (20) and (21) for \( I_1 \) and \( I_2 \) we have

\[ I_1 = B + \frac{1}{2} F_a (1 + \cos \theta_r), \]  

(22)

\[ I_2 = B - \frac{1}{2} F_a (1 - \cos \theta_r). \]  

(22')

Equations (14) and (15) in their averaged form are

\[ \frac{a^2}{4r^2} \frac{dI_1}{dr} = \frac{1}{2} \kappa \rho (B - I_1)(1 - \cos \theta_r), \]  

(23)

\[ \frac{a^2}{4r^2} \frac{dI_2}{dr} = \frac{1}{2} \kappa \rho (I_2 - B)(1 + \cos \theta_r). \]  

(23')

Substituting (22) and introducing the new variable \( \tau \) defined by

\[ d\tau = -\kappa dr, \]  

(24)

we find that

\[ \frac{dI_1}{d\tau} = \frac{dI_2}{d\tau} = F_a, \]  

(25)
or

$$I_1 = \frac{1}{2}F_a + B_0 + F_a \tau, \quad (26)$$

$$I_2 = -\frac{1}{2}F_a + B_0 + F_a \tau, \quad (26')$$

where $B_0$ is a constant of integration. Let $\tau = 0$ occur at $r = R$. Then to
determine $B_0$ we use the boundary condition that at $r = R$, $\tau = 0$ (cf. equa-
tion (9)),

$$B(R, 0) = \frac{1}{2} F(R) = \frac{1}{2} F_a \cdot \frac{a^2}{R^3}. \quad (27)$$

This determines $B_0$. We find

$$B_0 = \frac{1}{2} F_a \cos \theta_R + \sin^2 \theta_R. \quad (28)$$

From (26) and (21) we then have

$$I_1 = \frac{1}{2} F_a (1 + \cos \theta_R + \sin^2 \theta_R + 2 \tau), \quad (29)$$

$$I_2 = \frac{1}{2} F_a (\cos \theta_R + \sin^2 \theta_R - 1 + 2 \tau), \quad (29')$$

$$B = \frac{1}{2} F_a [\sin^2 \theta_R + (\cos \theta_R - \cos \theta_r) + 2 \tau]. \quad (29'')$$

(29'') now corresponds to Schwarzschild's solution

$$B = \frac{1}{2} F(1 + 2 \tau), \quad (30)$$

where $F$ now is the constant net flux. If $T_0$ is the boundary temperature
at $r = R$, then (29'') relates $T_0$ to the effective temperature $T_e$ (defined by
$\pi F_a = \sigma T_e^4$) by the equation

$$T_0 = \sqrt{\frac{a^2}{2R^3}} T_e. \quad (31)$$

We may finally notice that if $R$ is very large compared with $a$ then we
have

$$I_1 = F_a (1 + \tau); \quad I_2 = F_a \tau, \quad (32)$$

$$B = \frac{1}{2} F_a [(1 - \cos \theta_r) + 2 \tau]. \quad (32')$$

(32) is identical with the corresponding solution when the curvature is
neglected.

§ 7. A Second Approximation.—We first define the two quantities $K_1$
and $K_2$ as follows:

$$K_1 = \frac{1}{2} \int_0^{\theta_r} I \sin \theta \cos \vartheta \, d\vartheta \quad ; \quad K_2 = \frac{1}{2} \int_{\theta_r}^{\pi} I \sin \theta \cos \vartheta \, d\vartheta. \quad (33)$$

Multiply the equation of transfer by $\frac{1}{2} \sin \theta \cos \vartheta \, d\vartheta$ and integrate from
$0$ to $\theta_r$. On the L.H.S. we have, after partially integrating the second term,

$$\frac{1}{2} \int_0^{\theta_r} \frac{\sin \theta \cos \vartheta \, \partial}{\partial \vartheta} (r^2 I) d\vartheta - \frac{1}{2r} \int_0^{\theta_r} I \sin \theta (1 - \cos^2 \vartheta) d\vartheta - \frac{a^2}{2r^3} \cos \theta_r I(\theta_r, \theta_r). \quad (34)$$

Again, partially integrating the first integral and using (10'), (13) and noting
(33) the above reduces to

$$\frac{r^2}{r^2 dr} K_1 + \frac{1}{r} (K_1 - J_1) = \frac{dK_1}{dr} + \frac{1}{r} (3K_1 - J_1), \quad (35)$$

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
we have finally
\[
\frac{dK_1}{dr} + \frac{i}{r}(3K_1 - \mathcal{J}_1) = \frac{\kappa \rho}{4} \left( \frac{Ba^2}{r^2} - F_1 \right).
\]

Similarly
\[
\frac{dK_2}{dr} + \frac{i}{r}(3K_2 - \mathcal{J}_2) = \frac{\kappa \rho}{4} \left( -\frac{Ba^2}{r^2} - F_2 \right).
\]

Adding (36) and (37) and using the flux integral we have
\[
\frac{d(K_1 + K_2)}{dr} + \frac{i}{r}(3(K_1 + K_2) - (\mathcal{J}_1 + \mathcal{J}_2)) = -\frac{\kappa \rho}{4} \mathcal{A} \cdot \frac{a^2}{r^2}.
\]

§ 8. As before, let \( I_1 \) and \( I_2 \) be the mean intensities in the two ranges. Then we have
\[
K_1 = \frac{i}{2} I_1(1 - \cos \theta_r); \quad K_2 = \frac{i}{2} I_2(1 + \cos \theta_r).
\]

From (39), (18) and (19) we deduce that
\[
3(K_1 + K_2) - (\mathcal{J}_1 + \mathcal{J}_2) = \frac{i}{2} \frac{a^2}{r^2} \cos \theta_r \cdot (I_1 - I_2),
\]
\[
= \frac{i}{2} \frac{a^2}{r^2} \cos \theta_r \cdot \mathcal{A}.
\]

Substituting (40) in (38) we have
\[
\frac{d(K_1 + K_2)}{dr} = -F_a \cdot \frac{a^2}{2r^3} \cos \theta_r - F_a \cdot \frac{a^2 \kappa \rho}{4r^2}.
\]

On integration this yields
\[
K_1 + K_2 = -\frac{i}{2} F_a \cos \theta_r + \frac{F_a}{4} \int_0^\tau \sin^2 \theta_r d\tau + \frac{1}{3} K_0,
\]
where \( K_0 \) is the constant of integration and \( d\tau \) is defined as in (24). Hence by (16) and (40) we have
\[
B = -\frac{i}{2} F_a \cos \theta_r + \frac{3}{2} F_a \int_0^\tau \sin^2 \theta_r d\tau + K_0.
\]

The constant \( K_0 \) is determined as before. We find
\[
B = \frac{i}{2} F_a \left\{ \sin^2 \theta_R + (\cos \theta_R - \cos \theta_r) + \frac{3}{2} \int_0^\tau \sin^2 \theta_r d\tau \right\},
\]
where at \( r = R, \tau = 0 \). The above solution is analogous to the solution in the "plane problem" in the form first obtained by Milne. If R is very large compared with \( a \), we have
\[
B = \frac{i}{2} F_a \left\{ (1 - \cos \theta_r) + \frac{3}{2} \int_0^\tau \sin^2 \theta_r d\tau \right\}.
\]

Relation (31) continues to be true in this order of approximation.

§ 9. We now consider the case where the type of averaging we have

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
used in the preceding sections is not valid. As has been pointed out we
now use the unmodified Milne-Eddington type of approximation. Writing
\[ \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2; \quad F = F_0/r^2; \quad K = K_1 + K_2, \]
(46)
we now have setting \( 3K = \mathcal{J} \) (cf. equation (38))
\[ \frac{dK}{d\tau} = \frac{F_0}{4r^2}, \]
(47)
or
\[ K = \frac{1}{2} F_0 \int_0^{\tau} \frac{d\tau}{r^2} + \frac{1}{3} K_0, \]
(48)
where \( K_0 \) is a constant of integration. Let \( \tau = 0 \) occur at \( r = R \). Using the
usual boundary condition we derive that
\[ B = \frac{1}{2} F_0 \left\{ \frac{1}{R^2} + \frac{3}{2} \int_0^{\tau} \frac{d\tau}{r^2} \right\}, \]
(49)
or more simply in terms of the flux \( \pi F_R \) at \( r = R \)
\[ B = \frac{1}{3} F_R \left\{ 1 + \frac{3}{2} \int_0^{\tau} \frac{R^2}{r^2} d\tau \right\}. \]
(50)
If \( R \) is very large we have the simpler formula
\[ B = \frac{3}{3} F_0 \int_0^{\tau} \frac{d\tau}{r^2}. \]
(51)

§ 10. Variation of Intensity.—So far we have not attempted to determine
\( I \) as a function \( r \) and \( \theta \). Having now determined \( B \) as a function of \( r \) we
have to introduce this in the equation of transfer and solve it for \( I \). The
formal solution can be easily written down. If \( I(\theta, p) \) refers to the intensity
at a point \( P \) in a direction \( OA \) which is at a perpendicular distance \( p \) from
the centre, then one easily verifies that
\[ I(\theta, p) = e^{-\kappa p \csc^2 \theta} \int_0^\pi \int_0^{\kappa p \csc^2 \theta} B(\theta) B(\theta) \theta d\theta d\theta. \]
(52)
We already know \( B \) as a function of \( r \), and by means of the substitution
\( r = \rho \csc \theta \) we can transform it into a function of \( \theta \). Thus (52) formally
represents the solution to the problem. In practice, however, to perform
the integrations we need to know the variation of \( \kappa p \) with \( r \), and for this
reason it does not seem worth while to go into details over this, but we shall
illustrate the general trend of the calculations corresponding to the solution
(51) for \( B \) and when \( \kappa p \) varies as some inverse power of \( r \).

Let us then suppose that
\[ \kappa p = cr^{-n} = \rho^{-n} \sin^n \theta. \]
(53)
By (51) we now have
\[ B = \frac{3F_0 c}{4(n + 1) \rho^{n+1}} \sin^{n+1} \theta. \]
(54)
Introducing this in (52) we have

\[
I(\theta, p) = \frac{3F_0 e^2}{4(n+1)b^2} \int_0^\theta \sin^{n-2} \theta d\theta \int_0^\pi \sin^{2n-1} \theta d\theta. 
\]

(55)

Writing \(a = c p^{-n+1}\)

we have

\[
I(\theta, p) = \frac{3F_0 a}{4(n+1)b^2} \int_0^\theta \sin^{n-2} \theta d\theta \int_0^\pi \sin^{2n-1} \theta d\theta. 
\]

(57)

We are interested only in the "emergent radiation," i.e. when \(\theta = 0\). Then we have

\[
I(p) = \frac{3F_0 a}{4(n+1)b^2} \int_0^\pi \sin^{2n-1} \theta d\theta. 
\]

(58)

§ 11. We will consider two special cases of (58).

Case I \((n = 3)\).—Now \(a = c p^2\), and (58) simplifies to

\[
I(p) = \frac{3F_0 a^2}{16c} \int_0^\pi e^{-a(1 - \cos \theta)(1 - \cos^2 \theta) \sin \theta d\theta. 
\]

(59)

Introducing the new variable \(a \cos \theta = x\) we have

\[
I(p) = \frac{3F_0 e^{-a}}{2c} \int_{-a}^{a} e^{(1 - x^2)/a^2} dx. 
\]

(60)

On performing the integration we find that

\[
I(p) = \frac{3F_0 e^{-a}}{a^2 c} \left[ (a^2 + 3) \sinh a - 3a \cosh a \right], 
\]

where \(a = c/p^2\).

The following limiting forms may be noted

\[
I(p) \sim \frac{F_0 e^2}{5b^4} \text{ for } p \text{ large,} 
\]

(62)

and

\[
I(p) \sim \frac{3F_0}{2c} \left( 1 - \frac{3p^2}{c} \right) \text{ for } p \text{ small.} 
\]

(63)

Case II \((n = 2)\).—Now \(a = c/p\), and (58) simplifies to

\[
I(p) = \frac{F_0 a^4}{4c^2} \int_0^\pi e^{-a \theta} \sin^{3} \theta d\theta, 
\]

\[
= \frac{3F_0}{2c^2} \frac{a^4(e^{-a\pi} + 1)}{(a^2 + 9)(a^2 + 1)}. 
\]

(64)

We now have

\[
I(p) \sim \frac{F_0 e^2}{3b^4} \text{ for } p \text{ large,} 
\]

(65)
and

\[ I(p) \sim \frac{3F_0}{2c^2} \text{ for } p \text{ small.} \]  

(66)

Comparing (65) with (62) we notice that the more rapidly \( \kappa p \) falls off with distance the more rapidly does \( I(p) \) fall off with \( p \). This is quite a physically understandable result.

§ 12. A Purely Scattering Atmosphere.—When we are dealing with a purely scattering atmosphere there is no interchange of energy between the different frequencies, and we shall therefore consider each frequency separately. Let \( s_\nu \) be the coefficient of scattering, \( \sigma_\nu \) the optical thickness \( \int s_\nu \rho dr \). Assuming the scattered radiation to be equal in all directions the equation of transfer is

\[ \cos \theta \frac{\partial I_\nu}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I_\nu}{\partial \vartheta} = s_\nu \rho (I_\nu - \frac{1}{2} \int_0^\pi I_\nu \sin \theta d\theta). \]  

(67)

We see that formally equation (67) is identical in form with the equations (1) and (2) taken together. Hence the analysis of the preceding sections apply to this case provided we replace \( \tau \) and \( \kappa \) wherever they occur by \( \sigma_\nu \) and \( s_\nu \), respectively, and also add a suffix \( \nu \) to all the symbols.

§ 13. The Schuster Problem.—The problem which is specifically associated with the name of Schuster is the following *: "A layer of gas scattering monochromatic radiation of frequency \( \nu \) is placed in front of a bright background radiating with a given flux \( \pi G_\nu \). Given the optical thickness \( \sigma_\nu \) of the scattering material, to determine the emergent flux." In our case we have now to reformulate the problem in the following terms:—

"Enveloping a bright spherical surface radiating monochromatic radiation of frequency \( \nu \) with a mean intensity \( i_\nu \), is an atmosphere of gas scattering monochromatic radiation of frequency \( \nu \). Given the radial optical thickness \( \sigma_\nu \) of the surrounding atmosphere and given also that the bright spherical surface is of radius \( a \) and that the atmosphere extends to \( r = R \), to determine the mean intensity \( I_\nu (a) \) of the radiation emergent through a tangent cone drawn from a point \( r = R \) to the sphere \( r = a \)."

It is clear from this formulation that we have now to adopt the type of averaging we have used in §§ 5–8. We shall use the results of the first approximation. From (29) we now have

\[ I_1 = \frac{1}{2} F_\nu (a) (1 + \cos \theta_R + \sin^2 \theta_R + 2 \sigma_\nu). \]  

(68)

At \( r = a \), \( I = i_\nu \) and we have

\[ i_\nu = \frac{1}{2} F_\nu (a) (1 + \cos \theta_R + \sin^2 \theta_R + 2 \sigma_\nu). \]  

(69)

While at \( r = R \), \( I_1 = I_\nu (a) \) and by (68)

\[ I_\nu (a) = \frac{1}{2} F_\nu (a) (1 + \cos \theta_R + \sin^2 \theta_R). \]  

(70)

Hence we have the following solution for the "Schuster's problem":—

\[ r_\nu = \frac{I_\nu (a)}{i_\nu} = \frac{1 + \cos \theta_R + \sin^2 \theta_R}{1 + \cos \theta_R + \sin^2 \theta_R + 2 \sigma_\nu}. \]  

(71)

(71) replaces the usual Schuster's formula for \( r_v \), namely:

\[
r_v = \frac{1}{1 + \sigma_v}.
\]

(72)

As was to be expected, the value of \( r_v \) depends not only on the optical thickness \( \sigma_v \), but also on the extent to which the scattering material is spread out. But this dependence is not very pronounced. Thus for a given \( \sigma_v \), \( r_v \) is a minimum when \((1 + \cos \theta_R + \sin^2 \theta_R)\) is a maximum which it attains when \( \theta_R = 60^\circ \). When \( \theta_R = 60^\circ \), (71) takes the form

\[
r_v = \frac{1}{1 + (8/9)\sigma_v}.
\]

(73)

\( \theta_R = 60^\circ \) corresponds to the scattering material being spread out to a distance \( 2/\sqrt{3} \) (= 1.1547) times the radius of the original "photospheric surface."

Now it is known that in the more exact form of the Schuster's formula we have in the denominator of (72), \( 3/4 \cdot \sigma_v \) instead of simply \( \sigma_v \). For us to obtain the corresponding second approximation we really need to evaluate the integral (52) with \( B \) given by (44), for which we require the exact variation of \( s_{\nu \rho} \) with \( r \). As this is a rather complicated matter we shall not go into it here.

III. The Formation of Absorption Lines *

§ 14. In considering the formation of absorption lines it is not quite clear as to the method of approximation that has to be adopted. For the sake of simplicity we shall treat the problem on the un-modified Eddington type of approximation. The other type of averaging leads to far too complicated equations.

Let

\( \kappa_v \) be the coefficient of continuous absorption applicable to both inside and immediately outside the absorption line,
\( s_v \) (a function of \( \nu \)) the coefficient of line absorption,

\( I'_v(r, \theta) \) the intensity of the radiation in the frequency range \( \nu \) to \( \nu + d\nu \) within the line at a point distant \( r \) from the centre and in a direction making an angle \( \theta \) with the positive direction of the radius vector and \( I(r, \theta) \) the corresponding intensity if there had been no scattering (i.e.) outside the line.

In addition to the quantities \( F, \mathcal{J}, K \), we now introduce a corresponding set of quantities \( F'_v, \mathcal{J}'_v, K'_v \) defined in exactly the same way but with the \( I'_v \)'s replacing the \( I \)'s in the respective integrals.

The equation of transfer is (cf. Eddington, loc. cit., equation (4))

\[
\cos \theta \frac{\partial I'_v}{\partial r} - \frac{\sin \theta}{r} \frac{\partial I'_v}{\partial \theta} = -\rho(\kappa_v + s_v)I'_v + \rho(1 - \epsilon)s_v\mathcal{J}'_v + \epsilon s_v B_v + \rho \kappa_v B_v, \quad (74)
\]

where $\epsilon$ is the fraction of the absorbed scattered radiation which is lost in collisions of the second kind, and $B_\nu$ as usual corresponds to the Planck-function.

Multiplying (74) by $\frac{1}{2} \sin \theta d\theta$ and $\frac{1}{2} \sin \theta \cos \theta d\theta$, and integrating between 0 to $\pi$, we have (cf. equations (14), (15) and (38))

$$\frac{1}{4r^2} \frac{d}{dr} r^2 F_\nu' = -\rho (\kappa_\nu + \epsilon s_\nu)(\mathcal{F}_\nu' - B_\nu),$$

(75)

$$\frac{dK_\nu'}{dr} = \frac{1}{r} (3K_\nu' - \mathcal{F}_\nu') = -\frac{1}{\rho} (\kappa_\nu + s_\nu) F_\nu'.$$

(76)

Let

$$\eta = s_\nu/\kappa_\nu,$$

(77)

and defining the optical thickness $\tau_\nu$ by

$$d\tau_\nu = -\kappa_\nu dr,$$

(78)

we have our fundamental system of differential equations

$$\frac{1}{4r^2} \frac{d}{d\tau_\nu} r^2 F_\nu' = (1 + \epsilon \eta)(\mathcal{F}_\nu' - B_\nu),$$

(79)

$$\frac{dK_\nu'}{d\tau_\nu} = -\frac{1}{\rho \kappa_\nu} (\mathcal{F}_\nu' - 3K_\nu') + \frac{1}{\rho} (1 + \eta) F_\nu'.$$

(80)

Our approximation is now to set $\mathcal{F}_\nu' = 3K_\nu'$. Hence

$$\frac{1}{4r^2} \frac{d}{d\tau_\nu} r^2 F_\nu' = (1 + \epsilon \eta)(\mathcal{F}_\nu' - B_\nu),$$

(81)

$$\frac{d\mathcal{F}_\nu'}{d\tau_\nu} = \frac{3}{4} (1 + \eta) F_\nu'.$$

(82)

§ 15. Assume now that $\eta$ and $\epsilon$ are both constants as a function of $\tau_\nu$. It is doubtful whether this approximation can be justified to the extent to which it can be justified in the case where the matter is stratified in parallel planes (cf. Milne (2), loc. cit., p. 5). However, we shall adopt it to see the nature of the results that arise out of this assumption. From (81) and (82) we now have

$$\frac{1}{r^2 \tau_\nu} \frac{d}{d\tau_\nu} \left( r^2 \frac{d\mathcal{F}_\nu'}{d\tau_\nu} \right) = q^2 (\mathcal{F}_\nu' - B_\nu),$$

(83)

where

$$q^2 = 3(1 + \eta)(1 + \epsilon \eta).$$

(84)

We will now consider the case where $B_\nu$ has its "equilibrium value," i.e., the radiation just outside the line has its equilibrium density. We have then to adopt our earlier solution in the form (50) for $B_\nu$:

$$B_\nu = \frac{1}{2} F_R, v \left[ 1 + \frac{1}{\rho} \int_0^{\tau_\nu} \frac{R^2}{r^2} d\tau_\nu \right].$$

(85)

For the sake of brevity we shall hereafter suppress the suffix $\nu$ to $F_R$, but it
has to be understood that $F_R$ always stands for the net flux of radiation in the frequency range $(\nu, \nu + d\nu)$ at the boundary $(r = R)$ of the star.

From (83) and (85) it follows that

$$\frac{1}{r^2} \frac{d}{d\tau} \left\{ r^2 \frac{d(\mathcal{J}_\nu - B_\nu)}{d\tau} \right\} = q^2(\mathcal{J}_\nu - B_\nu),$$

or differently as

$$\frac{d^2(\mathcal{J}_\nu - B_\nu)}{d\tau^2} + \frac{2}{r} \frac{d(\mathcal{J}_\nu - B_\nu)}{d\tau} - q^2(\mathcal{J}_\nu - B_\nu) = 0.$$  (87)

It is seen that to solve (87) we need the variation of $\tau_\nu$ with $r$. We shall examine the case when $\tau_\nu$ varies as some inverse power $m$ of $r$, i.e.

$$\tau_\nu = er^{-m} \text{ (say)}. $$  (88)

Consistent with this assumption we should now rewrite (85) in the form

$$B_\nu = \frac{3}{2} F_0 \int_0^{r_\nu} \frac{d\tau_\nu}{r^2},$$

where $F_0$ now stands for $F_{0,\nu}$. Introducing the following new quantities

$$n = \frac{m+2}{2m}, \quad \mathcal{J}_\nu - B_\nu = Q \tau_\nu^n; \quad Z = q \tau_\nu,$$

the equation (87) combined with the relation (88) leads to the following differential equation for $Q$:

$$Z^2 \frac{d^2 Q}{dZ^2} + Z \frac{dQ}{dZ} - (n^2 + Z^2)Q = 0.$$  (91)

(91) is just Bessel’s equation with the purely imaginary argument in. We need a solution of (91) which tends exponentially to zero for large values of $Z$, since $\mathcal{J}_\nu$ must tend to its equilibrium value at great distances. Hence the appropriate solution we have to consider is the following:

$$Q(Z) = K_n(Z) = \frac{1}{2\pi} \left( I_{-n}(Z) - I_n(Z) \right) \cot n\pi. $$  (92)

Hence our solution is

$$\mathcal{J}_\nu = \frac{3}{2} F_0 \int_0^{r_\nu} \frac{d\tau_\nu}{r^2} + A(q \tau_\nu)^n K_n(q \tau_\nu),$$  (93)

where $A$ is a constant at present undetermined. Finally, by (82), we have for the flux $\pi F_\nu'$,

$$F_\nu' = \frac{4}{3(1+\eta)} \left\{ \frac{3 F_0}{4 r^2} - Aq(q \tau_\nu)^n K_{n-1}(q \tau_\nu) \right\}.$$  (94)

A is now determined from the boundary condition that at $\tau_\nu = 0$, i.e. as $r \to \infty$, $F_\nu' - 2\mathcal{J}_\nu' \to 0$. From physical considerations both must separately

tend to zero as \( r \to \infty \). We will not go into the details here, but one can easily prove that if \( n > \frac{1}{2} \) (i.e. if \( m \) is finite) then \( Z^n K_{n-1}(Z) \to 0 \) as \( Z \to 0 \), while \( Z^n K_n(Z) \) tends to some limiting value. Hence our boundary condition can be satisfied if, and only if, \( A = 0 \), in which case we simply have

\[
F' = \frac{F \rho r^{-2}}{1 + \eta}.
\]

On the other hand, for the flux \( F \), just outside the line,

\[
F = F \rho r^{-2}.
\]

Hence

\[
\frac{F'}{F} = \frac{1}{1 + \eta}.
\]

For the case where the material is stratified in parallel planes the formula corresponding to (97) is (cf. Eddington, loc. cit., equation (17))

\[
\frac{F'}{F} = \frac{1 + \frac{3}{2} q}{1 + \eta + \frac{3}{2} q}.
\]

Thus for the case we have considered the introduction of the curvature far from introducing complications essentially simplifies the problem. Though the result (97) has been obtained only for a strictly infinite atmosphere, we can expect it to be valid if only the atmosphere be sufficiently extended.

**IV. The Radiative Equilibrium of a Planetary Nebula**

§ 16. This problem has already been considered in some detail by Milne.* But he neglected the curvature of the layers. It is, however, quite easy to find the appropriate solutions, taking into account the curvature as well.

Let \( \pi S \) be the incident flux per cm.² due to the central star on the inner boundary of the nebular shell. The equation of radiative equilibrium then takes the form

\[
2 \int_0^{\pi} (B - I) \sin \theta d\theta = \frac{S r_1^2 e^{-(\tau_1 - \tau)}}{r^2},
\]

where the optical thickness \( \tau \) is measured from the outer boundary of the nebular shell the total radial optical thickness of which is taken to be \( \tau_1 \). Further, let \( r_1 \) and \( r_2 \) be the inner and the outer radii of the planetary nebula.

The equation of transfer is the same as before (equation (1)). Multiplying the equation of transfer by \( \sin \theta d\theta \) and integrating from 0 to \( \pi \) and using (99), we find that

\[
\frac{d}{d\tau} r^2 F = -S r_1^2 e^{-(\tau_1 - \tau)},
\]

which on integration yields

\[
r^2 F = \text{constant} - S r_1^2 e^{-(\tau_1 - \tau)}.
\]

---

At $\tau = \tau_1$, $r = r_1$, we have $F = 0$. This is precisely equivalent to the boundary condition discussed by Milne (loc. cit., p. 102). From this we deduce that

$$F = \frac{S r_1^2}{r^2}(1 - e^{-(\tau_1 - \tau)}), \quad (102)$$

which is now our flux integral. Multiplying the equation of transfer by $\frac{1}{2} \sin \theta \cos \theta d\theta$ and integrating from $0$ to $\pi$ and setting $3K = \mathcal{F}$, we have

$$\frac{dK}{d\tau} = \frac{S r_1^2}{4r^2}(1 - e^{-(\tau_1 - \tau)}), \quad (103)$$

which leads to

$$\mathcal{F} = \frac{3}{2} \int_{0}^{\tau} \frac{S r_1^2}{r^2}(1 - e^{-(\tau_1 - \tau)}) d\tau + K_0, \quad (104)$$

where $K_0$ is a constant of integration to be determined from our usual boundary condition that at $r = r_2$, $\tau = 0$, $F = 2\mathcal{F}$. We find that

$$K_0 = \frac{S r_1^2}{r_2^2}(1 - e^{-\tau_1}). \quad (105)$$

We finally obtain from (99), (104) and (105) that

$$B = \frac{1}{2} S \left\{ \frac{r_1^2}{r_2^2}(1 - e^{-\tau_1}) + \frac{\tau r_1^2}{r_2^2} e^{-(\tau_1 - \tau)} - \frac{3}{2} \int_{0}^{\tau} \frac{\tau r_1^2}{r^2}(1 - e^{-(\tau_1 - \tau)}) d\tau \right\}. \quad (106)$$

If the radius of curvature be neglected, i.e. $r_1 = r_2 = r =$ constant, then we easily derive from (106) that

$$B = \frac{1}{2} S \left( 1 + \frac{3}{2} e^{-\tau} - e^{-(\tau_1 - \tau)} + \frac{3}{2} \tau \right), \quad (107)$$

which is Milne's result (loc. cit., equation (28)). The ratio of the boundary temperatures at $r = r_2$ given by (107) and (106) is seen to be $\sqrt{r_1/r_2}$, which might be appreciable for many planetaries.

Summary

In this paper the standard problems in the theory of radiative equilibrium which arise in the theory of stellar atmospheres are rediscussed without neglecting the curvature of the outer layers of the star.

The solution for the Schuster problem shows that the residual intensity depends not only on the optical thickness of the scattering material but also on the extent to which it is spread out. For a given optical thickness the minimum absorption ratio occurs when the tangent cone from a point on the boundary of the scattering atmosphere has a semivertical angle of 60°.

The formation of absorption lines has also been discussed in detail for the case when $\tau_y$ varies as some inverse power $m$ of $r$. It is shown that if $m$ is finite then we always have

$$\frac{F_y'}{F_y} = \frac{1}{1 + \eta}.$$

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
(\(F_r\)' being the flux of radiation in the line, while \(F_r\) is the corresponding flux just outside the line and \(\eta = \frac{s_r}{\kappa_r}\)).

Finally, the radiative equilibrium of the planetary nebula is briefly considered. It is shown that if \(r_1\) and \(r_2\) are the inner and the outer radii of the nebular shell, then the temperature at the boundary is \(\sqrt{r_1/r_2}\) times the value that would be predicted if we had neglected the curvature.

*Trinity College, Cambridge: 1934 March 2.*

THE CONTOURS OF THE POTASSIUM RESONANCE LINES IN ABSORPTION.

J. F. Heard, Ph.D., 1851 Exhibition Student.

(Communicated by Prof. A. Fowler, F.R.S.)

Introduction

Minkowski * and Korff † have shown that, neglecting second order quantities, the classical dispersion theory of Lorentz ‡ and Voigt § for the distribution of energy within a spectral line which has arisen from absorption by a finite thickness of vapour at a moderate temperature yields an expression, the form of which, as given by Korff, is

\[
(\Delta \lambda)^2 \log_e \frac{I}{I_0} = -\frac{2\pi e^4 \lambda^2}{3m^2c^4} N
\]

where \(\Delta \lambda\) is the distance of a given frequency within the line from the resonance frequency and

- \(I\) is the intensity of light transmitted at the given frequency,
- \(I_0\) is the intensity of light transmitted at a frequency far from resonance, i.e. the intensity of the continuous background,
- \(e\) is the charge of the electron,
- \(\lambda\) is the wave-length of the line under consideration,
- \(m\) is the mass of the electron,
- \(c\) is the velocity of light,
- \(N\) is the number of linear oscillators in 1 sq. cm. section in the line of sight.

The resonance lines of alkali arc spectra lend themselves to a test of the classical theory. The number of linear oscillators corresponding to each member of the doublet may be taken as \(N = nzf\), where \(n\) is the number of atoms per c.c., \(z\) the length of the absorbing column, and \(f\) the oscillator strength or the proportion of atoms being raised by absorption to the particular doublet state involved. From a consideration of the statistical