The numerical work of § 13 on the "model one-constituent atmospheres" is hardly affected. To get the more "exact" values for $\log N_0^{(s)}$ and $\log N_1$ we should subtract $\log \xi$ and $\frac{1}{2} \log \xi$ from the values tabulated in § 13. These corrections are hardly appreciable. For instance, in the case of the hydrogen atmosphere the corrections for $\log N_0^{(s)}$ and $\log N_1$ at 10,000 degrees are only $-0.06$ and $-0.03$, respectively. At the lower temperatures the "corrections" are considerably less.

MODEL STELLAR PHOTOSPHERES.

S. Chandrasekhar.

(Communicated by Professor Milne)

1. The physical structure of the photospheric layers has been studied by Woltjer, Jeans and Milne among others.* For our purposes here the treatment given by Milne in his Bakerian Lecture is the most convenient. In the analysis Milne assumes a law $\kappa \propto P/T^4$, and in his numerical work to evaluate the temperature, temperature gradient, pressure, density, etc., he uses for the constant of proportionality in the law $\kappa \propto P/T^4$, that derived by him "astrophysically" from the maxima of zinc lines.

But McCrea † has recently pointed out that another profitable approach to the problems associated with the stellar atmospheres is to work with simplified "models" and to work out fully the consequences of the physical theory. It is proposed therefore to apply here this general scheme of advance to the problem of the physical structure of the photosphere on the basis of the author's recent work on the stellar absorption coefficient.‡

2. Statement of the Problem.—The problem is to determine the distribution of pressure, density, temperature, temperature gradient and opacity through the photospheric layers (in terms of the optical thickness $\tau$ and depth $h$) of a one-constituent atmosphere where the first stage ionization has set in. In the solution of this problem we make (following Milne) the simplifying assumption of using a mean value for the ionization, instead of treating this as a variable with the "depth." The formula for the absorption coefficient is (a special case of equation (63) of my paper)

$$\kappa = \frac{40}{\pi^4 \sqrt{3}} \frac{e^{9} h^4}{cm(2\pi m)^3} \chi_1 \left(1 + \frac{243 \chi_1}{kT} \right) \frac{P\xi}{(kT)^{\frac{3}{2}}} \text{atomic mass},$$

(1)

* Woltjer, B.A.N., 2, 171 (No. 66), 1924; Jeans, M.N., 85, 199, 1925; Milne, M.N., 85, 199, 1925; Milne, M.N., 90, 17, 1929, §§ 20, 21 and 22; Milne, Phil. Trans., A, 228, 421, 1929, § 16 (Bakerian Lecture).

† W. H. McCrea, "Model Stellar Atmospheres," M.N., 91, 836, 1931. This kind of approach to stellar problems in general has been very often urged by Professor Milne on various occasions. See, for instance, his remarks in Zeits. für Astrophysik, 1, 98, 1930, where he compares stellar problems to those in geometry.

where \( P = \text{electron pressure}, \)
\( \chi_1 = \text{the first "ionization potential" in ergs}, \)
\( \bar{\chi} = \text{the fraction of the atoms ionized}. \)

Since for the temperatures usually met with in the photospheric layers \( (\sim 10^4 \text{ degrees}) \) the second term in the brackets in equation (1) is by far the most important, we take as our basic formula in the analysis

\[
\kappa = \alpha P \bar{\chi} / T^4, \tag{2}
\]

where

\[
\alpha = \frac{40}{\pi^4 \sqrt{3}} \frac{243 \chi_1^2}{\text{cm}(2\pi m)(h)^{1/2}} \times 5.62 \times 10^{19} \times \frac{\bar{\chi}}{a}, \tag{2'}
\]

where \( \bar{\chi} = \text{is the ionization-potential in electron-volts} \);
\( a = \text{the atomic weight (not mass)}. \)

3. Solution of the Problem.—Let \( p' \) denote the radiation-pressure at depth \( h \) in the photospheric layers measured from some convenient reference level. A net flux \( \pi F \) traversing a layer \( \rho dh \), of absorption coefficient \( \kappa \), communicates a momentum \( \pi F \kappa \rho dh / c \). Hence

\[
\frac{dp'}{dh} = \frac{\kappa \rho \pi F}{c}. \tag{3}
\]

Milne has proved that the relation \( p' = \frac{1}{3} a T^4 \) holds to the extent to which \( T^4 = \frac{1}{3} T_1^4 (1 + \frac{3}{2} T) \) is an accurate solution of Schwarzschild’s problem. The equation of mechanical equilibrium is

\[
\frac{dp}{dh} + \frac{dp'}{dh} = \pi g. \tag{4}
\]

Dividing (4) by (3),

\[
1 + \frac{dp}{dp'} = \frac{\pi g}{\kappa \pi F}. \tag{5}
\]

Let \( M \) be the mass of the star, \( L \) its bolometric magnitude; then

\[
g = GM/r_1^2 \quad \text{and} \quad \pi F = L / 4\pi r_1^2, \quad \text{i.e.} \quad g / \pi F = 4\pi GM / L.
\]

Further,

\[
\kappa = \frac{\alpha P \bar{\chi}}{T^4} = \alpha P \bar{\chi} \left( \frac{1}{\rho'} \right)^{1/4}, \tag{6}
\]

where \( \bar{\chi} \) is the average number of free electrons per atom in the photospheric layers, i.e.

\[
\frac{P}{\rho} = \frac{\bar{\chi}}{1 + \bar{\chi}}. \tag{7}
\]

Hence (5) becomes

\[
1 + \frac{dp}{dp'} = \frac{4\pi eGM}{L} \left( \frac{1 + \bar{\chi}}{\bar{\chi}} \right) \left( \frac{p' + \frac{1}{p}}{p} \right). \tag{8}
\]

Make the substitution

\[
\rho = u \left( \frac{p'}{p} \right)^{1/4}, \tag{9}
\]
where

\[ p_1' = \frac{3}{4} a T_1^4. \]  

(10)

(8) now becomes

\[ \left(\frac{p_1'}{p'}\right)^{\frac{3}{5}} + \frac{19}{16} \frac{d u}{d p'} = \frac{4\pi G M}{L} \left(\frac{1 + \bar{x}}{\bar{x}} a\right)^{\frac{3}{5}} \frac{1}{u} \]

\[ = \left[ \frac{4\pi G M}{L} \left(\frac{1 + \bar{x}}{\bar{x}} a\right)^{\frac{3}{5}} T_1^\frac{3}{5} \right] \frac{1}{u} \]

\[ = \frac{A}{u} \quad \text{(say)}. \]  

(11)

Now \((p_1'/p')^{\frac{3}{5}}\) deviates from unity only very little, and we replace it by unity. Hence we have

\[ \frac{d p'}{p'} = -\frac{u d u}{A - u - \frac{1}{4} u^2}. \]  

(12)

Write

\[ A - u - \frac{1}{4} u^2 = -\frac{1}{4} (u - u_1) (u + u_2). \]  

(13)

From (13) we see immediately that

\[ A = (\frac{1}{2}) u_1 u_2; \]

\[ \frac{1}{2} = u_2 - u_1. \]  

(13')

Integration of (12) gives us now

\[ \log \left(\frac{p'}{p_0}\right) = -\frac{1}{2} \left[ u_1 \log \left(1 - \frac{u}{u_1}\right) + u_2 \log \left(1 + \frac{u}{u_2}\right) \right], \]  

(14)

remembering the "boundary condition" that, when \(u = 0\),

\[ p' = p_0' = \frac{3}{4} a T_0^4. \]

Since as the solution of Schwarzschild's problem we have

\[ \frac{p'}{p_0} = 1 + \frac{3}{5} \tau, \]  

(15)

we see that as \(\tau \) increases, \(p'/p_0'\) increases and \(u\) rapidly increases and approaches the limit \(u_1\). Hence by (9) the ratio of gas-pressure to radiation-pressure increases slowly according to \(u_1(p'/p_1)^{\frac{3}{5}}\). Thus

\[ \frac{p}{p'} \quad \text{ultimately increases as} \quad T^\frac{3}{5}. \]  

(16)

It may be remarked here that the corresponding result on the Jeans-Milne analysis is that "\(p'/p\) ultimately increases as \(T^\frac{3}{5}\)." Now equations (14), (15) and (9) determine \(p', \tau \) and \(p\) for a given \(u, p'\) in turn determining the temperature \(T\). It now remains to determine the heights to which these values of \(u\) refer. In equation (3) substitute \(\rho = \rho/(R/\mu)T\) and insert also the formula (6) for \(\kappa\). We find

\[ \frac{d p'}{d h} = \frac{a\bar{x}^2 \left(\frac{3}{4} a\right)^{\frac{7}{5}}}{1 + \bar{x}/p'} \frac{p^2}{(R/\mu)T} \frac{g L}{4\pi G M'}. \]  

(17)
or by (9),
\[ \frac{1}{T} \frac{dT}{dh} = \frac{1}{4} \frac{dp'}{dh} = \frac{1}{4} \frac{dx^2}{h^2} \frac{d}{dx} \left( \frac{\dot{p}}{p} \right) = \frac{\dot{p}}{p} \frac{2}{R/\mu} \frac{g}{4 \pi c GM} = \frac{1}{4} \frac{g}{A(R/\mu)T} \frac{u^2}{u^2} ; \] \tag{17'}

or by (13'),
\[ \frac{dT}{dh} = \frac{1}{R/\mu} \frac{u^2}{(1/\mu)u_1 u_2} . \] \tag{18}

We observe that as \( u \to u_1 \),
\[ \frac{dT}{dh} \to \frac{1}{R/\mu} \frac{u_1}{(1/\mu)u_2} . \] \tag{19}

or again by (13'),
\[ \left( \frac{dT}{dh} \right)_{\infty} = \frac{1}{R/\mu} \frac{1}{(1/\mu) + u_1 u_2} . \] \tag{19'}

From (17') we get, after some minor reductions,
\[ \frac{gdh}{(R/\mu)T} = - \frac{u_1 u_2 du}{u(u - u_1)(u + u_2)} . \] \tag{20}

(20) is identical with Milne's equation (80) of his Bakerian Lecture. (Of course our \( u_1 \) and \(-u_2\) are the roots of a slightly different quadratic equation in \( u \) and our variable \( u \) itself is differently defined.)

Denoting a mean value of \( T \) by \( \bar{T} \) we find that (Bakerian Lecture, equation (81))
\[ h - h_0 = \frac{(R/\mu)T}{g} \left[ \log \left( \frac{u - u_2}{u_1 + u_2} \right) - \frac{u_1}{u_1 + u_2} \log \left( \frac{1 - u}{u_1} \right) - \frac{u_1}{u_1 + u_2} \log \left( \frac{1 + u}{u_2} \right) \right] . \] \tag{21}

We have now obtained the complete solution to our problem, with, of course, certain simplifying assumptions.

4. Numerical Applications.—We use the above formulae to derive the physical characteristics of the photospheric layers of the "Sun," i.e. of a "model" whose mass, luminosity, surface gravity, \( T_1 \) etc., are the same as those of the Sun. Now Russell* estimates that the "level of ionization" in the solar atmosphere is such that the atoms of ionization-potential 8.3 volts are 50 per cent. ionized. Hence in our numerical work we take \( \bar{x} = 8.3 \) volts and \( \bar{x} = 1/\mu \). Also, to determine the value of \( a \) in (2), we need in addition a mean atomic weight. We assume this to be 1.5, to be in conformity with the estimates of Russell and Unsöld regarding the hydrogen-abundance in stellar atmospheres. Also, for "\( \mu \)" in equation (18) we take \( 0.75 H \).

The results of the calculation are given below:
\[ a = 2.58 \times 10^{21} ; \]
\[ \bar{x} = 1/\mu ; \]
\[ \mu = 0.75 m_H ; \]
atomic weight = 1.5 ;
\[ \bar{x} = 8.3 \) volts ; \]
\[ g = 2.74 \times 10^4 \text{ cm. sec.}^{-2} ; \]
\[ T_1 = 5740 \text{ degrees}. \]

Using these values,

\[ A = \frac{4\pi c G_\odot M T_\odot^{\frac{3}{2}}}{L} \left( \frac{1}{3} a \alpha \right) = 6582; \]

\[ u_1 = 74.02; \]
\[ u_2 = 74.86; \]
\[ u_1 + u_2 = 148.88. \]

The physical structure of the photospheric layers calculated with the above data and the formulæ of the previous section is tabulated in Table I.

It is seen that the physical characteristics thus calculated offer a highly plausible description of the state of the Sun's photospheric layers, the one exception being that the pressures appear to be perhaps rather large. This is a consequence of the low absorption coefficient consequent on the 50 per cent. ionization assumed in the calculations. To illustrate the effect of ionization, therefore, we repeat the calculations with the following data:

\[ \bar{x} = 7.812 \text{ volts}; \]
\[ \bar{x} = 1; \]
\[ \mu = 0.5 m_H; \]
atomic weight = 1;
\[ \alpha = 3.43 \times 10^{21}; \]
\[ A = 1650; \]
\[ u_1 = 36.86; \]
\[ u_2 = 37.70; \]
\[ u_1 + u_2 = 74.56. \]

The results of the calculations are contained in Table II. We note the general reduction in pressure and the increased absorption coefficient.

<table>
<thead>
<tr>
<th>Optical Thickness</th>
<th>Radiation-pressure, ( p' ), in Dynes cm.(^{-2} )</th>
<th>Gas-pressure, ( p ), in Dynes cm.(^{-2} )</th>
<th>Temperature Gradient ( dT/dh ), Degrees</th>
<th>Temperature</th>
<th>Depth ( h ), Km.</th>
<th>Density ( \rho ), Gms. cm.(^{-3} )</th>
<th>Absorption Coefficient ( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 0.0 0.0058 1.399 12.3</td>
<td>8481 0.0945 447.6 2.309 \times 10^{-11}</td>
<td>28.43</td>
<td>0.0 0.0 0.0121 1.432 25.27</td>
<td>4868 0.3781 591.6 4.716 \times 10^{-11}</td>
<td>57.07</td>
<td>0.0 0.0 0.0193 1.788 82.38</td>
<td>5147 2.364 846.6 1.453 \times 10^{-10}</td>
</tr>
</tbody>
</table>

On comparing Tables I and II (particularly Table II) with table IX of Milne's Bakerian Lecture, it will be observed that there is practically no
difference between the *predicted* distribution of temperature, density, pressure, etc., in the photospheric layers as \( \tau \) varies, and that deduced by Milne on the basis of his empirical law for the absorption coefficient. The physical structure of the photospheric layers as deduced by Milne has been generally regarded as representing what is "likely," and it is satisfactory that our results, based on a purely physical theory of the absorption coefficient, should be in general agreement.

**Table II**

<table>
<thead>
<tr>
<th>( u )</th>
<th>( \tau )</th>
<th>Radiation-pressure, ( \rho' ) ( \text{in Dynes cm}^{-2} )</th>
<th>Gas-pressure, ( \rho ) ( \text{in Dynes cm}^{-2} )</th>
<th>Temperature ( T ) ( \text{Degrees} )</th>
<th>Temperature Gradient ( dT/dh ) ( \text{Degrees Km}^{-1} )</th>
<th>Depth ( h ) ( \text{Km} )</th>
<th>Density ( \rho \times 10^{12} ) ( \text{Gms cm}^{-3} )</th>
<th>Absorption Coefficient ( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.386</td>
<td>0</td>
<td>4820</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.0052</td>
<td>1.397</td>
<td>6.146</td>
<td>4839</td>
<td>0.0629</td>
<td>468.2</td>
<td>7.693</td>
<td>57.15</td>
</tr>
<tr>
<td>10</td>
<td>0.213</td>
<td>1.432</td>
<td>12.04</td>
<td>4865</td>
<td>0.2514</td>
<td>607.5</td>
<td>15.71</td>
<td>113.2</td>
</tr>
<tr>
<td>15</td>
<td>0.522</td>
<td>1.494</td>
<td>19.06</td>
<td>4921</td>
<td>0.567</td>
<td>844.3</td>
<td>24.57</td>
<td>169.0</td>
</tr>
<tr>
<td>20</td>
<td>1.027</td>
<td>1.606</td>
<td>28.63</td>
<td>5008</td>
<td>1.006</td>
<td>943.2</td>
<td>34.97</td>
<td>222.3</td>
</tr>
<tr>
<td>25</td>
<td>1.937</td>
<td>1.789</td>
<td>41.20</td>
<td>5147</td>
<td>1.571</td>
<td>1065</td>
<td>48.48</td>
<td>268.6</td>
</tr>
<tr>
<td>30</td>
<td>3.805</td>
<td>2.180</td>
<td>62.36</td>
<td>5409</td>
<td>2.263</td>
<td>1223</td>
<td>69.98</td>
<td>314.2</td>
</tr>
<tr>
<td>35</td>
<td>6.380</td>
<td>2.712</td>
<td>89.15</td>
<td>5712</td>
<td>2.738</td>
<td>1374</td>
<td>94.51</td>
<td>332.7</td>
</tr>
<tr>
<td>40</td>
<td>10.861</td>
<td>3.644</td>
<td>134.3</td>
<td>6137</td>
<td>3.080</td>
<td>1568</td>
<td>132.2</td>
<td>333.5</td>
</tr>
<tr>
<td>45</td>
<td>1.731</td>
<td>4.985</td>
<td>200.4</td>
<td>6651</td>
<td>3.26</td>
<td>1783</td>
<td>182.5</td>
<td>323.7</td>
</tr>
<tr>
<td>50</td>
<td>2.794</td>
<td>7.196</td>
<td>314.3</td>
<td>7292</td>
<td>3.35</td>
<td>2046</td>
<td>261.0</td>
<td>306.3</td>
</tr>
<tr>
<td>55</td>
<td>6.741</td>
<td>15.40</td>
<td>781.2</td>
<td>8819</td>
<td>3.40</td>
<td>2677</td>
<td>536.5</td>
<td>274.2</td>
</tr>
</tbody>
</table>

But our results differ in one point from those of Milne, in that our linear depths (in kms.) are uniformly about 40 times larger. This is due to the fact that we have assumed in our calculations a very low mean molecular weight, 0.75\( m_H \) as against Milne's 20\( m_H \).

The predicted pressures and the absorption coefficients are of the expected order, though it should be remembered that this is largely connected with the low mean atomic weight used in our calculations.

5. Alternative Treatment.—It will be noticed that for the observed order of magnitudes for \( L, M, T, \) etc., \( A \) is very large—of order 1000. In fact we have

\[
A = 1097 \left( \frac{M}{M_\odot} \right) \left( \frac{L}{L_\odot} \right) \left( \frac{T_1}{5740} \right)^{\frac{3}{2}} \left( \frac{c'}{a} \right)^{1 + \frac{x}{a^2}} \ (22)
\]

where \( c' \) is the coefficient in the expression \( (2') \) for \( \kappa \), calculated with \( x = 8.3 \) volts and \( a = 1.5 \). From \( (13') \) we easily obtain for \( u_1 \) and \( u_2 \),

\[
u_1 = \sqrt{\beta^2 + 0.18} - 0.42;
\]

\[
u_2 = \sqrt{\beta^2 + 0.18} + 0.42;
\]

where

\[
\beta = \frac{1}{16} A \ (22')
\]

Since \( \beta \) is very large we see that \( u_1 \) and \( u_2 \) are both practically equal to \( \beta \). Hence we can rewrite \( (12) \) as

\[
\frac{19 dp'}{16 p'} = \frac{udu}{\beta^2 - u^2} \ (22'')
\]
Mr. S. Chandrasekhar,

(22") integrates immediately, and we obtain, remembering that when \( u = 0 \), \( p' = p_0' \),

\[
p' = p_0' \left( 1 - \frac{u^2}{\beta^2} \right)^{-\frac{3}{2}} \tag{23}\]

By (9) we have for the gas-pressure,*

\[
p = up_0' \left( 1 - \frac{u^2}{\beta^2} \right)^{-\frac{3}{2}} 2^{-\frac{3}{4}}. \tag{23'}\]

In obtaining (23') we have used the relation, \( p_0' = \frac{1}{2} p_1' \).

To determine the temperature gradient we proceed as before, and by (17') we have

\[
\frac{dT}{dh} = \frac{1}{A(R/\mu)} u^2. \tag{24}\]

By (23), on the other hand,

\[
1 - \frac{u^2}{\beta^2} = \left( \frac{T_0}{T} \right)^{\frac{3}{2}}; \tag{25}\]

or

\[
u^2 = \beta^2 \left[ 1 - \left( \frac{T_0}{T} \right)^{\frac{3}{2}} \right]. \tag{25'}\]

Inserting (25') in (24) and using the relation (22') we obtain

\[
\frac{dT}{dh} = \frac{4}{19} \frac{g}{R/\mu} \left[ 1 - \left( \frac{T_0}{T} \right)^{\frac{3}{2}} \right]. \tag{26}\]

Now as we descend into the photospheric layers the second term in brackets in the above equation very soon becomes negligibly small because of the high power to which a quantity less than unity is raised. Thus, if \( T = 1.8 T_0 \), \( (T_0/T)^{\frac{3}{2}} = 0.2 \), and if \( T = 1.275 T_0 \), \( (T_0/T)^{\frac{3}{2}} = 0.1 \). Hence we have, to a good approximation, for layers not immediately near the surface and for which \( T \geq 1.25 T_0 \),

\[
\frac{dT}{dh} = \frac{4}{19} \frac{g}{R/\mu} \tag{27}\]

or

\[ h - h_0 = \frac{19}{4} \frac{R/\mu}{g} T. \tag{28}\]

Comparing (27) with (19'), and remembering that \( u_1^{-1} \ll 1 \), we find that what we have called previously " \( \frac{dT}{dh} \) " is in fact a very good approximation for layers not immediately near the boundary. Equation (26) in fact shows the extent to which we have to descend in the photospheric layers

* From (23) and (23') we have

\[
p/p' \propto u \left( 1 - \frac{u^2}{\beta^2} \right)^{-\frac{3}{2}} \propto u p_0' \propto u T_0^4. \]

But as \( T \) increases, \( u \to \beta \) and is practically constant. Hence that " \( p/p' \) ultimately varies as \( T_0^4 \) " comes out from this treatment also.
before our approximation becomes valid. The temperature will have to increase by about 25 per cent. over the boundary temperature. In fig. 1 the results of Tables I and II are plotted—the temperature against the depth—and we see at once how soon the linear relation between \( h \) and \( T \) sets in—

![Graph](image)

**Fig. 1.**—Physical structure of the photospheric layers of the "Sun." The plot of temperature against depth to exemplify the linear relation between the temperature \( T \) and the linear depth \( h \) for layers not immediately near the boundary.

- **I**—Corresponds to the data of Table I of this paper.
- **II**—Corresponds to the data of Table II of this paper.
- **M**—Corresponds to the data of Table X of Milne's Bakerian Lecture.

which equations (26) and (27) jointly predict. (The dotted curve is the one obtained by using Milne's tabulated values. The linear relation is exemplified here also.*)

Again, (28) can be rewritten, with the help of (23), in the form

\[
h - h_0 = \frac{19}{4} \frac{(R/\mu)}{g} T_0 \left(1 - \frac{\nu^2}{\beta^2}\right).\tag{29}\]

* If \( \kappa \propto P/T^\beta \) then the relation (26) will be replaced by

\[
\frac{dT}{dh} = \frac{4}{17} \frac{g}{(R/\mu)} \left[1 - \left(\frac{T_0}{T}\right)^\beta\right].\tag{26'}\]

and instead of (28) we have

\[
h - h_0 = \frac{17}{4} \frac{(R/\mu)}{g} T.\tag{28'}\]
Equation (29) is of course true only after we have descended some distance into the photospheric layers.

The solution for \( h \) for layers near the boundary (\( T < 1.25 T_0 \)) can be obtained by a procedure analogous to that employed in § 3. Now since \( T \) deviates only a little (less than 25 per cent.) from the boundary value, the use of a "mean" temperature \( \overline{T} \) is amply justified. The solution is (cf. equation (21))

\[
\frac{h-h_0}{g} = \frac{(R/\mu)\overline{T}}{g} \left[ \log u - \frac{1}{2} \log \left( \frac{1 + \frac{u_2}{\beta^2}}{1 - \frac{u_2}{\beta^2}} \right) \right],
\]

(30)

where \( \overline{T} \), as before, represents some mean value of \( T \) for the layers traversed. (30) can be rewritten in a more convenient form. By (23'),

\[
\frac{h-h_0}{g} = \frac{(R/\mu)\overline{T}}{g} \left[ \log \left( \frac{p}{p_0'} \right) - \log 2.1 \right],
\]

(31)

or

\[
\frac{h-h_0}{g} = \frac{(R/\mu)\overline{T}}{g} \left[ \log p - \log p_0' + 0.130 \right].
\]

(32)

Equations (23), (23'), (32), and (29) give the complete solution to our problem.

From (28) we can obtain one other useful piece of information. If we consider two stars with the same \( \mu \) and boundary temperature (or \( T_0 \)), then, since

\[
\frac{p}{p_0} = 1 + \frac{3}{2} \tau,
\]

it is clear that for both these stars equality of \( \tau \) signifies equality of \( p' \) and therefore of \( T \) at the corresponding points. Then relation (28) shows that if we neglect the arbitrary additive constant, the \( h \)'s for the two stars should be inversely as the surface gravities, or, to put it more precisely, for two stellar atmospheres of the same molecular weight \( \mu \) and boundary temperature \( T_0 \), the thicknesses of the photospheric layers between places of equal temperature \( T' \) and \( T'' \) (\( T' < T'' \)) in the two stars are inversely proportional to the surface gravities if \( T' > 1.25 T_0 \).

It is clear that even if \( T' \) is not greater than \( 1.25 T_0 \), the proportionality of the "thicknesses" to \( g^{-1} \) will still be approximately valid if \( T'' \) is sufficiently large. Thus if (following Milne) we adopt \( \tau = 4 \) as defining the greatest depth to which we can see at the centre of the disc, it follows that the thicknesses between \( \tau = 0.01 \) (say) and \( \tau = 4 \) should be roughly proportional to \( g^{-1} \). This result has previously been noted by Milne from his numerical calculations contained in his Bakerian Lecture, but the above gives the result a precise analytical form and shows that the proportionality \( (h'' - h') \propto g^{-1} \) should strictly hold good if the boundary temperatures are the same and the temperature \( T' \) (corresponding to \( h' \), the upper level of the two) is appreciably greater than \( T_0 \).

6. General Remarks.—It will be noted that the above calculations are not

* Actually \( \tau = 4 \) corresponds to \( T'' = 1.626 T_0 \) and \( \left( \frac{T_0}{T} \right)^\mu = 0.01 \).
strictly deduced from the physical theory alone. Thus we have used in our numerical calculations values for $\bar{x}$ near that deduced by Russell for the level of ionization from different considerations. This is not, however, an intrinsic defect in the method, and is essentially due to the fact that to secure simplicity in the analysis we had to introduce a mean value $\bar{x}$ for the degree of ionization instead of treating this as a variable as we ought to have done; and the penalty to be paid for this is that we have to submit to our method still having a semi-empirical air about it. For, the fact that we assume beforehand a mean value for $\bar{x}$ presupposes that we have already a knowledge of this from other sources. Thus for temperatures of the order of 6000°, if we take $\chi = 13.54$ volts (say) then it is clear a priori that $\bar{x}$ can only be too small and the results of the calculations cannot surely be regarded as "representative." For, if $\bar{x} \to 0$ then $\kappa \to 0$, $A \to \infty$, and $dT/dh \to 0$, and the photospheric layers tend to have an infinite extension. Hence to be "reasonable" we must have values of $\bar{x}$ not too small and which should not be inconsistent with our choice of $\bar{x}$ and the physical conditions—a knowledge of which is presupposed when we assume $\bar{x}$.*

7. Summary.—(1) The physical structure of the photospheric layers is studied on the basis of an accurate theory of the absorption coefficient.

(2) It is found that ultimately the ratio of the gas-pressure to the radiation-pressure increases as $T^4$.

(3) Numerical applications are made to two typical cases, and though the values are derived purely on the basis of the physical theory and certain plausible assumptions regarding the "level of ionization," there seems nothing "impossible" about the predicted physical characteristics of the photospheric layers. The pressures and the absorption coefficients are of the expected order.

(4) An alternative treatment of the photospheric problem is given which shows that the linear depths are proportional to the temperatures for layers not immediately near the boundary.

(5) It is also pointed out that for two stars with the same $\mu$ and $T_0$ at points of equal $\tau$ (not too small) the linear depths are inversely as the surface gravities.

In conclusion I wish to record my thanks to Professor E. A. Milne, at whose suggestion the calculations presented here were undertaken.

* The consistency of the calculations can be tested as follows: To start the calculation we assume a mean value for $\bar{x}$. Then the method of the paper provides a complete march of $T$ and $p$ down the photospheric layers, and with these calculated values of $T$ and $p$ we could calculate $x$ at each level. The mean of these $x$'s should be equal to the initial value of $\bar{x}$ assumed. If it is different the calculation should be repeated with a new starting value of $\bar{x}$. 